Assignment 1: due Thursday February 12

1. Write up a complete proof of term-by-term differentiabilty, starting from:
   Let $\epsilon > 0$. We must find $\delta > 0$ such that $|w - z| < \delta$ implies
   \[
   \left| \frac{f(w) - f(z)}{w - z} - g(z) \right| < \epsilon.
   \]
   Here, $f(z) = \sum_0^\infty c_n z^n$ has radius of convergence $R > 0$, $|z| < R$, $|w| < R$, and $g(z)$ is given by the derived series.

2. Definition of $\partial_\xi$ (and $\partial_z$). Among all expressions (vector fields in the plane with $\mathbb{C}$-valued coefficients) $L = \phi(x, y) \partial_x + \psi(x, y) \partial_y$, there is exactly one with
   a) $L(f) = 0$ wherever $f$ is complex-differentiable,
   b) $L(\overline{\zeta}) = 1$.
   This $L$ is $\partial_\xi$. Determine the coefficients $\phi(x, y)$ and $\psi(x, y)$.

3. Assume that $f(z) = \sum_0^\infty c_n z^n$ and $g(z) = \sum_1^\infty a_n z^n$ (so $g(0) = 0$) have positive radii of convergence $R$ and $r$ respectively.
   a) Determine the power series for $f \circ g$ in powers of $z$.
   b) Let $g(z) = \alpha z$ ($\alpha \in \mathbb{C}$). What is the radius of convergence of the series for $f \circ g$?
   c) Answer the question in b) for general $g$.

4. a) Show that for a sequence of real numbers $q_n$,
   \[
   \limsup_{n \to \infty} q_n = \lambda
   \]
   if and only if the following pair of statements hold:
   i) For every $\epsilon > 0$, the set of $n$ for which $q_n \geq \lambda + \epsilon$ is a finite set.
   ii) There is a subsequence $\{q_{n(k)}\}$ of $\{q_n\}$ that converges to $\lambda$.
   b) Use this characterization of $\limsup$ to derive the formula for the radius of convergence of a power series:
   \[
   R^{-1} = \limsup_{n \to \infty} |c_n|^{1/n}.
   \]

5. Suppose that $\sum_{n=0}^\infty c_n z^n$ has radius of convergence $R > 0$.
   a) If $|z_0| = r < R$, show that the double summation
   \[
   \sum_{n=0}^\infty c_n \sum_{m=0}^n \binom{n}{m} z_0^{n-m}(z - z_0)^m
   \]
   converges absolutely whenever $|z - z_0| < R - r$.
   b) Give a counterexample to the assertion: The radius of convergence of
   \[
   \sum_{m=0}^\infty \left\{ \sum_{n=m}^\infty c_n \binom{n}{m} z_0^{n-m} \right\} (z - z_0)^m
   \]
   is $R - r$.  