No books, no notes, no calculators or other devices! Write legibly, and show all relevant work and explain clearly—or risk losing credit.

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110.202 Calculus III Professor Zucker
Name: ___________________
Final Exam: December 12, 2003
Soc. Sec. No.: ______________
Time allowed: 3 hours
Section Number: ______________

[20] 1. Let $D$ denote the unit disc in the $(x, y)$-plane. Determine the maximum and minimum values of $x + y^2$ on $D$.

[15] 2. Let $C$ be the graph $y = \sin(\pi/x)$, with $x \in [1, 2]$, traversed according to increasing values of $x$. Evaluate

$$\int_C 2xydx + x^2dy.$$
3. This problem is about the differential
\[ \omega = x^2 \, dy. \]
a) Let \( C \) be the unit circle, oriented counterclockwise. Apply Green’s Theorem to convert \( \int_C \omega \) to a double integral. Then evaluate the double integral.

b) True or false: \( \int_E \omega \) has the same value for all closed curves \( E \) that surround the origin (i.e., for which the origin lies inside \( E \))? Explain.

4. A vector field in space is given by the formula \( \mathbf{v}(x, y, z) = x^2 \mathbf{i} + y^3 \mathbf{j} \). Determine the flux of \( \mathbf{v} \) out of the rectangular solid \( 0 \leq x \leq 1, \ 0 \leq y \leq 2, \ 0 \leq z \leq 3. \)
5. Let \( \Omega = \{(x, y) : y \geq 0, 3 \leq x^2 + y^2 \leq 4\} \). Determine \((\bar{x}, \bar{y})\), the centroid of \(\Omega\).

6. Let \(C\) be the arc of the unit circle in the first quadrant, oriented counterclockwise.
   a) Evaluate \(\int_C xyj \cdot dr\).

b) Evaluate: \(\text{curl}(xyj) = \)

c) Use Stokes’ Theorem to show that if \(S\) is any surface in space having the \textit{whole} unit circle in the \((x, y)\)-plane as its boundary, then

\[
\iint_S yk \cdot n \, d\sigma = 0.
\]

\[ f(x, y) = \begin{cases} \frac{x^3}{\sqrt{x^2+y^2}} & \text{if } (x, y) \neq (0,0), \\ 0 & \text{if } (x, y) = (0,0). \end{cases} \]

Let \( r \) denote, as usual, distance from the origin.

a) Show that for all points \((x, y)\) other than the origin,

\[ \left| \frac{x}{r} \right| \leq 1 \quad \text{and} \quad \left| \frac{y}{r} \right| \leq 1. \]

b) Show that

\[ \lim_{(x,y) \to (0,0)} r^{-1}f(x, y) = 0. \]

c) Calculate \( f_x(x, y) \) at all points where this partial derivative is defined.

d) Is \( f \) differentiable at the origin? \textbf{Explain.}
8. Let $T$ be the solid $x^2 + y^2 + z^2 \leq 4$, and let $B$ denote its boundary.

a) Determine whether $(1, 0, \sqrt{3})$ is an interior point of $T$.

b) Given equation(s) for a (non-constant) curve in space that is tangent to the boundary $B$ at $(0, \sqrt{2}, \sqrt{2})$, but is not a curve in the boundary.

c) Evaluate $\int \int_B z^3 d\sigma$.

d) Evaluate $\int \int_B y^5 d\sigma$. 
9. a) Let $S$ be the lower half of the ellipsoid $x^2 + 4y^2 + z^2 = 4$. Give an effectively one-to-one parametrization of $S$.

b) Determine the upward unit normal to $S$ at the point $(\sqrt{3}, 0, -1)$. 