10.3.3

Since $e^{i\theta} = e^{-i\theta}$, if $f(x)$ is real, $\overline{f(x)} = f(x)$.

So

$$\overline{F(\omega)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{i\omega x} dx$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{i(-\omega)x} dx = \overline{f(-\omega)}$$

10.4.4

(a) Let $\hat{u} = \mathcal{F}(u(x,t))$, then by (10.4.13) and (10.4.16), we have

$$\hat{u}_t = (-\lambda^2 - \mu^2)\hat{u}, \quad \therefore \hat{u}(0,t) = \hat{f}(\omega) e^{(-\lambda^2 - \mu^2)t}$$

where we used the initial condition $\hat{u}(\omega,0) = \hat{f}(\omega)$. We may write

$$u(x,t) = \mathcal{F}^{-1}(\hat{u}(\omega, t))$$

$$= \mathcal{F}^{-1}(\hat{f}(\omega) e^{(-\lambda^2 - \mu^2)t})$$

$$= \mathcal{F}^{-1}\left( \hat{f}(\omega) e^{-\lambda^2 t} e^{-\mu^2 t} \right)$$

$$= e^{-\lambda^2 t} \mathcal{F}^{-1}(\hat{f}(\omega) e^{-\mu^2 t})$$

$$= e^{-\lambda^2 t} f(x)$$

where $g(x,t) = \mathcal{F}^{-1}(e^{-\mu^2 t}) = \frac{\pi}{\sqrt{2\pi}} e^{-\frac{x^2}{4\mu^2 t}}$. Thus
\[ U(x,t) = e^{-rt} \int_{-\infty}^{\infty} f(\tilde{x}) \frac{\exp\left(-\frac{(x-\tilde{x})^2}{4kt}\right)}{2\sqrt{\pi} \sqrt{kt}} \, d\tilde{x} \]

(b) multiply the solution above by \( e^{xt} \) we have:

\[ U(x,t) = e^{xt} u(x,t) = \int_{-\infty}^{\infty} f(\tilde{x}) \frac{\exp\left(-\frac{(x-\tilde{x})^2}{4kt}\right)}{2\sqrt{\pi} \sqrt{kt}} \, d\tilde{x} = f(x)G \]

where \( G(x,t) = \frac{\sqrt{\pi}}{\sqrt{kt}} \exp\left(-\frac{x^2}{4kt}\right) \). Note that it is the solution to the equation:

\[ \frac{\partial}{\partial t} (e^{xt} u(x,t)) - k \frac{x}{\sqrt{2\pi} \sqrt{kt}} \exp\left(-\frac{x^2}{4kt}\right) \]

which transforms \( u(x,t) \) by \( e^{xt} \).

10.4.8

We take Fourier transform of \( u \) with respect to \( y \): \( \hat{u}(x,w) = \mathcal{F}(u(x,y)) \),

namely,

\[ u(x,y) = \int_{-\infty}^{\infty} \hat{u}(x,w)e^{-iwx} \, dw \]

Then we have

\[ \hat{u} + \left[ \frac{\partial^2 \hat{u}}{\partial x^2} + \frac{\partial^2 \hat{u}}{\partial y^2} \right] = \frac{\partial^2 \hat{u}}{\partial x^2} + (-i \omega) \hat{u} = \frac{\partial^2 \hat{u}}{\partial y^2} - \omega^2 \hat{u} = 0 \]

with \( \tilde{u}(1;w) = G_1(w) \), \( \tilde{u}(L;w) = G_2(w) \). Therefore, we have

\[ \hat{u}(x,w) = A(w) \sinh(\omega x) + B(w) \cosh(\omega x) \]

and

\[ G_1(w) = \hat{u}(L;w) = B(w) \quad G_2(w) = \hat{u}(1;w) = A(w) \sinh(\omega L) + \]

\[ + G_1(w) \cosh(\omega L) \]
Then \( A(\omega) = \frac{G_2(\omega) - G_1(\omega) \cosh(\omega L)}{\sinh(\omega L)} \)

Thus
\[
\hat{u}(x, \omega) = G_2(\omega) \frac{\sinh(\omega x)}{\sinh(\omega L)} + G_1(\omega) \left[ \frac{\cosh(\omega x) - \cos(\omega L) \sinh(\omega x)}{\sinh(\omega L)} \right]
\]

