4.1.27 (page 226) Use the method of reduction of order to solve the given differential equation

\[(2 - t)y''' + (2t - 3)y'' - ty' + y = 0, t < 2; \ y_1(t) = e^t.\]

**Solution.** Since \(t < 2\) we can divide both sides of the question by \(2 - t\) so we have

\[y''' + \left(\frac{2t-3}{2-t}\right)y'' - \frac{t}{2-t}y' + \frac{1}{2-t}y = 0\]

Now assume \(y = v \cdot e^t\), then use the formula given in the problem we obtain directly

\[e^t v''' + (3e^t + \frac{2t-4}{2-t}e^t)v'' + (3e^t + 2(\frac{2t-3}{2-t})e^t - \frac{t}{2-t}e^t)v' = 0\]

Simplifying this equation we obtain

\[v''' + \left(\frac{t-3}{t-2}\right)v'' = 0\]

Now we let \(u = v''\) then we have

\[\frac{du}{u} = (-\frac{1}{t-2} - 1)dt\]

Integrating this differential equation we have

\[u = c_1(t-2)e^{-t}\]

Now we are facing a second-order differential equation

\[v'' = c_1(t-2)e^{-t}\]

Integrating both sides of this equation to obtain

\[v' = -c_1te^{-t} - c_1e^{-t} + 2c_1e^{-t} + c_2 = -c_1te^{-t} + c_1e^{-t} + c_2\]

Integrating again

\[v = c_1te^{-t} + c_1e^{-t} - c_1e^{-t} + c_2 e^{-t} + c_3\]

Then the final solution would be \(y = ve^t\)
6.1.6(page 311) Find the Laplace transform of \( f(t) = \cos at \), where \( a \) is a real constant.

Solution. We have

\[
\mathcal{L}(\cos at) = \int_0^\infty e^{-st} \cos at \, dt
\]

\[
= \frac{1}{a} \int_0^\infty e^{-st} d\sin at
\]

\[
= \frac{1}{a} e^{-st} \sin a \bigg|_0^\infty + \frac{s}{a} \int_0^\infty e^{-st} \sin at \, dt
\]

\[
= \frac{s}{a} \mathcal{L}(\sin at)
\]

\[
= \frac{s}{s^2 + a^2}.
\]
6.1.13 (page 311) Find the Laplace transform of \( f(t) = e^{at} \sin bt \).

**Solution.** We have

\[
\mathcal{L}(e^{at} \sin bt) = \int_0^\infty e^{-st} e^{at} \sin bt \, dt
\]

\[
= -\frac{1}{b} \int_0^\infty e^{-(s-a)t} d(cos bt) \quad \text{(recall } s > a)\]

\[
= -\frac{1}{b} e^{-(s-a)t} \cos bt \bigg|_0^\infty - \frac{s-a}{b} \int_0^\infty e^{-(s-a)t} \cos bt \, dt
\]

\[
= \frac{s-a}{b^2} \int_0^\infty d\sin bt
\]

\[
= -\frac{s-a}{b^2} e^{-(s-a)t} \sin bt \bigg|_0^\infty - \frac{(s-a)^2}{b^2} \int_0^\infty e^{-(s-a)t} \sin bt \, dt
\]

\[
= -\frac{(s-a)^2}{b^2} - \mathcal{L}(e^{at} \sin bt).
\]

So

\[
\mathcal{L}(e^{at} \sin bt) = \frac{b^2}{(s-a)^2 + b^2}
\]
6.1.18 (page 311) Use integration by parts to find the Laplace transform of $f(t) = t^n e^t$.

