Problem 1: Evaluate \( \int \frac{dx}{(ax^2 - b^2)^{3/2}} \) by using trigonometric substitution.

(sol) Let \( ax = b \sec \theta \) if \( a \neq 0 \) and \( b \neq 0 \)

\[ a \, dx = b \sec \theta \tan \theta \, d\theta \]

\[ dx = \frac{b}{a} \sec \theta \tan \theta \, d\theta \]

\[ \int \frac{dx}{(ax^2 - b^2)^{3/2}} = \int \frac{\frac{b}{a} \sec \theta \tan \theta \, d\theta}{[(b \sec \theta)^2 - b^2]^{3/2}} \]

\[ = \int \frac{\frac{b}{a} \sec \theta \tan \theta \, d\theta}{(b^2 \tan^2 \theta)^{3/2}} \]

\[ = \int \frac{\frac{1}{b^2} \frac{1}{\tan^2 \theta} \, d\theta}{\sec \theta} \quad \text{by} \quad \tan \theta = \frac{\sin \theta}{\cos \theta} \quad \sec \theta = \frac{1}{\cos \theta} \]

Now, notice that \( \frac{d}{d\theta} \left( \frac{-1}{\sin \theta} \right) = \frac{d}{d\theta} \left( \frac{\sin \theta}{\sin^2 \theta} \right) = \frac{\cos \theta}{\sin^2 \theta} \)

\[ \therefore \int \frac{dx}{(ax^2 - b^2)^{3/2}} = \frac{1}{ab^2} \left( \frac{-1}{\sin \theta} + C \right), \quad C \text{ is a constant} \]

but \( \sec \theta = \frac{ax}{b} \)

\[ \Rightarrow \sin \theta = \frac{\sqrt{(ax)^2 - b^2}}{ax} \]

\[ \therefore \int \frac{dx}{(ax^2 - b^2)^{3/2}} = \frac{-x}{b^2 \sqrt{(ax)^2 - b^2}} + \frac{c}{ab^2}, \quad \text{where} \ a \neq 0 \text{ and} \ b \neq 0. \]
If $a=0$, then

$$\int \frac{dx}{[(ax^2+b^2)^{3/2}] = \int \frac{dx}{(-b^3)^{3/2}}$$

doesn't make any sense.

If $b=0$, then

$$\int \frac{dx}{[(ax^2 + b^2)^{3/2}]} = \int \frac{dx}{(ax)^3} = \frac{1}{a^3} \int \frac{dx}{x^3} = \frac{-1}{2a^3 x^2} + C$$

for $a \neq 0$. 

Problem 2

(A.) Solve \( \frac{dv}{ds} = \frac{S+1}{sv+s} \)

(Sol.) \( \frac{dv}{ds} = \frac{S+1}{sv+s} = \frac{S+1}{s} \cdot \frac{1}{v+1} \) is separable.

\[ \int (v+1)\,dv = \int \frac{S+1}{s}\,ds = \int \left( 1 + \frac{1}{s} \right)\,ds \]

\[ \frac{v^2}{2} + v = S + \log(s) + C \]

\[ v^2 + 2v = 2(S + \ln(s) + C) \]

\[ (v+1)^2 = v^2 + 2v + 1 = 2S + 2\ln(s) + 2C + 1 \]

\[ v = -1 - \sqrt{2S + 2\ln(s) + 2C + 1} \]

where plus or minus sign depending on the initial data.

