Answers to Fall 2002 Final Exam

I was asked to issue the answers to last year’s Final, and I felt that the request was reasonable. I worked them out again. Then I had to write them up and prepare the alleged solutions in TEX, and then correct all typesetting errors. I hope there are no computational errors .... Please inform me of suspected mistakes.

It is of basic importance that you know how to go about answering the questions, and can carry out the methods for determining the answers.

1. Answer: \( \frac{5}{18} (2^{18} - 1) \)

2. a) Follow instructions; you are asked to do the problem in a particular way.
Answer: \( -\frac{\pi}{4} \).

   b) Answer: \( -\frac{1}{6} \).

3. b) Lots of freedom here. Easiest: seek an \( f \) that is identically zero on the coordinate axes, discontinuous along the axes.

4. a) Simplest choice: \( (x, y) = (0, 0) \). Answer (for that choice): \( \pi \).

   b) We’re asking for the directional derivative \( g_u \) at the given point \( R \), where \( u \) is the counterclockwise unit tangent vector at \( R \). Answer: \( -\sqrt{2}/4 \).

5. Extrema are seen to occur on the boundary. The Lagrange multiplier method gives that the extrema can only occur when \( x = 0, y = 0, x = 4y \). (It may help to notice that the function \( x^3 + y^3 \) is odd and the ellipse is rather symmetric.) Next?

6. The region: \( y \in [0, 3], 0 \leq x \leq \frac{3}{\sqrt{2}} \); or do it in the opposite order.
Answer: \( (\frac{1}{3} - \frac{1}{6})/4 \)

7. Recognize this as the usual parametrization of half of the unit sphere with the variables altered. Thus:

   a) Take \( f(x, y, z) = x^2 + y^2 + z^2 - 1 \), which is zero precisely on the unit sphere.

   b) No. \( S \) is only half of the sphere.

   c, d) Determine \( \mathbf{N} \). Then calculate \( |\mathbf{N}| = |\sin u| \). Answer: \( 2\pi/3 \).

8. First, sketch the curve!

   a) The boundary is the curve of revolution: \( x^2 + y^2 = \frac{1}{4}, z = 1 \).

   b) \( \nabla \times (y \mathbf{i}) \) is determined in space; don’t get confused by the fact that there are curves and surfaces being mentioned. Answer: \( -\mathbf{k} \).

   c) [I answered the wrong question earlier] The curve \( C \) is given by the equation

\[
y = z^8 \sin \left( \frac{\pi z^2}{4} \right),
\]

\( z \in [1, 2] \) (and \( x = 0 \)). A normal field is given by the two-variable gradient

\[
\nabla (y - z^8 \sin \left( \frac{\pi z^2}{4} \right))(y, z).
\]

d) Do you really want to parametrize the surface when the boundary is so nice? The with the curl in there, the situation is ripe for Stokes’ Theorem. Know the
statement of Stokes' cold, and you should see that. Answer: \( \pm \pi/2 \). (The sign happens to be \(-\).)

9. a) This is a straightforward, though tedious, surface area calculation. By fancy methods at the graduate level, I can see rather quickly that the answer is \( 8\pi^2 \). For last year's exam, I worked it out by the methods of Calc III. Did I get the same answer? Did you?

   b) Answer: the surface of a donut (or similar). Do you see the two revolutions from the parametric equations?