From the point of view of stable homotopy theory, twisted cohomology theories are represented by parametrized spectra. We will review this idea, and describe a way to implement it precisely using a monoidal model for $A_\infty$ multiplications. We then add in a group of equivariance $G$ and propose a general definition for $G$-equivariant twisted cohomology theories.

Let $R$ be a ring spectrum and let $GL_1 R$ be the $A_\infty$ space of units of $R$. We think of $GL_1 R$ as the space of equivalences of $R$-modules $R \rightarrow R$. The delooping $BGL_1 R$ classifies $R$-line bundles, meaning parametrized $R$-modules whose fibers admit equivalences to $R$. Starting from a map of spaces $\tau: X \rightarrow BGL_1 R$, we form the associated line $R$-bundle $L(\tau)$ over $X$ by the formula:

$$L(\tau) = R \wedge^{GL_1 R} \Sigma \infty X P(\tau),$$

where $P(\tau) \rightarrow X$ is the “principal $GL_1 R$-bundle” associated to $\tau$. The $\tau$-twisted $R$ homology and cohomology groups of $X$ are defined by

$$R^*_\tau = \pi^* r_! L(\tau) \quad \text{and} \quad R^*_{-\tau} = \pi_*(-)_* L(\tau),$$

where $r_!$ and $r_*$ are the left and right adjoints to the pullback functor $r^*$ from spectra to parametrized spectra over $X$. The functor $r_*$ admits a construction as the space of homotopy sections.

When $R = K$ is the complex $K$-theory spectrum, the space of units has the following homotopy type:

$$BGL_1 K \simeq K(\mathbb{Z}/2, 1) \times K(\mathbb{Z}, 3) \times BBSU\otimes_\emptyset.$$

Atiyah-Segal define twists of $K$-theory by realizing maps into the component $K(\mathbb{Z}, 3) = BP(\mathcal{H})$ as principal bundles with structure group the projective infinite unitary group $P(\mathcal{H})$ on a Hilbert space $\mathcal{H}$, then using such a bundle to twist the complex $K$-theory spectrum, modeled as a space of Fredholm operators.

In order to make this sketch precise, we need some sort of coherence machinery to make sense of the notion of a ‘principal $\Pi$-bundle’ when $\Pi$ is an $A_\infty$-space. The idea is to find a symmetric monoidal product $\boxtimes$ on the category of spaces whose monoids are $A_\infty$ spaces. If $\Pi$ is a $\boxtimes$-monoid, we define a principal $\Pi$-fibration over a space $X$ to be a $\Pi$-module $Y \rightarrow X$ over $X$ (where $X$ has the trivial $\Pi$-module structure) that is a Hurewicz fibration in the category of $\Pi$-modules and such that every fiber $Y_x$ admits an equivalence of $\Pi$-modules to $\Pi$. We can mimic the classical theory of the classification of fibrations to prove that the space $B^{\boxtimes}\Pi$ built using the symmetric monoidal product $\boxtimes$ as a two sided bar construction classifies principal $\Pi$-fibrations.

There are a few different options for the symmetric monoidal product $\boxtimes$, and each of them requires putting additional structure on the notion of topological space in use. One approach is the $L$-spaces of Blumberg and co.; these are spaces with an action of the first space $L(1)$ in the linear isometries operad. One them defines
the product $\boxtimes$ in analogy with the EKMM smash product of spectra. Monoids under $\boxtimes$ are the same thing as algebras over the (non-symmetric) linear isometries operad, and thus are $A_{\infty}$ spaces. Our model for spectra will be orthogonal spectra, not EKMM spectra, so we will use the following category instead. Let $I$ be the category of real inner-product spaces and linear isometries $V \rightarrow W$. An $I$-space is a continuous functor from $I$ to the category of spaces. We define the product $X \boxtimes Y$ of a pair of $I$-spaces as the left Kan extension of the external cartesian product $(V, W) \mapsto X(V) \times Y(W)$ along the direct sum functor $\oplus : I^2 \rightarrow I$. The functor $X \boxtimes Y$ is computed level-wise by the formula:

$$(X \boxtimes Y)(V) = \lim_{W \oplus W' \rightarrow V} X(W) \times Y(W').$$

Monoids under $\boxtimes$ are essentially filtered algebras over the (non-symmetric) linear isometries operad, and (a slight variant of) the colimit functor takes a $\boxtimes$-monoid to an $\mathcal{L}$-space. There is also a version of diagram spaces that fits with symmetric spectra. These are functors from Bökstedt’s indexing category $I$ of finite sets and injections into spaces.

