Name: ________________________________

Recitation section:
___ 1. Tuesday 3:00 (D. Ginsberg)
___ 2. Tuesday 4:30 (D. Ginsberg)
___ 3. Thursday 1:30 (P.Y. Chang)
___ 4. Thursday 3:00 (M. Farag)
___ 5. Thursday 3:00 (D. Seitova)

Work quickly and carefully, and write your solutions clearly. For clarity, it is recommended that you put your final answer in a box. However, please show your work and cite all the theorems or lemmas wherever applicable; partial credit will be given. Keep your cool and manage the time well.

Statement of ethics
I agree to complete this exam without unauthorized assistance from any person, materials, or device.

Signature: ________________________________ Date: ________________

<table>
<thead>
<tr>
<th>Problem</th>
<th>Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>/10</td>
</tr>
<tr>
<td>2</td>
<td>/10</td>
</tr>
<tr>
<td>3</td>
<td>/15</td>
</tr>
<tr>
<td>4</td>
<td>/10</td>
</tr>
<tr>
<td>5</td>
<td>/5</td>
</tr>
<tr>
<td>TOTAL</td>
<td>/50</td>
</tr>
</tbody>
</table>
Problem 1 (10 points). Solve the differential equation
\[ y' + y \ln x = \frac{1}{x^2}, \quad x > 0. \]

Hint: Rewrite \( \frac{1}{x^x} \) in terms of \( e^{something} \).

Solution. This is a first-order, linear ODE, and so we will solve it using an integrating factor. We start by setting:
\[ \mu(x) = e^{\int \ln x \, dx} = e^{x \ln x - x}, \]
where we have used \( \int \ln x \, dx = x \ln x - x \) (either by remembering this formula, or integrating by parts).

Multiplying the original equation by \( \mu \), we get:
\[ (1) \quad y' \mu + y \mu \ln x = (y \mu)' = \frac{\mu}{x^x}. \]
(Here we have used that \( \mu' = \mu \ln x \), by construction, as well as the product rule).

Now we use the hint to simplify the right-hand side. We note that:
\[ x^x = \left(e^{\ln x}ight)^x = e^{x \ln x}, \]
so that:
\[ \frac{\mu}{x^x} = \frac{e^{x \ln x} e^{-x}}{e^{x \ln x}} = e^{-x}. \]

We can now integrate both sides of (1):
\[ (y \mu)' = e^{-x} \implies y \mu = -e^{-x} + c \implies y = \frac{-e^{-x} + c}{e^{x \ln x - x}} = \frac{-e^{-x} + c}{x^x e^{-x}} \]
for some constant \( c \). Simplifying, we get:
\[ y = \frac{ce^x}{x^x} - \frac{1}{x^x} \]
is the general solution to this problem, when \( x > 0 \).
**Problem 2** \((2 \times 5 = 10\) points).

Compute the following integrals

- **(A)** \[ \int e^x \cos x \, dx \]

**Solution.** We integrate by parts, and set:

\[
\begin{align*}
&u = e^x, \quad dv = \cos x \, dx \\
\implies &du = e^x \, dx, \quad v = \sin x.
\end{align*}
\]

Then we have:

\[
\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx.
\]

We integrate the second term by parts, with:

\[
\begin{align*}
&u = e^x, \quad dv = \sin x \, dx \\
\implies &du = e^x \, dx, \quad v = -\cos x,
\end{align*}
\]

which gives:

\[
\int e^x \sin x \, dx = -e^x \cos x + \int e^x \cos x \, dx.
\]

Plugging this in to (2), we have:

\[
\int e^x \cos x \, dx = e^x \sin x + e^x \cos x - \int e^x \cos x \, dx
\]

\[
\implies 2 \int e^x \cos x = e^x (\sin x + \cos x),
\]

so that the most general antiderivative is:

\[
\int e^x \cos x \, dx = \frac{e^x}{2} (\sin x + \cos x) + C
\]

- **(B)** \[ \int \sin^3 x \cos^2 x \, dx \]

**Solution.** We write \(\sin^2 x = \sin x \sin^2 x = \sin x (1 - \cos^2 x) = \sin x - \cos x \sin x.\) The integral becomes:

\[ \int \sin^3 x \cos^2 x \, dx = \int \sin x \cos^2 x - \sin x \cos^4 x \, dx. \]

If we substitute \(u = \cos x,\) then \(du = -\sin x \, dx,\) and the above integral becomes:

\[ \int -u^2 + u^4 \, du = -\frac{1}{3}u^3 + \frac{1}{5}u^5 + C. \]

Subsituting back, the answer is:

\[ \int \sin^3 x \cos^2 x \, dx = -\frac{1}{3} \cos^3 x + \frac{1}{5} \cos^5 x + C. \]
Problem 3 (15 points). Using integration by parts, prove the reduction formula

\[
\int \frac{1}{(1 + x^2)^n} \, dx = \frac{x}{2(n-1)(1 + x^2)^{(n-1)}} + \frac{2n-3}{2(n-1)} \int \frac{1}{(1 + x^2)^{(n-1)}} \, dx
\]

