1. Determine whether the following statements are true or false. Justify your answer (i.e., prove the claim, derive a contradiction or give a counter-example).

(a) (10 points) If $A \subset B$, and $B$ is countable, then $A$ is countable.

False. $A$ may be finite.

(b) (10 points) If $\mathcal{B}$ is an open cover of $(0,1]$, then $\mathcal{B}$ has a finite subcover.

False. The cover $\mathcal{B} = \{(2/(n+2), 2/n) : n \in \mathbb{N}\}$ cannot have a finite subcover. Indeed, if $\mathcal{B}'$ was a finite subcover, then, there would be an $N$ so that if $I \in \mathcal{B}'$, then $I = (2/(n+2), 2/n)$ for some $n < N$. This would mean that the value $2/(N+3) \in (0,1]$ was not in any element of $\mathcal{B}'$ – that is, $\mathcal{B}'$ could not itself be a cover of $(0,1]$. 
(c) (10 points) If \([0, 1] \supset I_1 \supset I_2 \supset \ldots \supset I_n \supset \ldots\) is a nested sequence of closed intervals, then 
\(\bigcap_{n=1}^{\infty} I_n\) is non-empty.

True. As each \(I_i \subset [0, 1]\) they are all bounded. Hence, the \(I_k\) are all closed and bounded
intervals and so compact. By definition a closed interval is of the form \(I = [a, b]\) for \(a \leq b\) and
so is non-empty. Hence, their intersection is non-empty.

(d) (10 points) For non-empty \(A, B \subset \mathbb{R}\), let \(A + B = \{x + y : x \in A, y \in B\}\). If \(A\) is open, then
\(A + B\) is open.

True. Pick \(z \in A + B\) and write \(z = x + y\). As \(A\) is open, there is an \(\epsilon\) so that \((x - \epsilon, x + \epsilon) \subset A\).
Hence, \((z - \epsilon, z + \epsilon) = (x - \epsilon, x + \epsilon) + \{y\} \subset A + B\). That is, \(A + B\) contains a neighborhood
of each of its points and so is open.
(e) (10 points) Given sequences \( \{x_n\} \) and \( \{y_n\} \), define a new sequence \( \{z_n\} \) by \( z_{2n} = x_n \) and \( z_{2n-1} = y_n \). The sequence \( \{z_n\} \) converges if and only if \( \lim_{n \to \infty} x_n = \lim_{n \to \infty} y_n \) – that is, both sequences converge and have the same limit.

True. If \( \{z_n\} \) converges, then all subsequences – such as, \( \{x_n\} \) and \( \{z_n\} \) – converge to the same limit. Conversely, if \( \lim_{n \to \infty} x_n = \lim_{n \to \infty} y_n = x \), then for any \( \epsilon > 0 \), there is an \( m \) so that \( m < n \), implies that \( |x_n - x| < \epsilon \) and \( |y_n - x| < \epsilon \). Hence, if \( 2(m+1) < n \), \( |z_n - x| < \epsilon \).
That is, \( \lim_{n \to \infty} z_n = x \).

(f) (10 points) If \( f: D \to \mathbb{R} \) is a continuous function with domain \( D \subset \mathbb{R} \), then for all \( x_0 \in \overline{D} \), the closure of \( D \), \( \lim_{x \to x_0} f(x) \) exists.

False. Consider \( D = (-1,0) \cup (0,1) \) and \( f(x) = \frac{1}{x} \), then \( f \) is continuous but \( \lim_{x \to 0} f(x) \) does not exist.
2. (20 points) Let \( \{a_n\} \) be a Cauchy sequence, with \( a_n \geq a > 0 \). Working directly from the definitions, show that \( \{a_n^{-2}\} \) is Cauchy.

We note that \[ \left| \frac{1}{a_n} - \frac{1}{a_k} \right| = \frac{|a_n - a_k||a_n + a_k|}{|a_n|^2|a_k|^2} \leq 2Na^{-4}|a_n - a_k| \] where \( N > 0 \) is some number so that \( |a_n|, |a_k| \leq N \) and we used that \( a_n, a_k \geq a > 0 \). As Cauchy sequences are bounded, there is an \( N \) so that, for all \( n \), \( |a_n| \leq N \). Now, given, any \( \epsilon > 0 \), as \( \{a_n\} \) is Cauchy, there is an \( m \), so that if \( m < n, k \), then \( |a_n - a_k| < \frac{1}{2}N^{-1}a^4\epsilon \). Hence, \[ \left| \frac{1}{a_n} - \frac{1}{a_k} \right| \leq 2Na^{-4}\left( \frac{1}{2}N^{-1}a^4\epsilon \right) = \epsilon \]. That is, \( \{a_n^{-2}\} \) is Cauchy.
3. (a) (5 points) Let $S = \{ x \in \mathbb{R} : x^3 < x \}$. Determine sup $S$ and inf $S$.

We note that if $x > 0$, then $x \in S$ if and only if $x^2 < 1$, that is $x \in (0, 1)$. Likewise, if $x < 0$, then $x \in S$ if only if $x^2 > 1$, that is $x \in (-\infty, -1)$. Clearly, $0 \not\in S$, so $S = (-\infty, -1) \cup (0, 1)$. Hence, sup $S = 1$ and inf $S = -\infty$.

(b) (15 points) Let $a_{n+1} = \frac{1}{2} \left( a_n + \frac{2}{a_n} \right)$, $a_1 = 1$. Set $A = \{ x \in \mathbb{R} : x = a_n, n \in \mathbb{N} \} \subset \mathbb{R}$. Determine, \( \limsup_{n \to \infty} a_n, \liminf_{n \to \infty} a_n, \inf A, \sup A \) and all limit points (if any) of $A$. (Hint: Show that, for $n \geq 1$, $2 \leq a_{n+1}^2$.)

We note that for $n \geq 1$, $a_{n+1}^2 \geq 2$ and that for $n \geq 2$, $a_{n+1} \leq a_n$. To see the former we note that $a_{n+1}^2 = \frac{1}{4} (a_n + \frac{2}{a_n})^2 = \frac{1}{4} (a_n^2 + \frac{4}{a_n^2} + 4) \geq 2$. The latter then follows from $\frac{2}{a_n} \geq a_n$ for $n \geq 2$. From this we conclude that $a_1 = 1$ and $a_2 = \frac{3}{2}$ are, respectively, upper and lower bounds for $A$ and so sup $A = \frac{3}{2}$ and inf $A = 1$. For $n \geq 2$, $a_n$ is a bounded monotone non-increasing sequence and hence $\limsup_{n \to \infty} a_n = \liminf_{n \to \infty} a_n = \lim_{n \to \infty} a_n = a$ for some $a \in \mathbb{R}$. One verifies, that $a^2 = 2$ and that $a_n \geq 0$ and so conclude that $a = \sqrt{2}$. The only possible limit point of $A$ is $\sqrt{2}$, this is indeed a limit point as each $a_n$ is necessarily rational and $\sqrt{2}$ is irrational and so there are points in $A$ different from $\sqrt{2}$ arbitrarily close to $\sqrt{2}$ but different from it.