Math 104 **Fall 2009**

Homework 7

Due Wednesday, December 2, 2009

Do five of the following exercises.

1. If A is an $n \times n$ matrix with characteristic polynomial

$$P(\lambda) = (\lambda - \lambda_1)^{d_1} \dots (\lambda - \lambda_k)^{d_k}$$

what is the trace of A? what is the determinant of A?

2. A skew Hermitian matrix is a matrix obeying $A^* = -A$.
   (a) Show that $A = U\Lambda U^*$, with Λ diagonal and for a unitary U. (Hint: this is not a difficult question and you should think about how you could get back to the case you know; that is, the case where the matrix is Hermitian.)
   (b) Show that the eigenvalues are imaginary and the eigenvectors orthogonal.
   (c) Show that $A + I$ is invertible.
   (d) Show that $(I - A)(I + A)^{-1}$ is an orthogonal matrix.

3. Suppose A is positive semidefinite. Can you a find a square root of this matrix? In other words, can you find a matrix B such that $B^2 = A$? If yes, explain how you would construct it. If no, explain why no such matrix exists.

4. Suppose you have n vectors $x_1, \dots, x_n$ in $\mathbb{R}^m$. In class, we have seen that the first principal component is the unit-normed vector $u \in \mathbb{R}^m$ so that the projections of those vectors onto u have maximum variance.

   Another way to look at this is as follows: consider a line $\mathcal{L}$ going through some point $x_0 \in \mathbb{R}^m$ and with some orientation $u \in \mathbb{R}^m$, $\|u\| = 1$ (the equation of this line is $x_0 + tu$ where t is a scalar). Now consider the line that is closest to the point in the sense that it minimizes

$$\sum_{i=1}^{n} |\text{distance}(x_i, \mathcal{L})|^2$$

   (the sum of squares of the distances between the x_i's and the line).
   (a) Show that the slope of the closest line is the first principal component.
   (b) Show that this line goes through the average vector $\bar{x} = \sum_{i=1}^{n} x_i$.

5. Problem 23.1 in Trefethen and Bau. (In this exercise U is the upper triangular factor so that with $L = U^*$, $A^*A = U^*U = LL^*$.)

6. Suppose A is positive semidefinite. Show that the maximum eigenvalue of A, denoted by $\lambda_{\max}$, is given by the so-called *Rayleigh quotient*

$$\sup_{w \neq 0} \frac{w^* A w}{w^* w}.$$