NOTES FROM A TALK ON PARAMETRIZED THOM SPECTRA

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An $n$-dimensional vector bundle $\xi: V \rightarrow X$ gives rise to a spherical fibration $S^\xi: S^V \rightarrow X$, and thus to a local coefficient system

$$\tilde{H}^*(S^V) \otimes \Pi_1X \rightarrow \text{grAb}$$

$p \in X \rightarrow \tilde{H}^*(S^p)$

An orientation of $\xi$ is an isomorphism of local coefficient systems $\tilde{H}^*(S^V) \cong \mathbb{Z}[n]$, where $\mathbb{Z}[n]$ is the constant system given by the integers in degree $n$ and 0 elsewhere. The Serre spectral sequence associated with $\tilde{H}^*(S^V)$ takes the form

$$H^p(X, \tilde{H}^q(S^V)) \Rightarrow H^{p+q}(S^V, X) \cong \tilde{H}^{p+q}(X^\xi)$$

where, $X^\xi = S^V/X$ is the Thom space of $\xi$.

An orientation of $\xi$ "untwists" this spectral sequence so that it converges to $H^{p+q-n}(X)$. The resulting isomorphism of $E_\infty$ terms is the Thom isomorphism in integral cohomology.

Let $\gamma(n)$ be the tautological $n$-plane bundle over $BO(n)$. Then the Thom space $BO(n)\gamma(n)$ is the $n$-th space of the Thom spectrum $MO$ representing unoriented cobordism theory. More generally, given a compatible system of maps $f_n: X_n \rightarrow BO(n)$, we may define a spectrum $Mf$ whose $n$-th space is $X_n\gamma(n)$. Using this method, we can construct $MSO, MSp, MSpin, MString, MU$, etc. If $\xi - \eta$ is a virtual vector bundle, we choose $\xi'$ such that $\eta \oplus \xi'$ is the trivial bundle $e_N$ of rank $N$, and then define:

$$X^{\xi - \eta} = \Sigma^\infty X^{\xi \oplus \xi'}.$$ 

Alternatively, we could pass to the colimit of classifying spaces. The space $BO = \text{colim}_n BO(n)$ classifies rank zero virtual vector bundles, and we define the Thom spectrum associated to a map $f: X \rightarrow BO$ to be the Thom spectrum $Mf$ associated with the system $f_n: f^{-1}BO(n) \rightarrow BO(n)$ of maps into the finite skeleta.

In fact, the construction of Thom spectra only depends on the associated spherical fibration of a vector bundle. Let $h\text{Aut}(S^n)$ be the monoid of based homotopy equivalences $S^n \rightarrow S^n$. Then the classifying space $Bh\text{Aut}(S^n)$ classifies fibrations with fiber $S^n$. The $J$-homomorphism $J: BO(n) \rightarrow Bh\text{Aut}(S^n)$ is induced by one-point compactification. We may think of the space $\Omega^\infty S = \text{colim}_n \Omega^CS^n$ of stable self-maps of spheres as the space $\text{Hom}_S(S,S)$ of $S$-module endomorphisms of the sphere spectrum. The subspace $\text{colim}_n h\text{Aut}(S^n) \subset \Omega^\infty S$ corresponds to the space $GL_1S$ of $S$-module automorphisms of $S$.

Suppose we are given a (not necessarily commutative) $S$-algebra $R$ with unit map $\eta: S \rightarrow R$. There is an $A_\infty$ space of units $GL_1R \subset \Omega^\infty R$ corresponding to the subspace of $\text{Hom}_R(R,R)$ consisting of $R$-module automorphisms. Composing the $J$-homomorphism with the map of units induced by $\eta$, we have the diagram

$$BO \rightarrow \text{colim}_n Bh\text{Aut}(S^n) = BGL_1S \rightarrow BGL_1R$$

We will now extend the construction of Thom spectra to accept maps $f: X \rightarrow BGL_1R$ as input.

To this end, we define a universal principal $GL_1R$-bundle $EGL_1R \rightarrow BGL_1R$ in terms of a two-sided bar construction

$$B(*, GL_1R, GL_1R) \rightarrow B(*, GL_1R, *).$$

In order to do this truthfully, one needs to make a choice of model for $A_\infty$ spaces as monoids in some category with a symmetric monoidal structure $\boxtimes$. One then forms the bar construction in the usual way but with respect to $\boxtimes$ instead of the cartesian product. See [5,7,9] for a few different approaches to the required technology.
Writing $B$ for $B\text{GL}_1 R$, we may form the parametrized suspension spectrum $\Sigma_B^\infty E\text{GL}_1 R$, whose fibers are of the form $\Sigma^\infty_E \Sigma^\infty_{\text{GL}_1 R}$. In particular, this is a right $\Sigma^\infty_{\text{GL}_1 R}$-module. The spectrum $R$ is a left $\Sigma^\infty_{\text{GL}_1 R}$-module. In analogy with the vector bundle associated to the universal principal $O(n)$-bundle, we define the universal rank 1 $R$-module bundle (a.k.a. line $R$-bundle) to be the parametrized $R$-module
\[ \Sigma_B^\infty E\text{GL}_1 R \wedge \Sigma^\infty_{\text{GL}_1 R} R. \]

