1. (10 pts each) True or false; justify as much as you can.
   a. If \( f(x), g(x) \) are continuous functions on \([0,1]\) which agree at every rational, then \( f = g \) on \([0,1]\).
   True. Given any \( \varepsilon > 0 \) and \( y \in (0,1) \) choose \( \delta > 0 \) so that \(|(f(x) - g(x)) - (f(y) - g(y))| < \varepsilon \) if \(|x - y| \leq \delta\). Now take \( x \) to be a rational in \((y - \delta, y + \delta) \cap [0,1]\). Then \(|f(y) - g(y)| < \varepsilon\).

   b. If \(|f(x)|\) is continuous at \(x_0\) then \(f(x)\) is continuous at \(x_0\).
   False. Take \(f(x) = \begin{cases} -1 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases}\)

   c. If \(f\) is a strictly monotone function on \([0,1]\) with range an interval, then \(f\) is one to one.
   True. In fact (see my notes on monotone functions) \(f\) is continuous and one to one.

   d. Let \(f\) be continuous on \(\mathbb{R}\). Then the inverse image of an open interval is an open interval.
   False. The inverse image is open but not necessarily an interval. Take for example \(f(x) = \sin x\).

   e. If \(f(x)\) is uniformly continuous on \(\mathbb{R}\) and \(\{x_n\}\) is a Cauchy sequence, then so is \(\{f(x_n)\}\).
   True. \(|f(x_j) - f(x_k)| < \varepsilon\) if \(|x_j - x_k| < \delta(\varepsilon)\). So if \(j, k > N(\delta) = N(\varepsilon)\) then \(|x_j - x_k| < \delta\), i.e the sequence \(\{f(x_n)\}\) is Cauchy.

   f. There exists a continuous bijection map \(f: [0,1) \to \mathbb{R}\).
   False. The image of \(f([0, \frac{1}{2}])\) is compact, say contained in \([-N, N]\). Hence the inverse image of the points \(-(N + 1)\) and \(N + 1\) lie in \((\frac{1}{2}, 1)\). By the intermediate value theorem, the inverse image of the interval \((- (N + 1), N + 1)\) also lies in \((\frac{1}{2}, 1)\) so \(f\) cannot be one to one.

2. Let \(f: [0,1] \to [0,1]\) be continuous. Show that the equation \(f(x) = x\) has at least one solution in \([0,1]\).
   Let \(h(x) = f(x) - x\). Then \(h(0) = f(0) \geq 0\) and \(h(1) = f(1) - 1 \leq 0\). By the intermediate value theorem there is an \(x\) such that \(h(x) = 0\).

3. Let \(f(x)\) be a \(C^1\) function on \(\mathbb{R}^+\) and satisfy \(f'(x) > f(x), f(0) = 0\). Show that \(f(x) > 0\) for \(x > 0\).
   Let \(h(x) = e^{-x}f(x)\). Then \(h'(x) = e^{-x}(f'(x) - f(x)) > 0\) for \(x > 0\) and \(h(0) = 0\). Hence
h(x) > 0 for x > 0.

4. Let \( f(x) \) be strictly increasing and continuous on \([0, \infty)\) with \( f(0) = 0 \). Show that

\[
\int_0^a f(x)dx + \int_0^b f^{-1}(x)dx \geq ab.
\]

When does equality hold? Hint: Draw a picture and interpret geometrically.

The first integral is the area under the graph of \( y = f(x) \) from 0 to \( a \) and the second integral is the area bounded by the \( y \) axis and the graph from \( y = 0 \) to \( y = b \) ("area under the graph of \( f^{-1} \) from 0 to \( b^* \)). If \( a \neq f^{-1}(b) \), then the left hand side is strictly greater than the right hand side interpreted as the area of the rectangle with base \( a \) and height \( b \). If \( a = f^{-1}(b) \), we have equality.

5. Let \( f(x) \) be \( C^3 \) on an interval \( I \). Suppose \( a_0 < a_1 < a_2 \) are points of \( I \) and

\[
f(a_0) = f(a_1) = f(a_2) = f'(a_2) = 0.
\]

Show there is a point \( c \in I \) where \( f'''(c) = 0 \).

By the mean value theorem, there are points \( b_1 \in (a_0, a_1), b_2 \in (a_1, a_2) \) such that \( f'(b_1) = f'(b_2) = 0 \). Applying the mean value theorem again but this time to \( f'(x) \), there are points \( c_1 \in (b_1, b_2), c_2 \in (b_2, a_2) \) such that \( f''(c_1) = f''(c_2) = 0 \). By the mean value theorem applied to \( f''(x) \) we arrive

6. Let \( f(x) \) be continuous on \([0, \infty)\) and assume that \( L = \lim_{x \to +\infty} f(x) \) exists and is finite.

Show that \( f \) is bounded. (Recall \( L = \lim_{x \to +\infty} f(x) \) means that give \( \varepsilon > 0, \exists N = N(\varepsilon) \) such that \( x > N \) implies \( |f(x) - L| < \varepsilon \).)

By the definition of \( L \), there exists \( N \) such that \( x > N \) implies \( |f(x)| \leq L + 1 \) on \((N, \infty)\). Since \( f(x) \) is continuous on \([0, N+1]\), \( |f(x)| \leq M \) on \([0, N+1]\) for some \( M \). Hence \( f(x) \) is bounded by \( M+L+1 \) on \([0, \infty)\).

7. Let \( f(x) \) be Riemann integrable on \([0,1]\) and assume that \( f(x) = 0 \) when \( x \) is rational.

Show that \( \int_0^1 f(x)dx = 0 \). Note that \( f(x) \) is assumed bounded but nothing is assumed about the values of \( f(x) \) when \( x \) is irrational.

Since \( f \) is assume Riemann integrable, given \( \varepsilon > 0 \), there is a partition \( P \) such that

\[
S^+(f, P) - S^-(f, P) < \varepsilon.
\]

However any Cauchy sum \( S(f, P) = \sum f(x_k)(x_{k+1} - x_k) \) satisfies

\[
S^-(f, P) \leq S(f, P) \leq S^+(f, P).
\]

Therefore \( |\int_0^1 f(x)dx - S(f, P)| < \varepsilon \). Now choose each \( a_k \) to be rational so \( S(f, P) = 0 \). Then \( |\int_0^1 f(x)dx| < \varepsilon \) so \( \int_0^1 f(x)dx = 0 \).

8. Let \( f : [0, 1] \to \mathbb{R} \) be defined by \( f(x) = \begin{cases} 1 & \text{if } x = \frac{1}{n}, n \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases} \)

Show that \( f(x) \) is Riemann integrable.
Let $\varepsilon > 0$ be given and choose an integer $N$ so that $\frac{1}{N} < \varepsilon$. Let $I_n$ be a closed interval of length $2^{-(n+1)}\varepsilon$ centered at the point $\frac{1}{n}$, $n = 1, 2, \ldots, N$. Let $P$ be the partition of $[0, 1]$ consisting of the endpoints of the $I_n \cap [0, 1]$ and the points $0, \frac{\varepsilon}{2}, 1$. Then since $0 \leq f \leq 1$ and $f = 0$ on the open intervals $(\frac{1}{n}, \frac{1}{n+1})$, we have $|S^+(f, P) - S^-(f, P)| < \varepsilon$. Hence $f$ is Riemann integrable and as in problem 7 (here we choose $a_k$ to be irrational in the Cauchy sum) $\int_0^1 f(x)dx = 0$. 