(B.) Solve \( \begin{cases} x y' = y + x^2 \sin x \\ y(\pi) = 0 \end{cases} \)

(Sol.) \( \left( \frac{y}{x} \right)' = \frac{xy' - y}{x^2} = \frac{x^2 \sin x}{x^2} = \sin x \)

\[ \int_{\pi}^{x} \frac{(y(s))'}{s}\,ds = \int_{\pi}^{x} \sin(s)\,ds = -\cos(s) \bigg|_{\pi}^{x} = -\cos x - 1 \]

\[ \frac{y(x)}{x} - \frac{y(\pi)}{\pi} = -1 - \cos x \]

\[ \Rightarrow \frac{y(x)}{x} = -1 - \cos x \]

\[ \Rightarrow y(x) = -x - x\cos x \]
Problem 3

1. $(x, y) = (2, -2)$

\[ r = \sqrt{x^2 + y^2} = \sqrt{4 + 4} = 2\sqrt{2} \quad \text{and} \]

\[ \tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{y}{x} = -1 \]

\[ \therefore \theta = -\frac{\pi}{4} + k\pi \quad \text{where } k \text{ is an integer} \]

Hence we can have representations

\[ (r, \theta) = (2\sqrt{2}, -\frac{\pi}{4}) \quad \text{or} \quad (2\sqrt{2}, \frac{3\pi}{4}) \quad \text{etc.} \]

\[ \{(r, \theta) | r \geq 0, \quad -\frac{\pi}{4} \leq \theta \leq \frac{3\pi}{4}\} \]

(\theta = \frac{3\pi}{4})

\[ y (\theta = -\frac{\pi}{2}) \]

\[ y (\theta = \frac{\pi}{2}) \]

\[ x (\theta = 0) \]

The shaded region is the answer.
Problem 4
\[
\begin{align*}
\begin{cases}
  x(t) &= 2t + 2 - \pi \sin t \\
  y(t) &= 1 + \pi \cos t
\end{cases}
\end{align*}
\]

Find the equations of all tangent lines to the curve at \((2, 1)\).

\text{(sol)
First of all, we find } t \text{ such that }
\begin{align*}
\begin{cases}
  2t + 2 - \pi \sin t &= 2 \\
  1 + \pi \cos t &= 1
\end{cases}
\end{align*}

\Rightarrow \begin{cases}
  t = \frac{\pi}{2} \sin t \\
  \cos t = 0
\end{cases}

\text{When } \cos t = 0 \Rightarrow \begin{aligned}
  t &= \frac{\pi}{2} + k\pi, \quad k \text{ is an integer.}
\end{aligned}

\Rightarrow \begin{aligned}
  \sin t &= \sin \left( \frac{\pi}{2} + k\pi \right) = (-1)^k \\
  \Rightarrow \frac{\pi}{2} + k\pi &= t = \frac{\pi}{2} \sin t = \frac{\pi}{2} \cdot (-1)^k
\end{aligned}

\Rightarrow k\pi = \frac{\pi}{2} \cdot (-1)^k - 1

\text{Case (1) When } k \text{ is even } \Rightarrow 2k = 0 \Rightarrow k = 0

\text{Case (2) When } k \text{ is odd } \Rightarrow 2k = -2 \Rightarrow k = -1

\Rightarrow t = \frac{\pi}{2} \text{ or } -\frac{\pi}{2}

\text{Then } m = \text{slope } = \frac{dy}{dx} = \frac{y(t)}{x(t)} = \frac{\frac{\pi}{2} \sin t}{2 - \pi \cos t}

\Rightarrow m(\frac{\pi}{2}) = \frac{-\pi}{2}, \quad m(-\frac{\pi}{2}) = \frac{\pi}{2}

\therefore \text{ There are two tangents:}

y = -\frac{\pi}{2} (x-2) + 1 \quad \text{and} \quad y = \frac{\pi}{2} (x-2) + 1.
Problem 5 Determine the convergence by using comparison test:

\[ \int_{a}^{\infty} \frac{\tan^{-1}(x)}{2+e^x} \, dx \]  
(Forget the bounds?) \( a \in \mathbb{R} \).

\[
\text{Sol} \quad \text{Notice that} \quad \frac{-\pi}{2} \leq \tan^{-1}(x) \leq \frac{\pi}{2} \]

\[ \Rightarrow \quad \frac{-\pi}{2} \cdot \frac{1}{2+e^x} \leq \frac{\tan^{-1}(x)}{2+e^x} \leq \frac{\pi}{2} \cdot \frac{1}{2+e^x} \]

