Solutions to Math 104 Final Exam — Dec. 10, 2010

1. (20 points) Consider the following matrices that depend on a parameter $\lambda \in \mathbb{C}$:

$$A_\lambda = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ \lambda & -1 \end{bmatrix}$$

(a) For each λ, determine $||A_\lambda||_F$ the Frobenius norm of A_λ.

(Note: technically $\lambda \in \mathbb{C}$ which complicates things). The Frobenius norm of A_λ is the square-root of the sum of the squares (of the modulus) of the entries of A_F.

$$||A||_F = \sqrt{1^2 + 2^2 + (-1)^2 + |\lambda|^2} = \sqrt{6 + |\lambda|^2}$$

(b) For each λ, determine $||A_\lambda||_2$ the induced 2-norm of A_λ.

In order to compute the two norm of A_λ we compute the largest singular value σ_1 of A. This is because, by definition, $||A_\lambda||_2 = \sigma_1$. We have established in class that the singular values are the square roots of the eigenvalues of the matrix

$$B_\lambda = A_\lambda^* A_\lambda = \begin{bmatrix} 5 + |\lambda|^2 & -\bar{\lambda} \\ -\lambda & 1 \end{bmatrix}$$

The characteristic polynomial of this matrix is $p_B(z) = det(B - zI) = z^2 - (6 + |\lambda|^2)z + 5$ so by the quadratic formula the eigenvalues are

$$\alpha_\pm = \frac{6 + |\lambda|^2 \pm \sqrt{|\lambda|^4 + 12|\lambda|^2 + 16}}{2}$$

Hence, the largest singular value of A is

$$\sigma_1 = \sqrt{\frac{6 + |\lambda|^2 + \sqrt{|\lambda|^4 + 12|\lambda|^2 + 16}}{2}}$$

2. (20 points) Let $A \in \mathbb{C}^{m\times m}$ be hermitian (i.e. $A = A^*$). Let $P \in \mathbb{C}^{m\times m}$ be the matrix representing orthogonal projection onto $N(A)$. Please show that $X = A + P$ is invertible. (Hint: Think about the four fundamental subspaces of A).

In order to show that X is invertible is suffices to show that $N(X) = \{0\}$. To that end we note that $R(P) = N(A)$ by definition of P and that $R(A) = R(A^*)$ as A is hermitian. Now suppose that $\mathbf{x} \in N(X)$ so $X\mathbf{x} = 0$ then $(A + P)\mathbf{x} = 0$ so that $\mathbf{y} = A\mathbf{x} = -P\mathbf{x}$. In other words, $\mathbf{y} \in R(A)$ and $\mathbf{y} \in R(P)$ and so $\mathbf{y} \in N(A) \cap R(A^*)$. As $N(A)$ and $R(A^*)$ are complementary spaces this means that $\mathbf{y} = 0$. Hence, $\mathbf{x} \in N(A)$ and so $\mathbf{x} \in R(P)$. But then $\mathbf{x} = P\mathbf{x} = \mathbf{y} = 0$.

3. (20 points) Let $A \in \mathbb{R}^{4\times 2}$ be the matrix

$$A = \begin{bmatrix} 1 & -7 \\ 1 & 1 \\ 3 & -7 \\ 5 & -9 \end{bmatrix}$$

(a) Find a reduced QR factorization of A i.e. $A = \hat{Q}\hat{R}$.

