Final Exam Solutions

1. Let \( f(x) \) be a continuous function on \([a,b]\) with \( f(a) < 0 < f(b) \).

   a. (10 pts) Let \( x_0 \) be such that \( f(x_0) > 0 \). Show that there is an interval of the form \( I = (x_0 - \delta, x_0 + \delta) \) such that \( f(x) \geq \frac{f(x_0)}{2} \) on \( I \cap (a, b) \).

      By the uniform continuity of \( f \) with \( \varepsilon = \frac{f(x_0)}{2} \), there exists \( \delta = \delta(\varepsilon) \) such that \( |f(x) - f(x_0)| < \varepsilon \) if \( x \in I \cap (a, b) \). Hence
      
      \[
      f(x) \geq f(x_0) - \varepsilon = \frac{f(x_0)}{2} \text{ if } x \in I \cap (a, b).
      \]

   b. (10pts) Let \( S = \{ x \in [a, b] : f(x) > 0 \} \) and define \( c = \inf S \). Show that \( f(c) = 0 \).

      \( S \) is bounded below so \( c \) exists. (Note that \( S \) need not be connected but this does not matter.) By the continuity of \( f \), \( c \) is a point of \( \overline{S} \) so we have \( f(c) \geq 0 \). If \( f(c) > 0 \) then by the Intermediate value theorem, there is a point \( x_0 \in (a, c) \) where \( 0 < f(x_0) < f(c) \) contradicting the definition of \( c \). Thus \( f(c) = 0 \).

2. (20pts) Let \( f(x) \) be a function which is differentiable on \((-1,1)\) and continuous on \([-1,1]\). Suppose also that \( f'(x) \geq 0 \) for \( x \in (-1,0) \) and \( f'(x) \leq 0 \) for \( x \in [0,1) \). Show that \( f(x) \) has its global maximum at \( x = 0 \). Justify.

   Clearly \( f'(0) = 0 \). By the mean value theorem if \( x \in [-1,0] \), then \( f(x) - f(0) = xf'(c) \leq 0 \) since \( c \in (x,0) \) so \( f'(c) \geq 0 \). Similarly if \( x \in (0,1] \), \( f(x) - f(0) = xf'(c) \leq 0 \) since \( f'(c) \leq 0 \). Hence \( f(x) \) has its global max at \( x = 0 \).

3. Let \( f(x) \) be defined for \( x > 0 \) by

   \[
   f(x) = \int_1^x \frac{dt}{t}.
   \]

   a. (10pts) Compute the formal Taylor series of \( f(x) \) about \( x = 1 \).

      By the fundamental theorem I (since the integrand \( \frac{1}{t} \) is continuous for \( t > 0 \)), \( f(x) \) is \( C^1 \) and \( f'(x) = \frac{1}{x} \). Inductively, \( f^{(n)}(x) = (-1)^{n-1}(n-1)!x^{-n} \) and \( f(1) = 0 \), \( f^{(n)}(1) = (-1)^{n-1}(n-1)! \), \( n \geq 1 \). So formally
      
      \[
      f(x) = \sum_{n=1}^{\infty} (-1)^{n-1}(n-1)! \frac{(x-1)^n}{n!} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (x-1)^n.
      \]
b. (10pts) Find the radius of convergence of this series and justify.

We have \( \frac{1}{R} = \lim sup_{n \to \infty} |a_n|^{\frac{1}{n}} = \lim sup_{n \to \infty} (\frac{1}{n})^{\frac{1}{n}} = 1 \) since \( \log \frac{1}{R} = \lim_{t \to 0} t \log t = 0. \)

4. (20pts) State the Cauchy criterion for Riemann integrability and use it to show that any continuous function \( f \) on \([0,1]\) is Riemann integrable. You may state and use any of the basic properties of continuous functions on compact intervals.

The Cauchy criterion states that a function \( f \) on \([a,b]\) is Riemann integrable if given any \( \varepsilon > 0 \), there is a partition \( P \) of \([a,b]\) such that \( S^+(f,P) - S^-(f,P) \leq \varepsilon \) where
\[
S^+(f,P) = \sum_{i} M_i(x_i-x_{i-1}), \quad S^-(f,P) = \sum_{i} m_i(x_i-x_{i-1}), \quad M_i = \sup_{[x_{i-1}, x_i]} f, \quad m_i = \inf_{[x_{i-1}, x_i]} f.
\]

Let \( \varepsilon > 0 \) be given. Since \( f(x) \) is uniformly continuous (a continuous function on a compact set is uniformly continuous), there exists \( \delta > 0 \) such that \( |f(x) - f(y)| < \varepsilon \) whenever \( |x - y| < \delta \), \( x, y \in [0,1] \). Choose a partition \( x_i = \frac{i}{n}, \ i = 0, 1, \ldots, n \) with \( \frac{1}{n} < \delta \). Then \( M_i - m_i < \varepsilon \) since \( x_i - x_{i-1} = \frac{1}{n} < \delta \). Hence
\[
S^+(f,P) - S^-(f,P) \leq \sum_{i=1}^{n} (M_i - m_i)(x_i - x_{i-1}) < \varepsilon \sum_{i=1}^{n} (x_i - x_{i-1}) = \varepsilon.
\]

Hence \( f \) is Riemann integrable on \([a,b]\).

5a. (10pts) Define what it means for a family of functions \( F \) defined on \([a,b]\) to be equicontinuous.

A family \( F \) of functions defined on a common domain \( D \) (usually an interval) is said to be equicontinuous if given \( \varepsilon > 0 \) there exists \( \delta = \delta(\varepsilon) \) such that \( |f(x) - f(y)| < \varepsilon \) if \( |x - y| < \delta \) for all \( f \in F \).

5b. (10pts) Let \( N \) be a fixed positive integer and define \( F \) to be the family of all polynomials \( p(x) = \sum_{j=0}^{N} c_j x^j \) where \( |c_j| \leq 1 \). Show that \( F \) is equicontinuous on any \([a,b]\). Hint: What can you say about \( |p'(x)| \) ?

Observe that for any \( p(x) \in F \), \( |p'(x)| \leq \sum_{j=1}^{N} j M^{j-1} \leq C \) where \( C \) depends only on \( N, a, b \). In particular by the mean value theorem \( |p(x) - p(y)| \leq C |x - y| \ \forall x, y \in [a, b] \). This implies equicontinuity with \( \delta = \frac{\varepsilon}{C} \).
6a. (10pts) Define what it means for the series $\sum_{n=1}^{\infty} a_n$ to converge.

The series $\sum_{n=1}^{\infty} a_n$ converges if the sequence of partial sums $s_N = \sum_{n=1}^{N}$ converges.

6b. (10pts) Show that if $a_n \geq 0 \forall n$ and $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} a_{p}^{n}$ converges for all $p > 1$.

If $\sum_{n=1}^{\infty} a_n$ converges, then necessarily $a_n \to 0$ so we may choose $N$ such that $0 \leq a_n \leq \frac{1}{2}$ if $n \geq N$. In particular $0 \leq a_{p}^{n} < a_n$ for $n \geq N$. Therefore the series $\sum_{n=1}^{\infty} a_{p}^{n}$ converges since its partial sums (starting with $n = N$) are increasing and bounded.