Rewrite the equation above using \( \cosh(x) \sinh(y) - \cos(y) \sinh(x) = \sinh(y-x) \)

we have:
\[
\hat{u}(x, \omega) = G_2(\omega) \frac{\sinh(\omega x)}{\sinh(\omega L)} + G_1(\omega) \frac{\sinh(\omega L-x)}{\sinh(\omega L)}
\]

And:
\[
\hat{u}(x, y) = \int_{-\infty}^{\infty} \left[ G_2(\omega) \frac{\sinh(\omega x)}{\sinh(\omega L)} + G_1(\omega) \frac{\sinh(\omega L-x)}{\sinh(\omega L)} \right] e^{i\omega y} d\omega,
\]

with \( G_i(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g_i(x) e^{i\omega x} dx \)

10.5.1

A) We have
\[
\int_{-\infty}^{\infty} F(\omega) \sin(\omega x) d\omega = \int_{-\infty}^{\infty} e^{-\omega^2} \sin(\omega x) d\omega
\]

\[
= -\frac{1}{\beta} e^{-\omega^2} \sin(\omega x) \bigg|_0^\infty + \frac{x}{\beta} \int_{-\infty}^{\infty} e^{-\omega^2} \cos(\omega x) d\omega
\]

\[
= \frac{x}{\beta} [ -\frac{1}{\beta} e^{-\omega^2} \cos(\omega x) \bigg|_0^\infty - \frac{1}{\beta} \int_{-\infty}^{\infty} e^{-\omega^2} \sin(\omega x) d\omega ]
\]

Thus
\[
\left( \int_{-\infty}^{\infty} e^{-\omega^2} \sin(\omega x) d\omega \right) (1 + \frac{x^2}{\beta^2}) = -\frac{x}{\beta^2} (0 - 1)
\]
and \( \int_0^\infty e^{-\omega^2} \sin \omega x \, d\omega = \frac{x}{\beta^2 + x^2} \).

B Using the same method of integration by parts as above, we have

\[
\int_0^\infty F(\omega) \cos \omega x \, d\omega = \int_0^\infty e^{-\omega^2} \cos \omega x \, d\omega = \frac{\beta}{\beta^2 + x^2}
\]

10.5.12

We use the cosine transform

\( \tilde{\gamma}(w,t) = C[u(x,t)] = \frac{2}{\pi} \int_0^\infty u(x,t) \cos \omega x \, dx \)

Then \( C[\partial_t u] = \tilde{\nu}_t \) and use equation (10.5.26) to get \( C[u_{xx}] \)

\[
= \frac{-2}{\pi} \int_0^\infty \tilde{u}(x,0,t) - w^2 \tilde{\nu} = -w^2 \tilde{\nu}
\]

Thus \( \tilde{\nu}_t = -kw^2 \tilde{\nu} \) and \( \tilde{u}(w,0) = \tilde{u}(w,0) e^{-kw^2t} \).

Since we have

\( \gamma(w) = \tilde{f}(w) = C[f(x)] \)

\[
= \frac{2}{\pi} \int_0^\infty f(x) \cos \omega x \, dx
\]

which is even with respect to \( w \).

Thus \( u(x,t) = \int_0^\infty \tilde{f}(w) e^{-kw^2t} \cos \omega x \, d\omega \)

\[
= c^{-1}[\tilde{f}(w)e^{-kw^2t}]
\]

Use Table 10.5.1 on page 478, we have

\( c^{-1}[e^{-kw^2t}f(w)] = \frac{1}{4} \int_0^\infty f(y)[g(x-y) + g(x+y)] \, dy \),
where
\[ g(x) = C^{-1} \left[ e^{-kx^2} \right] = \sqrt{\frac{\pi}{4kt}} \exp \left[ \frac{-x^2}{4kt} \right] \]

Thus
\[ u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} f(y) \left( \exp \left[ \frac{-(x-y)^2}{4kt} \right] + \exp \left[ \frac{-(x+y)^2}{4kt} \right] \right) \, dy \]

10.6.10 Taking the double Fourier transform both sides yields
\[ \frac{\partial \tilde{u}}{\partial t} = (-k_1w_1^2 - k_2w_2^2) \tilde{u} \]

which implies
\[ \frac{d\tilde{u}}{dt} = (-k_1w_1^2 - k_2w_2^2) \tilde{u} \]

with \( \tilde{u}(w,0) = \tilde{f}(w) \)