To begin the QR factorization algorithm we normalize the first column of A this yields

$$\mathbf{q}_1 = \frac{1}{6}\begin{bmatrix} 1 \\ 1 \\ 3 \\ 5 \end{bmatrix}$$

and $r_{11} = 6$ the length of the first column. We next compute the inner product between $\mathbf{q}_1$ and the second column to obtain $r_{12} = -12$ then subracting $r_{12}\mathbf{q}_1$ from the second column yields

$$\hat{\mathbf{q}}_2 = \begin{bmatrix} -5 \\ 3 \\ -1 \\ 1 \end{bmatrix}$$

This has length 6 so that $r_{22} = 6$ and

$$\mathbf{q}_2 = \frac{1}{6}\begin{bmatrix} -5 \\ 3 \\ -1 \\ +1 \end{bmatrix}$$

Hence

$$\hat{Q} = \frac{1}{6}\begin{bmatrix} 1 & -5 \\ 1 & 3 \\ 3 & -1 \\ 5 & 1 \end{bmatrix}$$

and

$$\hat{R} = \begin{bmatrix} 6 & -12 \\ 0 & 6 \end{bmatrix}$$

and the reduced QR factorization of A is $A = \hat{Q}\hat{R}$.

(b) Solve the following overdetermined system in the sense of least squares:

$$A\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

By the QR factorization we know that orthogonal projection onto $R(A)$ is given by $\hat{Q}\hat{Q}^*$. In particular, since $A = \hat{Q}\hat{R}$ to solve $A\mathbf{x} = \mathbf{b}$ in the sense of least squares, we solve

$$\hat{Q}\hat{R}\mathbf{x} = \hat{Q}\hat{Q}^*\mathbf{b}$$

that is to solve

$$\hat{R}\mathbf{x} = \hat{Q}^*\mathbf{b}.$$

In our situation, the RHS is

$$\begin{bmatrix} 2/3 \\ -1 \end{bmatrix}$$

and solving by back substitution gives

$$\mathbf{x} = \begin{bmatrix} -2/9 \\ -1/6 \end{bmatrix}$$

4. (30 points) Let $A \in \mathbb{C}^{m\times m}$ be a square matrix. Order the singular values σ_i of A by $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_m \geq 0$ and order the eigenvalues λ_i of A so $|\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_m| \geq 0$.

   (a) Show that $\sigma_1 \geq |\lambda_1|$.

   Let $\mathbf{x} \in \mathbb{C}^m$ be the eigenvector associated to λ_1 so that $A\mathbf{x} = \lambda_1\mathbf{x}$. We may normalize $\mathbf{x}$ so that $||\mathbf{x}||_2 = 1$. We then have

$$||A\mathbf{x}||_2 = ||\lambda_1\mathbf{x}||_2 = |\lambda_1|||\mathbf{x}||_2 = |\lambda_1|$$

   As a consequence,

$$|\lambda_1| \leq \sup_{\mathbf{x}\neq 0} \frac{||A\mathbf{x}||_2}{||\mathbf{x}||_2} = ||A||_2 = \sigma_1$$

   (b) Show that $\sigma_m \leq |\lambda_m|$ (Hint: Write $\mathbf{x}_m$, an eigenvector associated to λ_m, in terms of the right singular vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m$ of A).

   Let $A = U\Sigma V^*$ be a SVD of A. We let $\mathbf{v}_1, \ldots, \mathbf{v}_m$ be the columns of V, i.e. the right singular vectors. Now suppose that $\mathbf{x}_m$ is a (non-zero) vector so that $A\mathbf{x}_m = \lambda_m\mathbf{x}_m$. We may normalize so that $||\mathbf{x}_m||_2 = 1$. Writing $\mathbf{x}_m$ in terms of the ONB given by the $\mathbf{v}_i$ one has

$$\mathbf{x}_m = \sum_{i=1}^{m} c_i\mathbf{v}_i$$

   Note that since $||\mathbf{x}_m||_2 = 1$ one has $\sum_{i=1}^m |c_i|^2 = 1$ by the Pythagorean thereom. As a consequence $A\mathbf{x}_m = \sum_{i=1}^m \sigma_i c_i \mathbf{u}_i$. Hence by the Pythagorean theroem

$$|\lambda_m|^2 = ||A\mathbf{x}_m||_2^2 = \sum_{i=1}^{m} \sigma_i^2|c_i|^2$$

   Since $\sigma_i \geq \sigma_1$ one obtains

$$|\lambda_m|^2 \geq \sum_{i=1}^{m} \sigma_1^2|c_i|^2 = \sigma_1^2.$$

(c) Using part a), show that if $||A||_2 < 1$ then $I + A$ is nonsingular. Here $||A||_2$ is the induced 2-norm and

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is the 3×3 identity matrix.

