FINAL FOR 110 405
FALL 2003

Answer all six questions. The first two questions are True/False. Fully justify your answer for the last four questions.

Question 1. (10 points; True/False). A function which is differentiable everywhere must also be continuous.

True.

Question 2. (10 points; True/False). The continuous image of a closed set is always closed.

False.

Question 3. (20 points). One of our theorems said that a continuous function on a compact set must be uniformly continuous. Prove this.

There are two approaches, using either limit points or open covers. We will do it by contradiction.

Suppose f is not uniformly cts. Then, for some $\epsilon > 0$, we get points x_n and y_n with

$$|x_n - y_n| < 1/n\,,$$

and

$$|f(x_n) - f(y_n)| \geq \epsilon\,.$$

By compactness, a subsequence x'_n converges to a point x in the set. The first equation above implies that y'_n goes to x as well. Since f is continuous, $f(x_n) \to f(x)$ and $f(y_n) \to f(x)$, contradicting the second equation.

Question 4. (20 points). Suppose that f is a continuous function defined on an interval. Show that the image is also an interval.

The defining property of an interval is that if a and b are in it, then so is every point between a and b.

Given a and b in the image, choose points x and y with $f(x) = a$ and $f(y) = b$. Suppose that c is between a and b. Since f is cts, the IMV theorem gives a z between x and y with $f(z) = c$, as desired.

Question 5. (20 points). Suppose that $f : \mathbf{R} \to \mathbf{R}$ satisfies $f \geq 0$ and $f'' \leq 0$ everywhere. Show that f must be constant.

Since $f'' \leq 0$ everywhere, f lies below its tangent lines. However, $f \geq 0$ then means that every one of these tangent lines must also be ≥ 0, i.e, must be horizontal (since any non-horizontal line eventually crosses the x-axis). This says that $f' = 0$ everywhere. By the meanvalue theorem, f is constant.

Question 6. (20 points). Suppose that f is a positive, continuous function on $[0, 1]$. First (10 points), prove that for any $\lambda > 0$ we have

$$2 \int_0^1 f \leq \lambda^2 + \lambda^{-2} \int_0^1 f^2 .$$

At each point x, we can write

$$0 \leq (\lambda^{-1} f(x) - \lambda)^2 = \lambda^2 + \lambda^{-2} \int_0^1 f^2 - 2 f(x) .$$

Integrating this gives the claim.

Second (the other 10 points), show that

$$\int_0^1 f \leq \left(\int_0^1 f^2 \right)^{1/2} .$$

The short answer is that this follows from the first part with $\lambda^2 = \left(\int_0^1 f^2 \right)^{1/2}$.

I will explain now how to guess the right value of λ:

Define a function $g(\lambda) = \lambda^2 + \lambda^{-2} \int_0^1 f^2$. It is easy to see that g is C^1 (it is a rational function) and $g \to \infty$ as $\lambda \to 0$ or $\lambda \to \infty$. It follows that the minimum (and not just the infimum) of g is achieved.

Differentiating g gives

$$g'(\lambda) = 2 \lambda - 2\lambda^{-3} \int_0^1 f^2 .$$

In particular $g'(\lambda) = 0$ exactly when

$$\lambda^4 = \int_0^1 f^2 .$$

It follows that the minimum of g is at $\lambda^2 = \left(\int_0^1 f^2 \right)^{1/2}$.