1. Perpendicularly bisecting $AC$ gives a diameter of the large circle. The two pieces of the diameter, separated by $AC$, happen to be the diameters of the two smaller circles. Thus, the sum of the radii of the smaller circles is $\frac{23}{2}$.

2. $\triangle ABC$ is a $30^\circ - 60^\circ - 90^\circ$ triangle, also $AE$, which equals $PS$, the width of the rectangle, is $AB \cdot \sqrt{3} = \sqrt{3}$. $\triangle PAF$ and $\triangle BQC$ are also $30^\circ - 60^\circ - 90^\circ$ triangles, giving $PA = \frac{1}{2} \cdot AF = BQ = \frac{1}{2}$, so $PQ = PA + AB + PQ = 2$, and the area of $PQRS$ is $2 \cdot \sqrt{3} = 2\sqrt{3}$.

3. Let $BD$ be the perpendicular to $AC$, with $D$ on the extension of $AC$. Since $\angle BDA = 90^\circ$ and $\angle DAB = 180^\circ - \angle BAC = 45^\circ$, $\angle BDA$ is a $45^\circ - 45^\circ - 90^\circ$ triangle. Thus, $BD = AB \cdot \sqrt{2} = \sqrt{2}$, and the area of $\triangle ABC$ is $\frac{1}{2} \cdot AC \cdot BD = \sqrt{2}$.

4. Drop perpendiculars $AF$ and $AE$ to $CD$, with $E$ and $F$ on $CD$. $\frac{1}{2} \cdot (AB + CD) \cdot AF = 36$, so $AF = \frac{36}{\frac{1}{2} \cdot 18} = 4$. Since $ABEF$ is a rectangle, $CE = DF = \frac{CD - AB}{2} = 3$, and $BC = \sqrt{BE^2 + CE^2} = 5$.

5. Since the rectangle $HJKL$ is rearranged from $\triangle ABC$, they have the same area, which is $\sqrt{3} \cdot \frac{AC^2}{4} = \sqrt{3} \cdot \frac{(AE + EC)^2}{4} = 16\sqrt{3}$.

6. Because $DF$ is parallel to $BC$, $\triangle ADF \sim \triangle ABC$, so $\frac{AD}{DF} = \frac{AB}{BC}$. Solving for $BC$ gives $BC = AB \cdot \frac{DF}{AD} = (AD + BD) \cdot \frac{DF}{AD} = (25 + 10) \cdot \frac{10}{25} = 14$.

7. Let the center of the circle be $O$. The region in question consists of $\triangle AOC$, $\triangle AOB$ and the minor sector $BOC$. Each of $\angle BOA$, $\angle AOC$, and $\angle COB$ is $120^\circ$, so the total area is $\frac{1}{2} \cdot OC \cdot OA \cdot \sin \angle COA + \frac{1}{2} \cdot OA \cdot OB \cdot \sin \angle AOB + \frac{120^\circ}{360^\circ} \cdot (\text{area of circle}) = \frac{18\sqrt{3}}{2} + \frac{18\sqrt{3}}{2} + \frac{1}{3} \cdot 36 = \frac{18\sqrt{3} + 12\pi}{3}$.

8. Without loss of generality, let $AB > AC$. $\triangle ABC$ and $\triangle DEC$ are both $30^\circ - 60^\circ - 90^\circ$ triangles, $2 = \frac{1}{4} BC = AC = AD + DC = ED + DC = ED + \frac{ED}{\sqrt{3}}$. Solving for $ED$ gives $ED = \frac{2}{\sqrt{3} + 1} = 3 - \sqrt{3}$.

9. Let the square be $ABCD$, with $AB$ on the hemisphere’s diameter, and let $O$ be the midpoint of the diameter. Then we have $OB = \frac{1}{2}$ and $BC = 1$, so the radius is $OC = \sqrt{OB^2 + BC^2} = \frac{\sqrt{5}}{2}$. Thus, the perimeter is $\frac{\pi \sqrt{5}}{2} + \frac{2\sqrt{5}}{2} = \sqrt{5} + \frac{\pi \sqrt{5}}{2}$. 


10. Since $\triangle DEC$ is an isosceles triangle, we have $\sqrt{2} \cdot ED = CD = \sqrt{2} \cdot AD$, so $ED = AD$ and $\triangle DAE$ is isosceles. Also, since $\angle EDC = 45^\circ$, $\angle ADE = 90^\circ - \angle EDC = 45^\circ$.

Thus, $\angle AED = \frac{180^\circ - \angle ADE}{2} = \angle BEC$ by symmetry. Finally, $\angle AEB = 360^\circ - \angle DEA - \angle CED - \angle BEC = 360^\circ - \frac{2(180^\circ - \angle ADE)}{2} - 90^\circ = 135^\circ$. 