In your own words, what is one-to-one and what is onto?
What is the actual definition of one-to-one? Of onto?

Theorem 2 is useful.
Suppose $\alpha : A \rightarrow B$ and $\beta : B \rightarrow A$ such that $\beta\alpha = 1_A$. Does this imply $\alpha\beta = 1_B$?

If yes, prove it. If no, give a counterexample.

What’s the least number of implications necessary to prove these three conditions are equivalent?

**Problem (0.3 #14).** See the book for the question.

**Proof.** Consider the mapping $\alpha : A \rightarrow B$ where $A$ and $B$ are nonempty.

(a $\implies$ b) Suppose $\alpha$ is one-to-one. By definition, $\alpha$ one-to-one means if $\alpha(a) = \alpha(a_1)$, then $a = a_1$. $(a, a_1 \in A)$

Define the map $\gamma : B \rightarrow A$ as follows.

Let $b \in B$. If $b \in \text{im}(\alpha)$, then there exists $a$ such that $\alpha(a) = b$. We show this $a$ is unique. If $\alpha(a_1) = b$ for some $a_1 \in A$, then $\alpha(a) = \alpha(a_1)$. But $\alpha$ is one-to-one, so $a = a_1$. Let $\gamma(b) = a$. On the other hand, if $b \notin \text{im}(\alpha)$, then send it to any element in the nonempty set $A$. Explicitly, choose $a_0 \in A$. ($a_0$ is read “a-nought” or “a-nought”). Otherwise, if $b \notin \text{im}(\alpha)$, then let $\gamma(b) = a_0$. By construction, the map is well-defined.

We check that $\gamma$ does what we want it to. Let $c \in A$. By associativity we have $\gamma\alpha(c) = \gamma(\alpha(c))$. By definition of $\gamma$, we have $\gamma(\alpha(c)) = a$ where $a$ is the unique element in $A$ with the property that $\alpha(a) = \alpha(c)$. Because we assumed $\alpha$ is one-to-one, then $a = c$. Therefore, $\gamma(\alpha(c)) = c$ and we have $\gamma\alpha(c) = c$ as desired.

(b $\implies$ c) Now we suppose the second statement. Then we can apply the existence of $\beta$ to the equality $\alpha\gamma = \alpha\delta$. Let $a \in C$. Then $\alpha\gamma(a) = \alpha\delta(a)$ and so $\gamma(a) = 1_A\gamma(a) = \beta\alpha\gamma(a) = \beta\alpha\delta(a) = 1_A\delta(a) = \delta(a)$. In conclusion, $\gamma = \delta$.

Remark: the final proof looks nice, but originally, I would have written on scratch paper that $\beta\gamma(a) = \beta\alpha\delta(a)$ and so $1_A\gamma(a) = 1_A\delta(a)$ and so $\gamma(a) = \delta(a)$. That is actually acceptable too, but I thought writing it in one line is also neat. Everybody has their own way of presenting a solution. As long as its neat and organized.

(c $\implies$ a) Assume the third statement. Let’s show the first. Suppose $\alpha(a) = \alpha(a_1)$ for $a, a_1 \in A$. Take any nonempty set $C$. For example, $C = \{0\}$. Then define $\gamma : C \rightarrow A$ by mapping all elements to $a$. Similarly, define $\delta : C \rightarrow A$ by mapping all elements to $a_1$. These maps are obviously well-defined (but we should get in the habit of checking). Then $\alpha\gamma(c) = \alpha(a) = \alpha(a_1) = \alpha\delta(c)$ for all $c \in C$. Thus $\alpha\gamma = \alpha\delta$ and therefore $\gamma = \delta$. Thus for any $c \in C$ we have $a = \gamma(c) = \delta(c) = a_1$.

Having proved $a = a_1$, we conclude $\alpha$ is one-to-one.

Could you directly prove the reverse implications? (i.e., $a \implies c$, $c \implies b$, and $b \implies a$)

Is the empty set an equivalence relation?

Date: 1/31/2014.
The Partition Theorem is amazing and can be a useful way to think about equivalence relations.