**Theorem.** Let $f : X \to B\text{GL}_1 R$ be a map of spaces. The parametrized Thom spectrum associated to $f$ is the base change of the universal rank 1 $R$-module bundle along the map $f$:
\[ M_X f = f^*(\Sigma_B^\infty E\text{GL}_1 R \wedge \Sigma^\infty_{\text{GL}_1 R} R) \]
\[ \cong \Sigma_X^\infty f^* E\text{GL}_1 R \wedge \Sigma^\infty_{\text{GL}_1 R} R. \]

$M_X f$ is a rank 1 $R$-bundle over $X$, i.e. a parametrized $R$-module spectrum over $X$ whose fibers are equivalent to $R$. The total Thom spectrum associated to $f$ is the $R$-module $Mf = r_! M_X f$. Here, $r_!$ is the right adjoint to the functor $r^*$ from spectra to parametrized spectra that gives the “untwisted” parametrized spectrum.

The association
\[ (f : X \to B\text{GL}_1 R) \mapsto M_X f \]
induces a bijection between the set of homotopy classes of maps $[X, B\text{GL}_1 R]$ and the set of fiberwise weak equivalence classes of parametrized rank one $R$-module spectra over $X$.

Suppose that $E$ is a parametrized spectrum with fiber $M$. We say that $E$ is trivializable if there is a weak homotopy equivalence of parametrized spectra $E \simeq r^* M$. It follows from the theorem that the Thom spectrum $M_X f$ is trivializable if and only if the the map $f : X \to B\text{GL}_1 R$ null-homotopic.

From the quasicategory point of view, there is a model for $B\text{GL}_1 R$ that suggests we take this theorem as a definition. In [1], a parametrized rank 1 $R$-module spectrum over $X$ is defined to be a map $f : X \to B\text{GL}_1 R$.

**Theorem (Mahowald-Ray).** Let $\xi : Y \to X$ be a spherical fibration over $X$, and let $f(Y) \to B\text{GL}_1 S$ be the map induced by the classifying map for $Y$. The spherical fibration $Y$ is $R$-orientable if and only if the parametrized Thom spectrum $M_X f(Y)$ is trivializable.

**Proof.** A Thom class $\mu \in R^n(X^\xi)$ may be represented by a map $\mu : r_! Y = X^\xi \to \Sigma^n R$ with adjoint $\bar{\mu} : Y \to r^* \Sigma^n R = R \wedge X^\xi S^n_X$. Composing with the multiplication of $R$ gives a map
\[ \psi : M_X f = R \wedge X^\xi Y \xrightarrow{id \wedge \bar{\mu}} R \wedge R \wedge X^\xi S^n_X \to R \wedge X^\xi S^n_X. \]

For each point $x \in X$, the class $\mu_x \in R^n(Y_x)$ is a unit if and only if the restriction $\psi_x$ of $\psi$ to the map of fibers over $x$ is a weak equivalence of $R$-modules. This proves the theorem.

When $\xi$ is $R$-oriented, we may deduce the Thom isomorphisms
\[ R_*(X^\xi) \cong R_{*+n}(X^\xi) \quad R^*(X^\xi) \cong R^{*+n}(X^\xi) \]
from the equivalences of spectra
\[ R \wedge X^\xi \cong R \wedge \Sigma^n X^\xi \quad F(X^\xi, R) \cong F(\Sigma^n X^\xi, R) \]
on obtained by applying $r_!$ to the equivalence of parametrized spectra given in the theorem.
Example. Let $R = H \mathbb{Z}$. Then $GL_1 H \mathbb{Z} = \mathbb{Z}/2$, and the composite
\[ w_2: BO \xrightarrow{J} BGL_1 S \rightarrow BGL_1 H \mathbb{Z} = K(\mathbb{Z}/2, 1) \]
represents the first Stiefel-Whitney class $w_2$. Let $\xi$ be a vector bundle, and let $f: X \rightarrow BO$ be the map induced by the map representing $\xi$. The vector bundle $\xi$ is $H \mathbb{Z}$-orientable if and only if the Thom spectrum $M_X f = X^e \wedge H \mathbb{Z}$ is trivializable. The latter condition is equivalent to the vanishing of the first Stiefel Whitney class $w_1(\xi) = [w_1 \circ f] \in H^1(X; \mathbb{Z}/2)$.