(Note that I should really bethe $m \times m$ identity otherwise the problem makes no sense.) Since $||A||_2 < 1$ we have that $\sigma_1 < 1$ and so by part a) we have that $|\lambda_1| < 1$. In particular, if $\lambda \in \Lambda(A)$ then $|\lambda| < 1$. Since $\lambda \in \Lambda(I+A)$ if and only if $\lambda - 1 \in \Lambda(A)$ and $1 > |\lambda - 1| \geq 1 - |\lambda|$ we see that $|\lambda| > 0$. In other words no eigenvalue of $I + A$ is zero. As a consequence $N(I + A) = \{0\}$ and so $I + A$ is invertible.

5. (20 points) Let $A \in \mathbb{C}^{3\times 3}$ be the matrix

$$A = \begin{bmatrix} -1 & 3 & -2 \\ 0 & 3 & 1 \\ 0 & 4 & 1 \end{bmatrix}$$

Find a unitary matrix $Q \in \mathbb{C}^{3\times 3}$ so that

$$QA = \begin{bmatrix} 1 & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix}.$$

Here $*$ represents an unspecified number.

Notice that the first column of A is $-\mathbf{e}_1$ and of QA is $\mathbf{e}_1$. Hence, we must have $Q(-\mathbf{e}_1) = \mathbf{e}_1$. That is the first column of $\mathbf{e}_1$ is $-\mathbf{e}_1$. For Q to be unitary it must have orthonormal columns and hence Q has the form

$$Q = \begin{bmatrix} -1 & 0 \\ 0 & Q' \end{bmatrix}$$

where $Q' \in \mathbb{C}^{2\times 2}$ is untary. In addition, we want to have

$$Q' \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}$$

Now for Q' to be unitary it must preserve length. In particular, one must have $|x| = \sqrt{3^2 + 4^2} = 5$. We take $x = 5$.

We now have a number of choices we could make in finding Q'. In the spirit of the Householder algorithm we take Q' to be a reflection. In this, case we set

$$\mathbf{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} - \begin{bmatrix} 5 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$$

be the vector normal to the line midway between $[3, 4]^\top$ and $[5, 0]^\top$. The orthononal projecteion onto $span(\mathbf{v})$ is given by

$$P = \frac{vv^*}{||v||_2^2} = \frac{1}{5} \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}$$

Then we have (as we saw in class or can easily convince ourselves) that Q' is given by

$$Q' = I - 2P = \frac{1}{5} \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$$

Which we verify has the desired properties. Hence

$$Q = \frac{1}{5} \begin{bmatrix} -5 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 4 & -3 \end{bmatrix}$$

6. (30 points) (a) Let $T \in \mathbb{C}^{m\times m}$ be upper triangular. Show that if T is unitary then T is diagonal. (Hint: Use the fact that columns are orthogonal and induct on m).

We prove the result by induction on m. When $m = 1$ then any matrix is diagonal so we are done. Assume the result is true for all $m \times m$ upper-triangular matrices. We wish to prove it true for $(m+1) \times (m+1)$ upper-triangular matrices.
To that end we note that if $T \in \mathbb{C}^{(m+1)\times(m+1)}$ is upper triangular and unitary. Then the first column of T must be of the form $\mu\mathbf{e}_1$ (as T upper triangular) and the length of the first column is 1 (as T unitary) so $|\mu = 1$. Since every other column of T is orthogonal to the first column (as T is unitary) T has the form

$$T = \begin{bmatrix} \mu & 0 \\ 0 & T' \end{bmatrix}.$$

Where $T' \in \mathbb{C}^{m\times m}$ is upper triangular and unitary. In particular, by the induction hypthosis T' is diagonal and hence so is T.

