LINEAR ALGEBRA (MATH 110.201)

HOMEWORK 6

Due date: Friday, 4 March at the beginning of the lecture. In order to make sure that graders can get assignments back to everyone on time and not derail the homework schedule, no late homework will be accepted.

Instructions: Be sure to write your name, section number, and TA’s name on your solution set. Please staple your solutions together before turning them in. When writing up your solutions, explain the logic behind your solutions, and write clearly (as if you are explaining your solution to a classmate). Points will be allocated for conceptual clarity, as well as accuracy in your calculations.

NOTE: In this homework, when we ask for a complete description of \( \text{Im}(A) \) (or \( \text{Ker}(A) \)), we mean that you should describe the vectors \( \vec{y} \) in \( \text{Im}(A) \) (or \( \vec{x} \) in \( \text{Ker}(A) \)) in terms of conditions on their coordinates. This is spelled out in the first exercise below, but you should use the same interpretation throughout the homework.

Exercise 1. Consider the linear transformation \( \text{Proj}_L : \mathbb{R}^2 \to \mathbb{R}^2 \), where \( L \) is the line passing through the origin and \( [2, -3] \). Determine the vectors \( \vec{y} = [y_1, y_2] \) that are in \( \text{Im}(\text{Proj}_L) \); give your answer in terms of conditions on \( y_1 \) and \( y_2 \). Determine the vectors \( \vec{x} = [x_1, x_2] \) that are in \( \text{Ker}(\text{Proj}_L) \); give your answer in terms of \( x_1 \) and \( x_2 \).

Exercise 2. Consider the following matrix:

\[
\begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

(1) Give a complete description of \( \text{Ker}(A) \) and \( \text{Ker}(A^2) \). How do they compare to each other?
(2) Show that it is always true that\(^1\) \( \text{Ker}(B) \subseteq \text{Ker}(B^2) \) when \( B \) is any square matrix of any size.

\(^1\)In class I explained that the symbol \( A \subseteq B \) means that everything in the set \( A \) is also in the set \( B \); for example, \( \{1, 2, \ldots\} \subseteq \mathbb{R} \) (every positive integer is also a real number).
Exercise 3. Consider again the same matrix $A$ as in exercise 2.

(1) Give a complete description of $\text{Im}(A)$ and $\text{Im}(A^2)$. How do they compare to each other?

(2) Can you say anything about how to compare $\text{Im}(B)$ and $\text{Im}(B^2)$ when $B$ is any square matrix of any size?

Exercise 4. Consider the following matrix:

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{pmatrix}$$

Find vectors spanning $\text{Im}(A)$.

Exercise 5. Consider again the same matrix as in exercise 4. Find vectors spanning $\text{Ker}(A)$.

Exercise 6. Is the following list of vectors linearly independent in $\mathbb{R}^3$? Justify your answer completely.

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

Exercise 7. Is the following list of vectors linearly independent in $\mathbb{R}^3$? Justify your answer completely.

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \quad \vec{v}_4 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

Exercise 8. Which of the following are bases of $\mathbb{R}^3$?

(1) The vectors $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$ from exercise 6.

(2) The vectors:

$$\vec{w}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \vec{w}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \vec{w}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

(3) The vectors:

$$\vec{w}_1 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \quad \vec{w}_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \quad \vec{w}_3 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

Exercise 9. Consider the following matrix:

$$\begin{pmatrix} 2 & 2 & -1 & 2 \\ 3 & 1 & 2 & 1 \\ -1 & 1 & -3 & 1 \end{pmatrix}$$

(1) Use the method introduced in the lecture to find a basis for $\text{Ker}(A)$.

(2) Use the method introduced in the lecture to find a basis for $\text{Im}(A)$. 
Exercise 10. Consider the following matrix:
\[
\begin{pmatrix}
1 & 2 & 3 & 4 \\
2 & 3 & 4 & 5 \\
3 & 4 & 5 & 6 \\
4 & 5 & 6 & 7
\end{pmatrix}
\]

(1) Use the method introduced in the lecture to find a basis for \(\text{Ker}(A)\).
(2) Use the method introduced in the lecture to find a basis for \(\text{Im}(A)\).
(3) What is \(\dim(\text{Ker}(A)) + \dim(\text{Im}(A))\)?

Exercise 11. Consider the vector \(\vec{x} = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}\) in \(\mathbb{R}^3\). Let \(W\) denote the set of all vectors \(\vec{w}\) in \(\mathbb{R}^3\) which are orthogonal to \(\vec{x}\): that is, the vectors for which \(\vec{v} \cdot \vec{w} = 0\), where \(\cdot\) denotes the dot product.

(1) Give a complete description of all vectors in \(W\).
(2) Is \(W\) a subspace of \(\mathbb{R}^3\)? If so, find a basis for \(W\) and state what \(\dim(W)\) is.

Exercise 12. Suppose that \(W \subset \mathbb{R}^n\) is a subspace. Define \(W^\perp\) to be the subset of vectors \(\vec{v}\) in \(\mathbb{R}^n\) with the property that \(\vec{v} \cdot \vec{w} = 0\) for all \(\vec{w}\) in \(W\) (here again, \(\cdot\) denotes dot product).

(1) Show that \(W^\perp\) is also a subspace of \(\mathbb{R}^n\).
(2) When \(W\) is the set of vectors lying on the plane defined by the equation \(2X - 5Y + 3Z = 0\), give a complete description of \(W^\perp\).