Problem 5
Consider polar coordinates \( f(r, \theta) = (r \cos \theta, r \sin \theta) \). Show that \( ||\frac{\partial f}{\partial r}|| = 1, ||\frac{\partial f}{\partial \theta}|| = r \), and \( \frac{\partial f}{\partial r} \cdot \frac{\partial f}{\partial \theta} = 0 \).

Problem 6
Consider spherical coordinates \( f(\rho, \theta, \phi) = (\rho \cos \theta \sin \phi, \rho \sin \theta \sin \phi, \rho \cos \phi) \). Show that:
\[
||\frac{\partial f}{\partial \rho}|| = 1 \\
||\frac{\partial f}{\partial \theta}|| = \rho \sin \phi \\
||\frac{\partial f}{\partial \phi}|| = \rho \\
\frac{\partial f}{\partial \rho} \cdot \frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial \rho} \cdot \frac{\partial f}{\partial \phi} = \frac{\partial f}{\partial \theta} \cdot \frac{\partial f}{\partial \phi} = 0
\]

Problem 7
Now we will consider spherical coordinates on \( \mathbb{R}^4 \). For any vector \( x \in \mathbb{R}^4 \) we may write \( x = (x', x_4) \) where \( x' \) is some vector in \( \mathbb{R}^4 \). Now note that both polar coordinates and spherical coordinates work by finding a parameterization of the set \( ||y|| = 1 \) and then multiplying by the distance away from the origin. Therefore to construct our coordinates we must parameterize \( ||x|| = 1 \) contained in \( \mathbb{R}^4 \).

First suppose we have a parameterization of the sphere \( ||y|| = 1 \) contained in \( \mathbb{R}^3 \) given by the function \( S(\theta_1, \theta_2) \).
where we set $\theta_1 = \theta$ and $\theta_2 = \phi$ from regular three dimensional spherical coordinates.

Now, note that for any $x \in \mathbb{R}^4$ with $||x|| = 1$ we have $||x'||^2 + x_4^2 = 1$. Therefore $(x_4, ||x'||)$ lies on the upper semicircle of $\mathbb{R}^2$ (since $||x'|| \geq 0$). So we may say $x_4 = \cos \theta_3$ and $||x'|| = \sin \theta_3$ for some $\theta_3 \in [0, \pi]$. And so we see that $||x|| = 1$ can be parameterized by $T(\theta_1, \theta_2, \theta_3) = (\sin (\theta_3)S(\theta_1, \theta_2), \cos \theta_3)$ and therefore our spherical coordinates for $\mathbb{R}^4$ are given by

$$g(r, \theta_1, \theta_2, \theta_3) = (r \sin (\theta_3)S(\theta_1, \theta_2), r \cos \theta_3)$$

Find

$$\frac{\partial g}{\partial \theta_j} \cdot \frac{\partial g}{\partial \theta_k}$$

$$\frac{\partial g}{\partial \theta_j} \cdot \frac{\partial g}{\partial r}$$

$$\frac{\partial g}{\partial r} \cdot \frac{\partial g}{\partial r}$$

for all values of $j$ and $k$. (Hint: Make life easier by using the results of Problem 6).