Example. Let $R = K$ be complex $K$-theory. Then $\Omega^\infty K = \mathbb{Z} \times BU$, and the space of units of $K$ decomposes as a product
\[ GL_1 K = \mathbb{Z}/2 \times BU(1) \times BU_\pi = \mathbb{Z}/2 \times K(\mathbb{Z}, 2) \times BU_\pi. \]
If we pass to the connected cover $SO$ of $O$, the map
\[ SO \xrightarrow{J} GL_1 S \rightarrow GL_1 K \]
factors through the connected cover $SL_1 K = K(\mathbb{Z}, 2) \times BU_\pi$ of $GL_1 K$. At the level of classifying spaces, the map
\[ w_2: BSO \xrightarrow{J} BGL_1 S \rightarrow BGL_1 K = K(\mathbb{Z}/2, 1) \times K(\mathbb{Z}, 3) \times BBU_\pi \xrightarrow{\beta} K(\mathbb{Z}/2, 1) \]
representing the first Stiefel-Whitney class is nullhomotopic. Let $Spin(n)$ be the universal cover of $SO(n)$, realized as a Lie group, and let $Spin^e(n) = Spin(n) \times \mathbb{Z}/2 U(1)$ be the associated principal $U(1)$-bundle over $SO(n)$. Then $Spin^e(n)$ is also a compact Lie group, and the colimit of the classifying spaces $BSpin^e = \text{colim}_n BSpin^e(n)$ is the fiber of the composite
\[ BSO \xrightarrow{w_2} K(\mathbb{Z}/2, 2) \xrightarrow{\beta} K(\mathbb{Z}, 3) = BU(1) \]
of the second Stiefel-Whitney class and the Bockstein $\beta$. Therefore we have the following commutative diagram.

\[ \begin{array}{ccc}
BSpin^e & \xrightarrow{J} & EGL_1 K \simeq \ast \\
\downarrow & & \downarrow \\
BSO & \xrightarrow{J} & BGL_1 S \\
& & \downarrow \pi \\
& & K(\mathbb{Z}, 3)
\end{array} \]

Since the map from $BSpin^e$ to $BGL_1 K$ is nullhomotopic, it follows that a real vector bundle $\xi$ is $K$-orientable if and only if it has a reduction of its structural group to $Spin^e$, i.e. if $w_1(\xi) = 0$ and $\beta w_2(\xi) = 0$. This is the result of Atiyah-Bott-Shapiro [3], who constructed Thom isomorphisms in $K$-theory for $Spin^e$-bundles.

Let $f: BSpin^e \rightarrow BGL_1 K$ be the composite in the diagram. Then $f$ is nullhomotopic, so the parametrized Thom spectrum $M_{BSpin^e} f$ is trivializable:
\[ M_{BSpin^e} f \simeq S_{BSpin^e} \wedge K. \]
Applying $r_1$ to the trivialization yields an equivalence of ring spectra (the Thom isomorphism):
\[ MSpin^e \wedge K \simeq BSpin^e_+ \wedge K. \]
In modern language, the Atiyah-Bott-Shapiro orientation is the map of ring spectra $MSpin^e \rightarrow K$ given by the composite
\[ MSpin^e \rightarrow MSpin^e \wedge K \simeq BSpin^e_+ \wedge K \rightarrow K \]
of the unit for $K$ theory with the projection of $BSpin^e$ to a point. See [6] for a direct construction of the map of ring spectra. We can think about the orientation map geometrically. The homotopy of $MSpin^e$ is the ring of bordism classes of manifolds equipped with $Spin^e$-structures on their tangent bundles, and the homotopy of the complex $K$-theory spectrum is given by equivalence classes of complex Hilbert spaces with an action of the Clifford algebra $\text{Cliff}(\mathbb{C}^n)$ and an odd skew-adjoint $\text{Cliff}(\mathbb{C}^n)$-linear Fredholm operator. The Atiyah-Bott-Sapiro orientation sends such a manifold $M$ to the Hilbert space of $L^2$ sections of the spinor
bundle, equipped with a Cliff($\mathcal{C}$)-action, along with the Fredholm operator given by the Dirac operator constructed from the connection associated to a choice of metric on $M$.

In general, if $A$ is a ring spectrum and $R$ is an $A$-algebra, we define an $R$-orientation of the Thom spectrum $Mf$ associated to $f: X \to BGL_1 A$ to be a choice of lift $\overline{f}$ in the following diagram:

$$
\begin{array}{c}
P \\
\downarrow \\
X \\
\downarrow f \\
BGL_1 A \\
\downarrow \\
BGL_1 R
\end{array}
\begin{array}{c}
\rightarrow \\
\overline{f} \\
\rightarrow \\
\rightarrow
\end{array}
\begin{array}{c}
EGL_1 R \\
\rightarrow \\
BGL_1 R
\end{array}
$$

In other words, $\overline{f}$ is a choice of trivialization of the rank one $R$-bundle associated to $f$. A choice of orientation is equivalent to a choice of a map of $A$-algebra spectra $Mf \to R$ that can be thought of as the “projection to the fiber.” This is the approach to orientations developed by Ando, Blumberg, Gepner, Hopkins, and Rezk [2,4] in order to calculate the space of $tmf$-orientations of $MString$.

References