However, \( e^x \leq 2+e^x \) and \( 2+e^x \leq 2e^x \) if \( x \geq 0 \)

\[ \Rightarrow \quad \frac{-\pi}{2} e^{-x} \leq \frac{\tan^{-1}(x)}{2+e^x} \leq \frac{\pi}{2} e^{-x} \]

\[ \therefore \quad \int_{a}^{\infty} \frac{\tan^{-1}(x)}{2+e^x} \, dx \] has the same convergence as \( \int_{a}^{\infty} e^{-x} \, dx = e^{-a} \)

Therefore, it is convergent.
Problem 6 \[ \sum_{j=0}^{\infty} \frac{(-1)^j \sqrt{j}}{j+5} \] Determine whether it's convergent or not.

(The original bound doesn't make sense when \( j = -1 \))

Consider the function

\[ f(x) = \frac{\sqrt{x}}{x+5} \text{ on } x \geq 0. \]

\[ f'(x) = \frac{(x+5)^{1/2} - \sqrt{x}}{(x+5)^2} = \frac{\frac{x+5}{2} - x}{(x+5)^2 \sqrt{x}} = \frac{5-x}{2 \sqrt{x} (x+5)^2} \]

\[ \therefore \text{ When } x > 5, \quad f'(x) \leq 0 \Rightarrow f(x) \text{ is decreasing when } x > 5. \]

Also,

\[ \lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{\sqrt{x}}{x+5} = \lim_{x \to \infty} \frac{1}{\sqrt{x} + \frac{5}{\sqrt{x}}} = 0 \]

Therefore, by Alternating Series Test,

\[ a_j = \frac{\sqrt{j}}{j+5} \text{ is decreasing to zero when } j \geq 5, \]

\[ \therefore \sum_{j=0}^{\infty} \frac{(-1)^j \sqrt{j}}{j+5} \text{ is convergent.} \]
Problem 7. Find a power series expansion for \( f(x) = \frac{x}{(1+4x)^2} \)

(sol) Notice that \( f(x) = \frac{x}{(1+4x)^2} = x \cdot (1+4x)^{-2} \)

and the binomial coefficient \( \binom{x}{k} = \frac{x(x-1)\ldots(x-k+1)}{k!} \)

for \( x \in \mathbb{R}, k \in \mathbb{N} \).

\[ (-2)_k = \frac{-2x(-2+1)\ldots(-2-k+1)}{k!} = \frac{(-2)(-3)\ldots(-k+1)}{k!} = \frac{(-1)^k(k+1)!}{k!} = (-1)^k \cdot (k+1) \]

\[ (1+4x)^2 = \sum_{k=0}^{\infty} \binom{2}{k} (4x)^k = \sum_{k=0}^{\infty} (-1)^k \cdot (k+1) \cdot 4^k \cdot x^k \]

\[ f(x) = x \cdot (1+4x)^{-2} = \sum_{k=0}^{\infty} (-1)^k \cdot (k+1) \cdot 4^k \cdot x^k+1 \]

\[ = \sum_{n=1}^{\infty} (-1)^{n-1} \cdot 4^n \cdot n \cdot x^n \quad \text{(Change index \( n = k+1 \))} \]

\[ \uparrow \quad k = n-1 \]
Problem 8 \[ a_1 = 1, \quad a_n = \sqrt{6 + a_{n-1}}, \quad n \geq 2 \]

1. Use Mathematical Induction to show \( a_n \leq 3 \) \( \forall n \).

\[ \text{(sol.)} \]
(1) For \( n = 1 \): \( a_1 = 1 \leq 3 \) is true.
(2) Suppose for \( n = k \), \( a_k \leq 3 \) is true,

then consider the case \( n = k+1 \):

\[ a_{k+1} = \sqrt{6 + a_k} \leq \sqrt{6 + 3} = \sqrt{9} = 3 \]

is true.

