Math 104 Final Exam — Dec. 10, 2010

Name: ____ **SUID#:** ______________

- Complete the following problems. In order to receive full credit, please *show all of your work* and *justify your answers*. You are free to use results from class or the course text book as long as you clearly state what you are citing.
- **You have 3 hours.** This is a closed-book, closed-notes exam. No calculators or other electronic aids will be permitted. If you finish early, you must hand your exam paper to a member of teaching staff.
- Please check that your copy of this exam contains 14 numbered pages and is correctly stapled.
- If you need extra room, use the back sides of each page. If you must use extra paper, make sure to write your name on it and attach it to this exam. Do not unstaple or detach pages from this exam.
- Please sign the following:

> "On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination."

Signature: ______________________________

The following boxes are strictly for grading purposes. Please do not mark.

Question	Points	Score
1	20	
2	20	
3	20	
4	30	
5	20	
6	30	
7	30	
8	30	
Total	200	

1. (20 points) Consider the following matrices that depend on a parameter $\lambda \in \mathbb{C}$:

$$A_\lambda = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ \lambda & -1 \end{bmatrix}$$

(a) For each λ, determine $||A_\lambda||_F$ the Frobenius norm of A_λ.

(b) For each λ, determine $||A_\lambda||_2$ the induced 2-norm of A_λ.

2. (20 points) Let $A \in \mathbb{C}^{m\times m}$ be hermitian (i.e. $A = A^*$). Let $P \in \mathbb{C}^{m\times m}$ be the matrix representing orthogonal projection onto $N(A)$. Please show that $X = A + P$ is invertible. (Hint: Think about the four fundamental subspaces of A).

3. (20 points) Let $A \in \mathbb{R}^{4\times 2}$ be the matrix

$$A = \begin{bmatrix} 1 & -7 \\ 1 & 1 \\ 3 & -7 \\ 5 & -9 \end{bmatrix}$$

(a) Find a reduced QR factorization of A i.e. $A = \hat{Q}\hat{R}$.

(b) Solve the following overdetermined system in the sense of least squares:

$$A\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

4. (30 points) Let $A \in \mathbb{C}^{m\times m}$ be a square matrix. Order the singular values σ_i of A by $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_m \geq 0$ and order the eigenvalues λ_i of A so $|\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_m| \geq 0$.

   (a) Show that $\sigma_1 \geq |\lambda_1|$.

   (b) Show that $\sigma_m \leq |\lambda_m|$ (Hint: Write $\mathbf{x}_m$, an eigenvector associated to λ_m, in terms of the right singular vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m$ of A).

(c) Using part a), show that if $||A||_2 < 1$ then $I + A$ is nonsingular. Here $||A||_2$ is the induced 2-norm and

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is the 3×3 identity matrix.

5. (20 points) Let $A \in \mathbb{C}^{3\times 3}$ be the matrix

$$A = \begin{bmatrix} -1 & 3 & -2 \\ 0 & 3 & 1 \\ 0 & 4 & 1 \end{bmatrix}$$

Find a unitary matrix $Q \in \mathbb{C}^{3\times 3}$ so that

$$QA = \begin{bmatrix} 1 & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix}.$$

Here $*$ represents an arbitary number.

6. (30 points) (a) Let $T \in \mathbb{C}^{m\times m}$ be upper triangular. Show that if T is unitary then T is diagonal. (Hint: Use the fact that columns are orthogonal and induct on m).

(b) Let

$$A = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & a_{mm} \end{bmatrix} \in \mathbb{C}^{m\times m}$$

be a diagonal matrix. Show that if A is unitary then $|a_{ii}| = 1$ for $1 \leq i \leq m$.

(c) Let $X \in \mathbb{C}^{m\times m}$ be unitary, use parts a), b) and the Schur factorization to show that X is unitarily diagonalizable (i.e. $\mathbb{C}^m$ has an orthonormal basis of eigenvectors) and that $\lambda \in \Lambda(X)$ implies $|\lambda| = 1$.

7. (30 points) Let $A \in \mathbb{R}^{3\times 3}$ be the matrix

$$A = \begin{bmatrix} -3 & 2 & -2 \\ 0 & 1 & 0 \\ 2 & -1 & 2 \end{bmatrix}$$

(a) Find the eigenvalues of A and give their algebraic multiplicity.

(b) Verify that A is diagonalizable and find a basis of eigenvectors.

(c) Determine the matrices X and Λ so that X is non-singular and Λ is diagonal and so one has a factorization:

$$A = X\Lambda X^{-1}$$

(d) Compute $A^n\mathbf{e}_1$ for $n \geq 1$ an integer. Please simplify your answer as much as possible. Here

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

8. (30 points) Let $A \in \mathbb{R}^{2\times 2}$ be the following matrix

$$A = \begin{bmatrix} 3 & 2 \\ 1 & -2 \end{bmatrix}$$

Let $S_p = \{\mathbf{x} \in \mathbb{R}^2 : ||\mathbf{x}||_p = 1\}$. Let $AS_p = \{A\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} \in S_p\}$. Here $1 \le p \le \infty$ and $||\cdot||_p$ is the p-norm on $\mathbb{R}^2$.

(a) Compute $\mu_1 = ||A||_1$ the induced 1-norm of A and $\mu_\infty = ||A||_\infty$ the induced ∞-norm of A. Remember to justify your computation.

(b) Determine all vectors $\mathbf{x}_1 \in S_1$ and $\mathbf{x}_\infty \in S_\infty$ so that $||A\mathbf{x}_1||_1 = \mu_1$ and $||A\mathbf{x}_\infty||_\infty = \mu_\infty$.

(c) Sketch S_1 and AS_1 and indicate the vectors $\mathbf{x}_1$ and $A\mathbf{x}_1$ on your picture.

(d) Sketch S_∞ and AS_∞ and indicate the vectors $\mathbf{x}_\infty$ and $A\mathbf{x}_\infty$ on your picture.

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