(b) Let

$$A = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & a_{mm} \end{bmatrix} \in \mathbb{C}^{m\times m}$$

be a diagonal matrix. Show that if A is unitary then $|a_{ii}| = 1$ for $1 \le i \le m$.

Since the columns of a unitary matrix must be of unit length it is straightforward to see that $|a_{ii}| = 1$.

(c) Let $X \in \mathbb{C}^{m\times m}$ be unitary, use parts a), b) and the Schur factorization to show that X is unitarily diagonalizable (i.e. $\mathbb{C}^m$ has an orthonormal basis of eigenvectors) and that $\lambda \in \Lambda(X)$ implies $|\lambda| = 1$.

Write the Schur factorization of X as

$$X = QTQ^*$$

where $Q \in \mathbb{C}^{m\times m}$ is unitary and $T \in \mathbb{C}^{m\times m}$ is diagonal. For X to be unitary one has $X^* = X^{-1}$. On the one hand

$$X^* = (QTQ^*)^* = (Q^*)^*T^*Q^* = QT^*Q.$$

on the other

$$X^{-1} = (QTQ^*)^{-1} = (Q^*)^{-1}T^{-1}Q^{-1} = QT^{-1}Q^*.$$

Hence T is unitary and so by a) and b) is diagonal with entries on the diagonal all of length 1. In all cases the diagonal entries of T are the eigenvalues of X and in this case the columns of Q are the eigenvectors of X and so we have proved the claim.

7. (30 points) Let $A \in \mathbb{R}^{3\times 3}$ be the matrix

$$A = \begin{bmatrix} -3 & 2 & -2 \\ 0 & 1 & 0 \\ 2 & -1 & 2 \end{bmatrix}$$

(a) Find the eigenvalues of A and give their algebraic multiplicity.

By expanding along the middle column we see that the characteristic polynomial of A is:

$$p_A(z) = \det(zI - A) = (z-1)\left((z+3)(z-2) - 2(-2)\right) = (z-1)(z^2 + z - 6 + 4).$$

By inspection (or using the quadratic formula) this can be factored as

$$p_A(z) = (z-1)(z-1)(z+2).$$

Thus, the eigenvalues are $\lambda = -2$ with algebraic multiplicity 1 and $\lambda = 1$ with algebraic multiplicity 2.

(b) Verify that A is diagonalizable and find a basis of eigenvectors.

As

$$A - (-2)I = \begin{bmatrix} -1 & 2 & -2 \\ 0 & 3 & 0 \\ 2 & -1 & 4 \end{bmatrix}$$

we can do Gaussian elimination to see that an eigenvector associated to $\lambda = -2$ is

$$\mathbf{x}_1 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

Similarly, as

$$A - I = \begin{bmatrix} -4 & 2 & -2 \\ 0 & 0 & 0 \\ 2 & -1 & 1 \end{bmatrix}$$

Gaussian elimination shows that one has (linearly independent) eigenvectors

$$\mathbf{x}_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

one verifies that $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ form a linearly independent set and since $dim \mathbb{C}^3 = 3$ they must form a basis.

(c) Determine the matrices X and Λ so that X is non-singular and Λ is diagonal and so one has a factorization:

$$A = X\Lambda X^{-1}$$

If we let

$$X = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Using Gaussian elimination one computes

$$X^{-1} = \frac{1}{3}\begin{bmatrix} -1 & 2 & -2 \\ 2 & -1 & 4 \\ -2 & 1 & -1 \end{bmatrix}$$

Hence using the fact that the columns of X are eigenvectors of A one has

$$A = \frac{1}{3}\begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}\begin{bmatrix} -1 & 2 & -2 \\ 2 & -1 & 4 \\ -2 & 1 & -1 \end{bmatrix}$$