Solution. We are essentially forced to pick:

\[
u = \frac{1}{(1 + x^2)^n}, \quad dv = dx
\]

\[\implies du = -\frac{2nx}{(1 + x^2)^{n+1}} \, dx, \quad v = x.
\]

Writing \( I_n = \int \frac{1}{(1 + x^2)^n} \, dx \), we have:

\[
I_n = \frac{x}{(1 + x^2)^n} + 2n \int \frac{x^2}{(1 + x^2)^{n+1}} \, dx.
\]

We want to re-write this last term so that it looks more like \( I_n \). If we write \( x^2 = 1 + x^2 - 1 \), the integral in the last term is:

\[
\int \frac{x^2}{(1 + x^2)^{n+1}} \, dx = \int \frac{1 + x^2}{(1 + x^2)^{n+1}} \, dx - \int \frac{1}{(1 + x^2)^{n+1}} \, dx
\]

\[= \int \frac{1}{(1 + x^2)^n} \, dx - I_{n+1}.
\]

Putting everything together, we see that we have proven:

\[
I_n = \frac{x}{(1 + x^2)^n} + 2nI_n - 2nI_{n+1}
\]

Of course, \( n \) is just a dummy variable, so we can replace \( n \) with \( n - 1 \) everywhere (or, we could have started with \( I_{n-1} \) in the first place). The above formula is then:

\[
I_{n-1} = \frac{x}{(1 + x^2)^{n-1}} + 2(n-1)I_{n-1} - 2(n-1)I_n \implies 2(n-1)I_n = \frac{x}{(1 + x^2)^{n-1}} + 2(n-1)I_{n-1} - I_{n-1}
\]

Simplifying, we get:

\[
I_n = \frac{x}{2(n-1)(1 + x^2)^{n-1}} + \frac{2n-3}{2(n-1)} I_{n-1}.
\]

This is exactly the formula we wanted to prove.
Problem 4 (10 points). Find the general solution of the differential equation
\[ y' + \sin \left( \frac{x + y}{2} \right) = \sin \left( \frac{x - y}{2} \right). \]

Hint: The following formulas could be useful
\[ \sin(A + B) = \sin A \cos B + \cos A \sin B \]
\[ \sin(A - B) = \sin A \cos B - \cos A \sin B. \]

Solution. We begin by applying these formulas to \( \sin \left( \frac{x + y}{2} \right) \) and \( \sin \left( \frac{x - y}{2} \right) \):
\[
\sin \left( \frac{x + y}{2} \right) = \sin \left( \frac{x}{2} \right) \cos \left( \frac{y}{2} \right) + \cos \left( \frac{x}{2} \right) \sin \left( \frac{y}{2} \right) \\
\sin \left( \frac{x - y}{2} \right) = \sin \left( \frac{x}{2} \right) \cos \left( \frac{y}{2} \right) - \cos \left( \frac{x}{2} \right) \sin \left( \frac{y}{2} \right).
\]
Plugging this in to the equation, it simplifies to:
\[ y' = -2 \cos \left( \frac{x}{2} \right) \sin \left( \frac{y}{2} \right) \implies \csc \left( \frac{y}{2} \right) dy = -2 \cos \left( \frac{x}{2} \right) dx \]
To integrate this, we will need to recall the formula:
\[ \int \csc u du = -\ln |\csc u + \cot u| + c \]
We arrive at the implicit solution:
\[ \ln \left| \csc \left( \frac{y}{2} \right) + \cot \left( \frac{y}{2} \right) \right| = -2 \sin \left( \frac{x}{2} \right) + c \]
If we exponentiate both sides, we can simplify a bit:
\[ \left| \csc \left( \frac{y}{2} \right) + \cos \left( \frac{y}{2} \right) \right| = e^{-2 \cos \left( \frac{x}{2} \right) + c} = Ce^{-2 \cos \left( \frac{x}{2} \right)}, \]
where \( C = e^c \).
**Problem 5** (5 points). Describe the motion of a particle with position \((x, y)\) as \(t\) varies in the given interval

\[
x = 3 + 2 \cos t, \quad y = 1 + 2 \sin t, \quad t \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]
\]

**Solution.** We start by noticing that:

\[(x - 3)^2 + (y - 1)^2 = 4 \cos^2 t + 4 \sin^2 t = 4,
\]

so that the particle is confined to the circle of radius 2, centered at \((x, y) = (3, 1)\).

At \(t = \frac{\pi}{2}\), the particle is at \((3, 3)\) (the “top” of the circle), and at \(t = \frac{3\pi}{2}\), the particle is at \((3, -1)\) (the “bottom” of the circle). Therefore:

| The particle moves along the circle of radius 2 centered at \((3, 1)\). It starts at the top of the circle and moves counterclockwise to the bottom. |

\[
\begin{array}{c}
\text{\(t\) from \(\frac{\pi}{2}\) to \(\frac{3\pi}{2}\)}
\end{array}
\]