**Solution.** We define $I_n = \mathcal{L}(t^n e^{at})$. Then

$$I_n = \int_0^\infty e^{-st} t^n e^{at} dt$$

$$= -\frac{1}{s-a} e^{-(s-a)t} \bigg|_0^\infty + \frac{n}{s-a} \int_0^\infty t^{n-1} e^{-(s-a)t} dt$$

$$= \frac{n}{s-a} I_{n-1}$$

So

$$I_n = \frac{n!}{(s-a)^n} I_0$$

And

$$I_0 = \mathcal{L}(e^{at}) = \frac{1}{s-a}$$
6.2.10(page 320) Find the inverse Laplace transform of the function

\[ F(s) = \frac{2s - 3}{s^2 + 2s + 10}. \]

**Solution.** It is easy to verify

\[
\begin{align*}
\frac{2s - 3}{s^2 + 2s + 10} &= \frac{2s}{(s + 1)^2 + 9} - \frac{3}{(s + 1)^2 + 9} \\
&= \frac{2}{s + 1} \frac{1}{(s + 1)^2 + 9} - \frac{5}{3} \frac{3}{(s + 1)^2 + 9}
\end{align*}
\]

Therefore

\[ F(s) = \mathcal{L}(2e^{-t} \sin 3t - \frac{5}{3} e^{-t} \cos 3t) \]
6.2.27 (a) (page 321) Using the Taylor series for $\sin t$

$$\sin t = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n + 1)!}$$

and assuming that the Laplace transform of this series can be computed term by term, verify that

$$\mathcal{L}\{\sin t\} = \frac{1}{s^2 + 1}, \quad s > 1.$$  

(Hint: you may use geometric series formula, and be sure to verify why $s > 1$)

**Solution.** Since we can take Laplace transform term by term, then

$$\mathcal{L}(\sin t) = \mathcal{L}\left(\sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n + 1)!}\right)$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n + 1)!} \mathcal{L}(t^{2n+1})$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (2n + 1)!}{(2n + 1)!} \frac{1}{s^{2n+2}}$$

$$= \frac{1}{s^2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(s^2)^n}$$

$$= \frac{1}{s^2} \left(1 + \frac{1}{s^2}\right) \quad \text{(because } s > 1)$$

$$= \frac{1}{s^2 + 1}$$
**Quizz 2**

**Bonus question**

6.2.27 (b) (page 321) Let

\[ f(t) = \begin{cases} \frac{\sin t}{t}, & t \neq 0 \\ 1, & t = 0 \end{cases} \]

Find the Taylor series for \( f \) about \( t = 0 \). Assuming that the Laplace transform of this function can be computed term by term, verify that

\[ \mathcal{L} \{ f(t) \} = \arctan(1/s), \quad s > 1 \]

Hint: If you have completed part (a) then you may realize the only non-trivial part of this problem is to compute the Taylor series of \( \arctan(t) \). But this is not trivial at first glance. Every Taylor series can be computed manually by the standard definition, but usually this could be the last thing you would like to do because it involves the calculation of arbitrary order derivatives. For example let us look at \( \arctan(t) \), if you manually compute its derivatives it will be a pain. But by the uniqueness of Taylor series we can always use any technique or method to expand \( \arctan(t) \) to a power series, which will exactly serve as its Taylor series as long as it converges. Starting from this philosophy, it is fairly easy to compute the first derivative of \( \arctan(t) \), which is

\[ \frac{d}{dt} \arctan(t) = \frac{1}{1 + t^2}. \]

Or in other words

\[ \arctan(t) = \int_0^t \frac{1}{1 + s^2} ds. \]

If you can recall the geometric series technique we used in part (a) then the rest thing should be quite accessible to you. The Taylor series expansion of \( \arctan(t) \) served as one of the oldest algorithm to compute \( \pi \), say

\[ \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots \]

Despite of its elegance, this series converges to \( \frac{\pi}{4} \) very slowly so it does not have too much practical importance in computation.

By the way, in case you may be interested in, the Taylor series for \( \tan(t) \) is

\[ \tan(t) = \sum_{n=1}^{\infty} \frac{B_{2n}(-4)^n(1 - 4^n)}{(2n)!} t^{2n-1}, \quad |x| \leq 1, \]

in which the Bernoulli numbers \( B_n \) are determined by

\[ \frac{t}{1 - e^{-t}} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}. \]