# Math 104 : Midterm

*Instructions:* Complete the following 4 problems. Remember to show all your work. No notes or calculators are allowed. Please sign below to indicate you accept the honor code.

Name: ________________________________________

SUID: ________________________________________

Signature: ________________________________________

| Problem | 1 | 2 | 3 | 4 | Total |
|---|---|---|---|---|---|
| Score | | | | | |

**Problem #1.** (20 pts) Let $\mathbf{w}_1$, $\mathbf{w}_2$ and $\mathbf{w}_3$ be three vectors in $\mathbb{C}^3$. Let

$$\begin{aligned}
\mathbf{v}_1 &= \mathbf{w}_1 - \mathbf{w}_3, \\
\mathbf{v}_2 &= \mathbf{w}_1 + \mathbf{w}_2, \\
\mathbf{v}_3 &= \mathbf{w}_1 + \lambda \mathbf{w}_3, \text{ and} \\
\mathbf{v}_4 &= 2\mathbf{w}_1 + \mathbf{w}_2 - \mathbf{w}_3.
\end{aligned}$$

Where here $\lambda \in \mathbb{C}$. For what value $\lambda_0$ is it always true that when $\lambda = \lambda_0$, $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ and $\mathbf{v}_4$ never span $\mathbb{C}^3$. Justify your answer. (Hint: Rewrite the problem using matrices).

**Answer:**

Let us set

$$V = \begin{bmatrix} \mathbf{v}_1 & |\mathbf{v}_2 & |\mathbf{v}_3 & |\mathbf{v}_4 \end{bmatrix}$$

and

$$W = \begin{bmatrix} \mathbf{w}_1 & |\mathbf{w}_2 & |\mathbf{w}_3 \end{bmatrix}$$

Then we have

$$V = WA$$

where

$$A = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & \lambda & -1 \end{bmatrix}$$

The $\mathbf{v}_i$ do not span $\mathbb{C}^3$ when and only when $dim R(V) \leq 2$. By the rank-nullity theorem this occurs when and only when $dim N(V) \geq 1$. Notice that $N(A) \subset N(V)$ and so $dim N(V) \geq dim N(A)$. Applying the Gaussian elimination algorithm to $A$ one arrives after a sequence of row operations to $A'$ with

$$A' = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & \lambda + 1 & 0 \end{bmatrix}$$

Notice if $\lambda + 1 = 0$ then $rref(A)$ has 2 pivots. Otherwise $rref(A)$ has 3 pivots. In the former case, $dim N(A) = dim N(rref(A)) = 4 - 2 = 2$ while in the latter $dim N(A) = dim N(rref(A)) = 4 - 3 = 1$. In particular, $\lambda_0 = -1$ always ensures that the $\mathbf{v}_i$ do not span. Notice that if the $\mathbf{w}_i$ are linearly independent and $\lambda_0 \neq -1$ then $dim N(V) = dim N(A) = 1$ and so $dim R(V) = 3$. In particular, the $\mathbf{v}_i$ would span $\mathbb{C}^3$ in this case.

**Problem #2.** (30 pts) Let

$$\mathbf{v} = \begin{bmatrix} 2\sin\theta \\ -2\cos\theta \end{bmatrix} \in \mathbb{R}^2$$

Let $A \in \mathbb{R}^{2\times 2}$ denote the matrix which gives orthogonal projection onto $span(\mathbf{v})$.

a) Determine $A$.

**Answer:**
As $\mathbf{v} \neq 0$, $span(\mathbf{v})$ is one dimensional. Hence there is a unit length vector $\mathbf{q} \in span(\mathbf{v})$ and $span(\mathbf{q}) = span(\mathbf{v})$. In particular, take

$$\mathbf{q} = \frac{1}{2}\mathbf{v} = \begin{bmatrix} \sin\theta \\ -\cos\theta \end{bmatrix}.$$

The projecter $A$ is then given by:

$$A = \mathbf{q}\mathbf{q}^* = \begin{bmatrix} \sin\theta & -\cos\theta \end{bmatrix} \begin{bmatrix} \sin\theta \\ -\cos\theta \end{bmatrix} = \begin{bmatrix} \sin^2\theta & -\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \cos^2\theta \end{bmatrix}.$$

b) Determine $N(A)$ and $R(A)$.

**Answer:**
As $A$ is a projector onto $span(\mathbf{v})$ one must have $R(A) = span(\mathbf{v})$ by definition. Because $A$ is an orthogonal projector one also has $R(A)^\perp = N(A)$. Notice that for

$$\mathbf{w} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

one has $\mathbf{w} \neq 0$ and $\langle \mathbf{w}, \mathbf{v} \rangle = 0$. So $\mathbf{w} \in R(A)^\perp = N(A)$ and is non-trivial. As $dimN(A) = 1$ (by the rank-nullity theorem for instance) one then has $N(A) = span(\mathbf{w})$.

c) Determine a full $QR$ factorization of $A$.