**Definition.** Given an equivalence relations \(\equiv\), the corresponding quotient set is defined as

\[ A_\equiv = \{[a] \mid a \in A\}. \]

One simple way to think of it is to view the equivalence class as becoming one object. Try to put it in your own words.

**Example.** Let’s consider the population of the United States as being partitioned into 50 states (for the sake of this example, assume the United States only consists of exactly the 50 states). This defines an equivalence relation by saying two people are equivalent if they belong to the same state. The quotient set would most be like just thinking of the 50 states. We know that there are many people contained in each state, but when you think of the quotient set, you just think of the states themselves. Now given any person in the United States, there is an obvious map to the quotient set. Just assign each person to the state which they’re in.

**Definition.** In general, we have the natural mapping (or canonical mapping):

\[ \varphi : A \to A_\equiv \]

given by

\[ \varphi(a) = [a] \]

for all \(a \in A\).

(In mathematics, there are often “natural” or “canonical” mappings. As the adjectives imply, the map should either be something most mathematicians would think about or somehow warrant being singled out compared to other possible mappings.)

Recall Example 4 in the section. Kernel equivalence is partitioning a set \(A\) by using a map which is defined on \(A\). The map partitions \(A\) based on where it maps elements in \(A\). Similar to how we could have started with the map of assigning people the states in which they belong in order to define an equivalence relation.

**Problem (0.4 #9).** \(\alpha : A \to B\).

Let \(\equiv\) be the kernel equivalence and let \(\varphi : A \to A_\equiv\) denote the natural mapping. Define

\[ \sigma : A_\equiv \to B \]

by

\[ \sigma([a]) = \alpha(a) \]

for all equivalence classes \([a]\) in \(A_\equiv\). Show many things (see book).

**Proof.** (a) \(\sigma\) is well defined: Suppose \([a] = [b]\). Then \(\sigma([a]) = \alpha(a) = \alpha(b) = \sigma([b])\).

\(\sigma\) is one-to-one: If \(\sigma([a]) = \sigma([b])\), then \(\alpha(a) = \alpha(b)\). Then \(a \equiv b\). By theorem, we have \([a] = [b]\).

\(\sigma\) is onto, if \(\alpha\) is onto: If \(\alpha\) is onto, then given \(b \in B\), there exists \(a \in A\) such that \(\alpha(a) = b\). Thus \(\sigma([a]) = \alpha(a) = b\).

(b) We have \(\sigma \varphi(a) = \sigma([a]) = \alpha(a)\). Thus \(\alpha = \sigma \varphi\).

(c) Suppose \(\alpha(A)\) is a finite set. Further, without loss of generality, suppose \(B = \alpha(A)\). Then \(\sigma\) is onto. By part (a), the map \(\sigma\) will also be onto. Since \(\sigma\) is
already one-to-one, we conclude \( \sigma \) is a bijection between \( A_\infty \) and \( \alpha(A) \). We conclude that \( |A_\infty| = |\alpha(A)| \).

(d)(i) We have the following chart:

\[
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 6 & 12 & & \\
1 & & 1 & 2 & 3 & 4 & 6 & 12 \\
2 & 2 & 1 & 3 & 4 & 6 & 12 & \\
3 & 3 & 2 & 1 & 6 & 12 & & \\
4 & 4 & 2 & 1 & 6 & 12 & & \\
6 & 6 & 3 & 2 & 1 & 12 & & \\
12 & 12 & 6 & 4 & 3 & 2 & 1 & \\
\end{array}
\]

I count 15 different elements in the image. Let’s generalize the problem. That is, fix a positive integer \( m \) and suppose \( U = \{a_1, a_2, \ldots, a_n\} \) where \( a_i \) are all the positive factors of \( m \). Find \( |A_\infty| \). Good luck!