Therefore, by Mathematical induction, \( a_n \leq 3 \) is true for all \( n \geq 1 \).

2. Show that \( a_n \) is increasing.

\[ \text{(sol.)} \]
Consider \( a_{n+1}^2 - a_n^2 = 6 + a_n - a_n^2 \]

\[ = (3 - a_n)(2 + a_n) \geq 0 \]

by \( 0 \leq a_n \leq 3 \) for all \( n \geq 1 \).

Thus, \( a_{n+1} - a_n \geq 0 \) \( \forall n \geq 1 \)

\[ \Rightarrow (a_{n+1} - a_n)(a_{n+1} + a_n) > 0 \quad \forall n \geq 1 \]

However, \( a_{n+1} + a_n \) is positive.

\[ \therefore a_{n+1} - a_n \geq 0 \Rightarrow a_{n+1} \geq a_n \quad \forall n \geq 1. \]

3. Now, notice that, \( a_n \) is bounded by 3 and increasing.

\[ \therefore \text{an is a convergent sequence. Let} \ a = \lim_{n \to \infty} a_n \]

\[ \Rightarrow a = \lim_{n \to \infty} a_n = \lim_{n \to \infty} \sqrt{6 + a_{n-1}} = \sqrt{6 + a} \Rightarrow a^2 - a - 6 = 0 \Rightarrow a = -2 \text{ or } 3 \]
Problem 9. Find the Taylor series expansion of $e^{-4x}$ around $x = -1$.

(sol.) \[ f(x) = e^{-4x} \Rightarrow f^{(k)}(x) = (-4)^k \cdot e^{-4x} \]
\[ \therefore f^{(k)}(-1) = (-4)^k \cdot e^4 \]

Then $T[f](x) = \text{the Taylor expansion of } e^{-4x} \text{ at } x = -1$
\[ = \sum_{k=0}^{\infty} \frac{f^{(k)}(-1)}{k!} \cdot (x+1)^k \]
\[ = \sum_{k=0}^{\infty} \frac{(-4)^k \cdot e^4}{k!} \cdot (x+1)^k \].
Problem 10 \quad f(x) = x \sin x. \quad \text{Find } T_4 f(x) \text{ and estimate } R_4(x) = |f(x) - T_4(x)| \text{ if } |x| < 1.

sol \quad \text{(I think the original problem is considering the Taylor series at } x = 0.)

\[ f(0) = 0 \]
\[ f'(x) = \sin x + x \cos x \Rightarrow f'(0) = 0 \]
\[ f''(x) = 2 \cos x - x \sin x \Rightarrow f''(0) = 2 \]
\[ f^{(3)}(x) = -3 \sin x - x \cos x \Rightarrow f^{(3)}(0) = 0 \]
\[ f^{(4)}(x) = -4 \cos x + x \sin x \Rightarrow f^{(4)}(0) = -4 \]

\[ \therefore \quad T_4 f(x) = \frac{f''(0)}{2!} x^2 + \frac{f^{(4)}(0)}{4!} x^4 = x^2 - \frac{1}{6} x^4 \]

Notice that \( f^{(5)}(x) = 5 \sin x + x \cos x \) and \( |x| < 1 \)

\[ \Rightarrow \quad |f^{(5)}(x)| \leq 5 \cdot |\sin x| + |x| \cdot |\cos x| \leq 6 \]

Hence, by the Taylor inequality:

Suppose \( |f^{(n+1)}(x)| \leq M \) for \( |x - a| \leq d \)

then \[ |R_n(x)| \leq \frac{M}{(n+1)!} |x - a|^{n+1} \]

\[ \Rightarrow \quad |R_4(x)| \leq \frac{6}{(4+1)!} |x - 0|^{4+1} = \frac{6}{120} |x|^5 = \frac{|x|^5}{20} < \frac{1}{20} \]