Johns Hopkins Mathematics Tournament  
April 23, 2005  

ALGEBRA SOLUTIONS

1. Rewrite the provided expression as  

\[ a^2 + b^2 - 6a = b^2 + (a - 3)^2 - 9. \]

Since the minimum value of a squared term is zero, the answer is \([-9]\).

2. Let the rectangle’s length be \(L\) and its width be \(W\). We have \(2L + 2W = 25\), so \(W = 12.5 - L\). Also, \(LW = 25\). Substituting \(W\) into the area equation gives \(12.5L - L^2 = 25\). Applying the quadratic formula and taking the larger value, we get \(L = \boxed{10}\).

3. If the roots are \(p\) and \(q\), we have  

\[(x - p)(x - q) = x^2 + 2bx + 1 = 0.\]

Expanding the product on the left shows that \(p + q = -2b\) and \(pq = 1\). Use these values to find an expression for \((p - q)^2\):

\[
(p - q)^2 = p^2 - 2pq + q^2 \\
= p^2 + 2pq + q^2 - 4pq \\
= (p + q)^2 - 4pq \\
= (-2b)^2 - 4 = 4b^2 - 4.
\]

The desired difference then follows by taking the square root of both sides.

\[p - q = \sqrt{4b^2 - 4} = \frac{2\sqrt{b^2 - 1}}{2} = \boxed{\sqrt{b^2 - 1}}\]

4. We will sum each term of \(f(x)\) separately. \(2^0 + 2^1 + \ldots + 2^8\) is a geometric series, with sum \(\frac{1 - 2^9}{1 - 2} = 255\); \(0 + 1 + \ldots + 8\) is the eighth triangular number \(8 \cdot \frac{9}{2} = 36\); finally, we subtract \(4 \cdot 9 = 36\), so the total sum is \(255 + 36 - 36 = \boxed{255}\).

5. **Solution 1:** \(496 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 31\). Thus, all factors are of the form \(31^a \cdot 2^b\), with \(a = 0\) or 1 and \(b = 0, 1, 2, 3,\) or 4. The sum of these factors is

\[
(1 + 31)(1 + 2 + 4 + 8 + 16) = 32 \cdot 31 = \boxed{992}.
\]

**Solution 2:** \(496\) is a perfect number, meaning that its factors (other than itself) sum to itself. Thus we add \(496\) to itself to find that the total sum is

\[496 + 496 = \boxed{992}\].
6. Summing the first three equations, we get \(9a + 2b - 5c + 2d = 6\). Subtracting the last equation yields \(a + b + c + d = \frac{2}{3}\).

7. The area of an equilateral triangle is \("x^2\) with \(x\) being its side length. Thus, since \(\frac{\sqrt{3}x^2}{4} = \sqrt{3}\), \(x = 2\). The \(y\)-coordinate of the third vertex is between the \(y\)-coordinates of the other two, 0 and 2, so it is 1. The \(x\)-coordinate corresponds to the triangle’s height, which equals \(2 \cdot \frac{\text{area}}{\text{side}} = \frac{2\sqrt{3}}{2} = \sqrt{3}\), so the coordinates are \((\sqrt{3}, 1)\).

8. Note that the sum of two of the three factors is always positive (2\(a\), 2\(b\), and 2\(c\)). Also, the differences between two of the three factors is nonzero since \(a\), \(b\), and \(c\) are distinct (±(2\(b\) − 2\(c\)), ±(2\(c\) − 2\(a\)), ±(2\(a\) − 2\(b\))). Therefore, we conclude that the three factors are distinct positive factors of 15. The only triplet that satisfies this is (1, 3, 5). Thus, 2\(a\), 2\(b\), and 2\(c\) are 1 + 3 = 4, 1 + 5 = 6, and 3 + 5 = 8 in some order, and \(abc = \frac{4}{2} \cdot \frac{6}{2} \cdot \frac{8}{2} = 24\).

9. Multiplying by 2\(ab\) and rearranging gives \((a - 2)(b - 2) = 4\). Thus, \(a - 2\) and \(b - 2\) are factors (not necessarily positive) of 4. The possible pairs are (1, 4), (2, 2), (4, 1), (−1, −4), (−2, −2), and (−4, 1). (−2, −2) translates into \(a = b = 0\), an extraneous solution; the other 5 possibilities are valid pairs.

10. The sum can be rewritten as

\[
\begin{align*}
= &\frac{2^2 - 1^2}{(1 \cdot 2)^2} + \frac{3^2 - 2^2}{(2 \cdot 3)^2} + \frac{4^2 - 5^2}{(3 \cdot 4)^2} + \ldots \\
= &\left(\frac{1}{1^2} - \frac{1}{2^2}\right) + \left(\frac{1}{2^2} - \frac{1}{3^2}\right) + \left(\frac{1}{3^2} - \frac{1}{4^2}\right) + \ldots \\
= &\frac{1}{1^2} - \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{4^2} + \ldots \\
= &1.
\end{align*}
\]