Integrating yields
\[ \tilde{u} = \tilde{f}(w)e^{(-k_1w_1^2 - k_2w_2^2)t} \]

\[ = \int_{\mathbb{R}^2} f(r_0) G(r-r_0, t) \, dr_0 \]

where \( r = (x, y) \) and
\[ G(r, t) = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{x^2}{4kt}} \frac{1}{\sqrt{4\pi \sqrt{k_1k_2}} e^{-\frac{r^2}{4\sqrt{k_1k_2}}}} \]

10.6.11 (C):

We take the cosine transform in \( x \) and sine transform in \( y \) on both sides of the equation. We thus have
\[ \frac{\partial \tilde{u}}{\partial t} = \frac{2}{\pi} w_1 u(0, y, t) - \frac{w_2^2}{4\pi} \frac{\partial}{\partial y} (x(0, t) - \omega_2^2 \tilde{u}(y, t)) \]
\[ \frac{2u}{2t} = - (w_1^2 + w_2^2) \hat{w} \]

By the BC: \( \frac{\partial u}{\partial y}(x_0, t) = 0, \)

\( u(0, y, t) = 0. \)

It follows that

\[ \hat{w}(w) = \hat{f}(w) e^{-k|w|} \]

where

\[ \hat{f}(w) = \frac{4}{\pi^2} \int_0^\infty \int_0^\infty f(x, y) \sin(w x + w_2 y) dy dx. \]

Thus

\[ u(x, y, t) = \int_0^\infty \int_0^\infty \hat{f}(w) e^{-k|w|} \sin(w x + w_2 y) dw_1 dw_2. \]

12.2.4.

If \( w(x, 0) = f(x) \), then, by (12.2.12), \( u = f(x - ct) \) is a solution.

If \( w(0, t) = h(t) \), \( f(x - ct) \), i.e. \( f(x - ct) = h(x - ct) \). Thus \( u(x, y, t) = h(t - \frac{x}{c}) \) is a solution.

12.3.5.

We obtain solutions by evaluating equation D'Alembert's equation (12.3.13):

\[ u(x, t) = \frac{f(x - ct) + f(x + ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s) ds. \]

Since \( f = 0 \), the first term is always 0.

There are two different sets of solutions for the two regions \( t < \frac{h}{c} \) and \( t > \frac{h}{c} \), corresponding respectively to the times before and after the pulses separate.

In the region \( t < \frac{h}{c} \), the solutions are
\[ u(t,x) = \frac{1}{2C} \left\{ \begin{array}{ll} \int_{x-ct}^{x+ct+h} f(x) \, dx & \text{for } x < -h-ct \\ \int_{x-ct}^{x+ct} f(x) \, dx & \text{for } -h-ct < x < h+ct \\ \int_{x-ct}^{x+ct-h} f(x) \, dx & \text{for } h-ct < x < h+ct \\ 0 & \text{for } h+ct < x \end{array} \right. \]

If \( t > \frac{h}{C} \), the solutions are:

\[ u(t,x) = \frac{1}{2C} \left\{ \begin{array}{ll} \int_{x-ct}^{x+ct} f(x) \, dx & \text{for } x < -h-ct \\ \int_{x-ct}^{x+ct} f(x) \, dx & \text{for } -h-ct < x < h+ct \\ \int_{x-ct}^{x+ct} f(x) \, dx & \text{for } h+ct < x \end{array} \right. \]

R.3.b

(a) The left hand side is \[ \frac{\partial^2 y}{\partial t^2} = \frac{1}{p^2} \frac{\partial^2 w}{\partial t^2}. \]

the right hand side \[ \frac{\partial^2 y}{\partial t^2} = C^2 \nabla^2 u \]

\[ = C^2 \frac{1}{p^2} \frac{\partial}{\partial t} \left( \frac{p^2 \partial u}{\partial t} \right) \]

\[ = C^2 \frac{1}{p^2} \left( \frac{p^2 \partial^2 u}{\partial t^2} - \frac{\partial w}{\partial t} \right) \]

\[ = C^2 \frac{1}{p^2} \left( \frac{\partial^2 \frac{\partial w}{\partial t}}{\partial p^2} + \frac{\partial w}{\partial p} - \frac{\partial w}{\partial t} \right) \]

\[ = C^2 \frac{\partial^2 w}{\partial p^2} \]