One can then carry out the program of classifying principal $\Pi$-fibrations when $\Pi$ is $\boxtimes$-monoid. The hard part is proving that the universal object $E^{\boxtimes}\Pi \rightarrow B^{\boxtimes}\Pi$ is a quasifibration of $I$-spaces.

We will now weave in an action of a compact Lie group $G$ into the story. Fix a $G$-universe $U$. We define the diagram category $I_G$ as follows. The objects of $I_G$ are $G$ inner product spaces $V$ that admit an isomorphism to some $G$-subspace of $U$. The morphisms in $I_G$ from $V$ to $W$ are the (not necessarily equivariant) linear isometries $V \rightarrow W$. The group $G$ acts on the morphism spaces $I_G(V, W)$ by conjugation, making the category $I_G$ enriched in the category $G\mathcal{U}$ of $G$-spaces and $G$-maps. Passage to fixed points of morphism spaces defines the category $G\mathcal{I} = (I_G)^G$ with the same objects as $I_G$, but whose morphisms are $G$-equivariant linear isometries.

An $I_G$-space is a continuous $G$-functor $X : I_G \rightarrow \mathcal{U}_G$. Concretely, this means that we have a $G$-space $X(V)$ for every object $V$ of $I_G$, along with $G$-equivariant maps

$$X : I_G(V, W) \rightarrow \mathcal{U}_G(X(V), X(W))$$

that encode the functoriality of $X$. Notice that left multiplication by an element $g \in G$ defines a linear isometry $\varphi_g : V \rightarrow V$ for any indexing space $V$ in $I_G$, and thus an automorphism $X(\varphi_g)$ of the space $X(V)$. This action does not necessarily coincide with the pre-existing action of $g$ on the $G$-space $X(V)$.

If $E$ is an orthogonal $G$-spectrum, then the $I_G$-space

$$\Omega^*)E : V \mapsto \Omega^V E(V)$$

should be thought of as a model for the underlying $G$-equivariant infinite loop space of the spectrum $E$. If $R$ is an orthogonal (commutative) ring $G$-spectrum, then $\Omega^*)R$ is a (commutative) $\boxtimes$-monoid, and we may form the group-like $\boxtimes$ monoid $GL_1 R \subset \Omega^*)R$ of units of $R$ by restricting to those components at each level which are invertible under the $\boxtimes$-multiplication.

We now seek to define a $G$-equivariant classifying space $B_G GL_1 R$ which will give rise to equivariant twists of $R$-theory. In the case where the bundle structure group $\Pi$ is an honest topological group, this works in the following way. We construct the universal $G$-equivariant principal $\Pi$-bundle $E_G\Pi \rightarrow B_G\Pi$ as the quotient map
\[ E(\mathcal{F}) \rightarrow E(\mathcal{F})/\Pi \text{ where } \mathcal{F} \text{ is the following family of subgroups of } \Gamma = \Pi \rtimes G: \]
\[ \mathcal{F} = \{ H < \Gamma \mid H \cap \Pi = e \}. \]

Elements of \( \mathcal{F} \) are determined by a subgroup \( H < G \) and a 1-cocycle \( \varphi \in C^1(H; \Pi) \).

The data
\[ (H, \varphi): \varphi: H \rightarrow \Pi, \quad \varphi(gh) = \varphi(g) \cdot \varphi(h) \]
corresponds to the subgroup
\[ H = \{(\varphi(h), h) \in \Pi \times G \mid h \in H\} \in \mathcal{F}. \]

Then by definition \( E(\mathcal{F}) = B(\mathcal{F}, \mathcal{O}_\Gamma, S) \), where \( \mathcal{O}_\Gamma \) is the orbit category, \( \mathcal{F}(H) \) is a point for \( H \in \mathcal{F} \) and empty otherwise, and \( S \) is the “realization” functor with values in \( \Gamma \)-spaces defined by \( S(|\Gamma/H|) = \Gamma/H \) and similarly for morphisms.

In our setting, \( \Pi \) is not a \( G \)-equivariant group but a grouplike \( G \)-equivariant \( \boxtimes \)-monoid in \( I \)-spaces. Let \( \Gamma = \Pi \rtimes G \) be the \( \boxtimes \)-monoid defined by \( \Gamma(V) = \Pi(V) \times G \), with multiplication
\[ \Pi(V) \times G \times \Pi(W) \times G \xrightarrow{G \text{ acts on } \Pi(W)} \Pi(V) \times \Pi(W) \times G \times G \rightarrow \Pi(V \oplus W) \times G, \]
where the second arrow is the multiplication of \( \Pi \) and \( G \).