(d)(ii) The answer here is a bit more immediate. You may realize the image will attain the values \{1, 2, \ldots, 18\}, but you need a proof. A chart suffices as a proof, but that’s a 10 by 10 chart! I write up a possible proof that doesn’t use a chart here. If \( a \in A \cup \{0\} \), then \( a = 10a_2 + a_1 \) where \( a_1, a_2 \in \{0, 1, \ldots, 9\} \). Thus \( \alpha(a) = a_1 + a_2 \). But since \( 0 \leq a_i \leq 9 \), then \( 0 \leq a_1 + a_2 \leq 18 \). Note, however, that \( \alpha(a) = 0 \) only if \( a_1 = a_2 = 0 \), so if \( a \in A \), then \( 1 \leq \alpha(a) \leq 18 \). We need to conclude that \( \alpha \) is surjective onto \{1, 2, \ldots, 18\}. Given \( b \in \{1, 2, \ldots, 9\} \) we choose \( a = b \) so that \( \alpha(a) = b \). Given \( b \in \{10, 11, \ldots, 18\} \), then \( b - 9 \in \{1, 2, \ldots, 9\} \) and we choose \( a = 10(b - 9) + 9 \) so that \( \alpha(a) = (b - 9) + 9 = b \). The great thing about this proof is that it would generalize well to the set \( A = \{n \in \mathbb{Z} \mid 1 \leq n \leq 10^k - 1\} \) for a fixed integer \( k \geq 1 \).

Three or more lines are said to be concurrent if they intersect at a single point.

**Problem (1.1 #19).**

*Proof.* By “region” we will mean the “interior.” For this exercise, this just means to not include the boundary. When you place that \((n+1)\)th line down, it passes through the other \( n \). This is guaranteed because no two lines are parallel. It passes these \( n \) lines at \( n \) distinct points, because no three lines are concurrent. Label the intersections \( v_1, v_2, \ldots, v_n \). These points along your \((n+1)\)th line divide the line into \((n+1)\) sections. Without the vertices \( v_i \), name the sections \( s_0, s_1, \ldots, s_n \). Intuitively, each \( s_i \) belongs to a single region, no existing region contains more than one \( s_i \), and each \( s_i \) separates the existing region into two new regions. Assuming these claims are true, then \( n + 1 \) new regions have been created and \( a_{n+1} = a_n + (n+1) \) as claimed by the exercise.

I find I’m unable to give a good proof of these claims without appealing to more topology and/or analysis. You may read what I have to say if you like, otherwise, an intuitive understanding is appropriate. To prove these three claims, it is convenient to first prove that the existing regions are necessarily convex. Well, the intersection of any collection of convex sets is convex and a single line divides the plane into two halves, both of which are convex. A given region is define by being the intersection of these different halves and we conclude every existing region is convex.

If \( s_i \) belonged to more than one region, it must necessarily cross a line that separates them, contradicting the construction of \( s_i \). That shows the first claim. If a region contained two sections \( s_i \) and \( s_j \), then by convexity it will also contain
points which join them. However, if \( s_i \) and \( s_j \) were distinct, this would imply the region contains a vertex \( v_k \) for some \( k \). Since the region doesn’t contain the boundary, we conclude \( i = j \) and that a region contains at most one section. Finally, for the last part, it is simplest to go back to the entire \((n+1)\)th line. A given region is either divided by the line, or lies to either side of the line.

This exercise is useful in proving 1.1 #6.

**Problem (1.1 #25).** This problem is almost a bit of a brain teaser, but if you think about it a while, it makes sense. I would rewrite it like this: Let \( p_n \) be a statement about \( n \) for each \( n \geq 1 \). Suppose \( p_n \implies p_{n-1} \) for each \( n \geq 2 \). Also suppose that \( p_n \) is (known to be) true for infinitely many values of \( n \). Prove that \( p_n \) is true for all \( n \geq 1 \).” In any case, you may rephrase the problem in a way that it makes sense to you, so long as you are sure you don’t change what the problem is asking. This is helpful in doing proof-based problems.

**Proof.** Let \( k \geq 1 \). Then there exists \( m \geq k \) such that \( p_n \) is true, because \( p_m \) is true for infinitely many values of \( n \). Then, by induction (I omitted details), \( p_i \) is true for \( i = 1, 2, \ldots, m \). In particular, \( p_k \) is true for \( k \). As \( k \) was an arbitrary integer greater than or equal to 1, we conclude \( p_n \) is true for all \( n \geq 1 \).