So \[ \frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial p^2}. \]
The general solution for the equation of \( w \) is given by
\[ u(p, t) = F(p - ct) + G(p + ct), \]
then the solution \( u = \frac{w}{p} \) is given by
\[ U(p, t) = \frac{F(p - ct) + G(p + ct)}{p}, \]
Since the distance from the origin, \( p \), is in the numerator, amplitude decays when \( p \) is large.

The solution is of the form
\[ u(x, t) = F(x - ct) + G(x + ct). \]
For \( x > 0 \), the initial condition yields
\[ F(x) = G(x) = 0 \quad x > 0, \]
For \( t > 0 \), the boundary condition yields
\[ h(t) = F(-ct) + G(ct), \quad t > 0. \]
Thus if \( x > ct \), then both \( F \) and \( G \) have positive arguments and thus \( F = G = 0 \). Therefore \( u(x, t) = 0 \).
If \( x < ct \), then both \( F \) and \( G \) have positive arguments and thus \( F = G = 0 \). Therefore \( u(x, t) = 0 \) if \( x < ct \).
However, if \( x < ct \), only the argument of \( F \) is negative. We then have:
\[ U(x,t) = F(x-ct) + G(x+ct) = h(t-\frac{x}{c}) - G(ct-x) + G(ct+x) \]

Since \( x < ct \) for both \( ct - x > 0 \) and \( ct + x > 0 \), thus:

\[ U(x,t) = \begin{cases} 
90 & x > ct \\
9(t - \frac{x}{c}) & 0 < x < ct 
\end{cases} \]

**12.5.2.**

We seek solutions of the form

\[ U(x,t) = F(x-ct) + G(x+ct). \quad \quad (1) \]

By IC, we have \( F(x) = G(x) = 0 \) \( 0 < x < L \). \quad (2)

By BC on \( x = 0, L \), we have

\[ G(x) = -F(2L-x), \quad x > L. \quad \quad (3) \]

\[ F(x) = h(-\frac{x}{c}) - G(-x), \quad x < 0. \quad \quad (4) \]

First, observing (3) and (4), we know that

\[ F(x) = h(-\frac{x}{c}) - G(-x) = h(-\frac{x}{c}) + F(x+2L), \quad x < -2L. \quad (5) \]

By (2) and (3), we have \( G(x) = 0 \) \( 0 < x < 2L \).

So \( F(x) = h(-\frac{x}{c}) \), \( -2L < x < 0 \). \quad \quad (6)

Now we can calculate all values of \( F \) for \( x < 0 \) by (5) and (6):

\[ F(x) = \sum_{j=0}^{[-\frac{x}{2L}]} h(-(x+2jL)), \quad x < 0 \quad (7) \]

where \( [-\frac{x}{2L}] \) is the integral part of the number \( -\frac{x}{2L} \).
Then by (3), we have:
\[ G(x) = -F(2L-x) = \sum_{j=0}^{[(x-2L)/2L]} h(-(2L-x+2jL)/c) = -\sum_{j=1}^{[x/2L]} h((x-2jL)/c) \tag{8} \]

In conclusion,
\[ F(x) = \begin{cases} 0 & 0 < x < L \\ \sum_{j=0}^{[-x/(2L)]} h(-(x+2jL)/c) , & x < 0 \end{cases} \]
\[ G(x) = \begin{cases} 0 & 0 < x < 2L \\ \sum_{j=1}^{[x/2L]} h((x-2jL)/c) , & x \geq 2L \end{cases} \]

Thus the solution \( u \) for the equation with \( 0 < x < L \) and \( t > 0 \) is:
\[ u(x, t) = \begin{cases} 0 & x - ct > 0 \\ h(t-x/c) , & x - ct < 0 , x + ct < 2L \end{cases} \]

and when \( x - ct < 0 , x + ct > 2L \)
we have:
\[ u(x, t) = \sum_{j=0}^{t-x/2L} h(-(x-ct+2jL)/c) = -\sum_{j=1}^{t/2L} h((x+ct-2jL)/c) \]