**Answer:**
The first column of $A$ is

$$\mathbf{a}_1 = \begin{bmatrix} \sin^2\theta \\ -\sin\theta\cos\theta \end{bmatrix} = \sin\theta \begin{bmatrix} \sin\theta \\ -\cos\theta \end{bmatrix} = \sin\theta\mathbf{q}_1.$$

Here $\mathbf{q}_1$ is a unit vector and $span(\mathbf{a}_1) \subset span(\mathbf{q}_1)$. Hence is the correct choice for the $QR$ factorization. In this case $r_{11} = \sin\theta$. We now set

$$r_{12} = \langle \mathbf{q}_1, \mathbf{a}_2 \rangle = -\sin^2\theta\cos\theta - \cos^3\theta = -\cos\theta\left(\sin^2\theta + \cos^2\theta\right) = -\cos\theta$$

Then with $\mathbf{a}_2$ the second column of $A$

$$\hat{\mathbf{q}}_2 = \mathbf{a}_2 - r_{12}\mathbf{q}_1 = 0$$

Thus we just need to pick $\mathbf{q}_2$ orthogonal to $\mathbf{q}_1$ and set $r_{22} = 0$ we determined such a $\mathbf{q}_2$ in b) Hence we have the full $QR$ factorization

$$A = QR = \begin{bmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{bmatrix} \begin{bmatrix} \sin\theta & -\cos\theta \\ 0 & 0 \end{bmatrix}$$

We note that the $Q$ term is unitary (as the columns are orthonormal) and the $R$ term is upper-triangular so this is indeed a $QR$ factorization.

**Problem #3.** (20 pts) Let $A, B \in \mathbb{C}^{m\times m}$ suppose that $AB = 0$ and $BA = 0$.

a) What, if any, is the relationship between the null space of $A$ and the column space of $B$? Justify your answer.

**Answer:**
One has $R(B) \subset N(A)$ as if $\mathbf{v} \in R(B)$ then there is a $\mathbf{w}$ so that $\mathbf{v} = B\mathbf{w}$ and then $A\mathbf{v} = AB\mathbf{w} = 0\mathbf{w} = 0$ and so $\mathbf{v} \in N(A)$. There is no other relationship since for instance if $A = 0$ and $B = I$ then $AB = 0 = BA$ but $R(B) = \mathbb{C}^m$ while $R(A) = \{0\}$.

b) Show that either $\dim N(A) \geq \frac{m}{2}$ or $\dim N(B) \geq \frac{m}{2}$.

**Answer:**

By part a) we have that $R(B) \subset N(A)$. Hence by the basis extension theorem we have $dimR(B) \leq dimN(A)$. By the rank-nullity theorem applied to $B$ $dimR(B) + dimN(B) = m$. Thus,

$$m = dimR(B) + dimN(B) \leq dimN(B) + dimN(A)$$

This implies either $dimN(B)$ or $dimN(A)$ is larger than $m/2$ as claimed.

**Problem #4.** (30 pts)

a) Suppose that $A, B \in \mathbb{C}^{m\times m}$ are unitary matrices. Verify that $A^*$ and $AB$ are also unitary. (Hint: Use the algebraic properties of the adjoint)

**Answer:**
Since $A$ is unitary one has $AA^* = I$ taking the adjoint of this implies $(AA^*)^* = I^* = I$. Thus $(A^*)^*A^* = I$ that is $(A^*)^{-1} = (A^*)^*$ which means $A^*$ is unitary. Similarly, $BB^* = I$ so $(AB)^*AB = B^*A^*AB = B^*IB = B^*B = I$ which implies $(AB)^{-1} = (AB)^*$ and so $AB$ is unitary.

b) Let

$$\mathbf{v}_1 = \frac{1}{5}\begin{bmatrix} 3 \\ 0 \\ -4 \end{bmatrix}, \mathbf{v}_2 = \frac{1}{5}\begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \in \mathbb{R}^3$$

Verify that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthonormal basis of $\mathbb{R}^3$. Justify your answer.

**Answer:**
By direct computation one verifies that $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = \delta_{ij}$ (i.e. is 1 when $i = j$ and otherwise 0). This implies the set is orthonormal. Any orthogonal set of vectors is automatically linearly independent and as there are three elements in the set and $\dim \mathbb{C}^3 = 3$ we must have that the $\mathbf{v}_i$ also span $\mathbb{C}^3$ and hence are a basis.

c) Let

$$\mathbf{w}_1 = \frac{\sqrt{2}}{2}\begin{bmatrix}1\\0\\1\end{bmatrix}, \mathbf{w}_2 = \begin{bmatrix}0\\-1\\0\end{bmatrix}, \mathbf{w}_3 = \frac{\sqrt{2}}{2}\begin{bmatrix}-1\\0\\1\end{bmatrix} \in \mathbb{R}^3$$

be a set of orthonormal vectors. Determine the orthogonal matrix $U \in \mathbb{R}^{3\times 3}$ so that $U\mathbf{v}_i = \mathbf{w}_i$ for $i = 1, 2, 3$ here the $\mathbf{v}_i$ are given in b). (Hint: Use part a) )

**Answer:**
Let us write

$$V = \begin{bmatrix}\mathbf{v}_1 | & \mathbf{v}_2 | & \mathbf{v}_3\end{bmatrix} \text{ and } W = \begin{bmatrix}\mathbf{w}_1 | & \mathbf{w}_2 | & \mathbf{w}_3\end{bmatrix}.$$

As both $\{\mathbf{v}_i\}$ and $\{\mathbf{w}_i\}$ are orthonormal bases of $\mathbb{C}^3$ both $V$ and $W$ are orthogonal matrices. We note that orthogonal and unitary are the same in this context as both matrices have real entries. Then we have

$$W = UV$$

so

$$U = WV^{-1} = WV^\top (= WV^*)$$

By part a) $U$ is unitary and has real entries so is orthogonal. We compute explicitly

$$U = \begin{bmatrix}3\sqrt{2}/10 & -\sqrt{2}/2 & -2\sqrt{2}/5\\ -4/5 & 0 & -3/5\\ 3\sqrt{2}/10 & \sqrt{2}/2 & -2\sqrt{2}/5\end{bmatrix}$$