(d) Compute $A^n \mathbf{e}_1$ for $n \geq 1$ an integer. Please simplify your answer as much as possible. Here

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

As $A = X\Lambda X^{-1}$ one computes that

$$A^n = X\Lambda^n X^{-1}.$$

That is

$$A^n = \frac{1}{3}\begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & (-2)^n \end{bmatrix}\begin{bmatrix} -1 & 2 & -2 \\ 2 & -1 & 4 \\ -2 & 1 & -1 \end{bmatrix}$$

Hence,

$$\begin{aligned} A^n \mathbf{e}_1 &= \frac{1}{3}\begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \\ (-2)^{n+1} \end{bmatrix} \\ &= \frac{1}{3}\begin{bmatrix} -1 + (-2)^{n+2} \\ 0 \\ 2 + (-2)^{n+1} \end{bmatrix} \end{aligned}$$

8. (30 points) Let $A \in \mathbb{R}^{2\times 2}$ be the following matrix

$$A = \begin{bmatrix} 3 & 2 \\ 1 & -2 \end{bmatrix}$$

Let $S_p = \{\mathbf{x} \in \mathbb{R}^2 : ||\mathbf{x}||_p = 1\}$. Let $AS_p = \{A\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} \in S_p\}$. Here $1 \leq p \leq \infty$ and $||\cdot||_p$ is the p-norm on $\mathbb{R}^2$.

(a) Compute $\mu_1 = ||A||_1$ the induced 1-norm of A and $\mu_\infty = ||A||_\infty$ the induced ∞-norm of A. Remember to justify your computation.

For a vector $\mathbf{x} \in S_1$ let us write $\mathbf{x} = \alpha\mathbf{e}_1 + (1-\alpha)\mathbf{e}_2$ where we assume $0 \leq \alpha 1$ (so we assume $\mathbf{x}$ in first quadrant, by symmetry this is enough). Then $A\mathbf{x} = (2+\alpha)\mathbf{e}_1 + (-2+3\alpha)\mathbf{e}_2$ then $||A\mathbf{x}||_1 = |2+\alpha| + |-2-3\alpha|$. When $\alpha > 2/3$ this is equal to $2+\alpha-2+3\alpha = 4\alpha$, while for $\alpha \leq 2/3$ this is equal to $2+\alpha+2-3\alpha = 4-2\alpha$. Notice that this is maximized for $\alpha = 0$ or $\alpha = 1$ and has maximum value $\mu_1 = 4$.

For a vector $\mathbf{y} \in S_\infty$ let us write $\mathbf{y} = x\mathbf{e}_1 + y\mathbf{e}_1$ where $x = 1$ and $0 \leq yleq1$ or $y = 1$ and $0 \leq x < 1$ (so again we are in the first quadrant). Then $A\mathbf{y} = (3x+2y)\mathbf{e}_1 + (x-2y)\mathbf{e}_2$. Then $||A\mathbf{y}||_\infty = \max\{|3x+2y|, |x-2y|\}$. By inspection one sees that the maximum value is $\mu_\infty = 5$.

(b) Determine all vectors $\mathbf{x}_1 \in S_1$ and $\mathbf{x}_\infty \in S_\infty$ so that $||A\mathbf{x}_1||_1 = \mu_1$ and $||A\mathbf{x}_\infty||_\infty = \mu_\infty$.

In the previous problem we see that $\mathbf{x}_1$ can be any of the following and no other vectors:

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

In the previous problem we see that $\mathbf{x}_\infty$ can be any of the following and no other vectors:

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix},$$

(c) Sketch S_1 and AS_1 and indicate the vectors $\mathbf{x}_1$ and $A\mathbf{x}_1$ on your picture.

(d) Sketch S_∞ and AS_∞ and indicate the vectors $\mathbf{x}_\infty$ and $A\mathbf{x}_\infty$ on your picture.