We may now make a “cocycle” description of the family \( \mathcal{F} \). Let \( H < G \) be a subgroup of \( G \). A 1-cocycle \( \varphi \in C^1(H; \Pi) \) is a map of \( I \)-spaces
\[ \varphi_V: H \rightarrow \Pi(V) \]
satisfying the cocycle condition
\[ \varphi_V(gh) = \varphi_V(g) \cdot \varphi_V(h) \]
\[ \varphi_V \mid_H \text{ is the restriction of the action of } G \text{ on } \Pi(W) \text{ to } H. \]

Coboundaries involve inverses, which is NOT rigidly presented in the data of a grouplike \( \boxtimes \)-monoid. One could perhaps make use of the existence of up to homotopy inverse to define the cohomology groups \( H^*(G; \Pi) \), but I have not pursued this idea very far.

Still, we can make sense of the orbit category \( O_\mathcal{F} \) associated to the family \( \mathcal{F} \). The data \( (H, \varphi) \) determine a sub \( \boxtimes \)-monoid
\[ H(V) = \{(\varphi_V(h), h) \in \Pi(V) \times G \mid h \in H\} \subset (\Pi \times G)(V) \]
The multiplication of \( H \) induces a (right) action of \( H \) on the \( I \)-space \( \Pi \rtimes G \), and we can form the quotient \( I \)-space
\[ (\Pi \rtimes G)/H = (\Pi \rtimes G) \boxtimes_H \ast. \]
The quotient $(\Pi \times G)/H$ is a $(\Pi \times G)$-module. Let $\mathcal{O}_\mathcal{F}$ be the category whose objects are the $(\Pi \times G)$-modules of the form $(\Pi \times G)/H$ for some $(H_0, \varphi)$ as above, and whose morphisms are maps of $(\Pi \times G)$-modules.

The category $\mathcal{O}_\mathcal{F}$ is enriched in $\mathcal{I}$-spaces, and there is an enriched functor $S: \mathcal{O}_\mathcal{F} \to (\mathcal{I}$-spaces) that sends $(\Pi \times G)/H$ to its underlying $\mathcal{I}$-space. We may now form the two sided bar construction $B^\otimes(*, \mathcal{O}_\mathcal{F}, S)$ with respect to the $\otimes$ product. In other words, $B^\otimes(*, \mathcal{O}_\mathcal{F}, S)$ is the $\mathcal{I}$-space arising as the geometric realization of the simplicial $\mathcal{I}$-space with $q$-simplices the $\mathcal{I}$-space

$$\bigoplus_{(H_0, \ldots, H_q)} [(\Pi \times G)/H_{q-1}, (\Pi \times G)/H_q] \mathcal{I} \cdots \mathcal{I} [(\Pi \times G)/H_0, (\Pi \times G)/H_1] \mathcal{I} ((\Pi \times G)/H_0$$

Here $[-, -]$ denotes the $\mathcal{I}$-space hom in the category $\mathcal{O}_\mathcal{F}$ and the coproduct is over sequences of data $H = (H, \varphi)$ determining an object of $\mathcal{F}$.

The upshot is that $E_G \Pi = E\mathcal{F} = B^\otimes(*, \mathcal{O}_\mathcal{F}, S)$ makes sense (as an $\mathcal{I}$-space), and we may form the quotient $\mathcal{I}$-space

$$E_G \Pi \to E_G \Pi /\Pi = B_G \Pi.$$

When $G$ is trivial, this looks like

$$E_G \Pi = B^\otimes(*, \Pi, \Pi) \to B(*, *, *) = B\Pi,$$

recovering the non-equivariant universal $\Pi$-quasifibration discussed earlier.

One question: we’d like $B_G \Pi$ to be an equivariant delooping of the grouplike $\otimes$-monoid $\Pi$, but it’s not clear exactly what this means. The non-trivial one-dimensional representation spheres are those of the form $S^\sigma$, where $\sigma: G \to \{\pm 1\}$ is a sign representation of $G$. Can we find deloopings $B_\sigma \Pi$ of $\Pi$ in each “$\sigma$-direction” within $B_G \Pi$?

From here, we can define $G$-equivariant (in fact, RO($G$)-graded) twists of $R$-theory. Letting $\Pi = GL_1 R$ with the induced $\otimes$ monoid structure coming from $R$, a map $\tau: X \to B_G GL_1 R$ gives rise to the pullback bundle $P(\tau) = \tau^* E_G GL_1 R$ over $X$ (perhaps after an approximation by a fibration). Then the homotopy sections of the $G$-equivariant parametrized spectrum

$$R \land_{\Sigma_\infty^{GL_1 R}} \Sigma_X^\infty P(\tau)$$

give the $\tau$-twisted $R$-cohomology of $X$. In future work, I intend to check if this agrees with the definitions of Hopkins-Freed-Teleman in the case of equivariant $K$-theory.