Some notes on the category of $p$-local harmonic spectra

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Introduction

In [9], Mark Hovey and Neil Strickland determine that the lattice of localizing subcategories of the $E(n)$-local stable homotopy category is isomorphic to the Boolean algebra of subsets of the set with $n$ elements. They also show that the lattice of localizing subcategories of the $K(n)$-local stable homotopy category isomorphic to the Boolean algebra $\mathbb{F}_2$. Since $E(n)$ is Bousfield equivalent to the wedge of the first $n$ Morava K-theories, it seems natural to the author to ask if this result can be extended to the infinite wedge of Morava K-theories, or even arbitrary wedges. Indeed, it is shown that for a wedge of Morava K-theories indexed by some $I \subseteq \mathbb{N}$, the Bousfield lattice of the stable homotopy category localized at that wedge is precisely the Boolean algebra of subsets of $I$. Note that this result only applies to Bousfield lattices, but can probably be extended to all localizing subcategories.

1 The Harmonic Lattice

Definition 1. We work in the stable homotopy category localized at a fixed prime $p \in \mathbb{N}$. We introduce the following notation to improve readability:

(i) For $I \subseteq \mathbb{N}$, let $K(I) = \bigvee_{i \in I} K(i)$, where $K(i)$ is the $i$th Morava K-theory. For instance, the harmonic spectrum $\bigvee_{n \in \mathbb{N}} K(n)$ will be denoted by $K(\mathbb{N})$.

(ii) The localization functor associated to $K(I)$ will be denoted by $L_I$, except when $I = \{0, 1, \ldots, n\}$, in which case we will write $L_n$ for the associated localization functor, as is traditional.

(iii) By $\mathcal{H}$ we mean the stable homotopy category localized at $K(\mathbb{N})$, i.e. the $p$-local harmonic category.

(iv) For an arbitrary spectrum $X$, let $\text{supp}(X) = \{n : K(n) \wedge X \not\simeq *\}$ and $\text{cosupp}(X) = \{n : K(n) \wedge X \simeq *\}$. 

1
(v) Let \( \langle X \rangle_\mathcal{H} \) denote the collection of \( Y \in \mathcal{H} \) such that \( X \wedge Y \simeq * \), i.e. the “harmonic” Bousfield class of \( X \).

**Remark.** From [8], Theorem 3.5.1, we know that \( \mathcal{H} \) has a natural structure satisfying the following:

(i) \( \mathcal{H} \) is a triangulated category with a closed symmetric monoidal structure compatible with the triangulation.

(ii) \( \mathcal{H} \) has a set \( G \) of strongly dualizable objects such that the smallest localizing subcategory of \( \mathcal{H} \) containing \( G \) is \( \mathcal{H} \) itself.

(iii) \( \mathcal{H} \) has all coproducts.

(iv) Every cohomology functor is representable on \( \mathcal{H} \).

We also know from [10] Theorem 3.1 (assuming Vopěnka’s principle, but most likely under weaker hypotheses) that the Bousfield classes of \( \mathcal{H} \) form a set rather than a proper class, hence the notion of Bousfield lattice is well-defined in this context (i.e. we’re allowed to quantify over arbitrary subsets of the lattice to form meets and joins, so on and so forth). It follows from [8], Theorem 3.7.3 that \( \langle K(n) \rangle_\mathcal{H} \) is minimal in the Bousfield lattice of \( \mathcal{H} \) for every \( 0 \leq n < \infty \).

**Lemma 1.** If \( X \) is harmonic then \( K(n) \wedge X \simeq * \) for every \( n \) if and only if \( X \simeq * \).

**Proof.** It is obvious that if \( X \simeq * \) then \( K(n) \wedge X \simeq * \) for every \( n \). Suppose then that \( K(n) \wedge X \simeq * \) for every \( n \) and \( X \) is harmonic. Then \( K(\mathbb{N}) \wedge X \simeq * \) so \( X \) is \( K(\mathbb{N}) \)-equivalent to *, but \( X \) is \( K(\mathbb{N}) \)-local, hence \( X \simeq * \).

**Proposition 1.** If \( X \) and \( Y \) are objects of \( \mathcal{H} \), then \( X \wedge Y \simeq * \) if and only if \( Y \wedge K(n) \simeq * \) for every \( n \in \text{supp}(X) \).

**Proof.** Suppose first that \( Y \wedge X \simeq * \), i.e. \( Y \in \langle X \rangle_\mathcal{H} \). We note that, since \( X \wedge K(n) \simeq \bigvee_d \Sigma \Sigma^d K(n) \) for every \( n \in \text{supp}(X) \), \( \langle X \rangle_\mathcal{H} \wedge \langle K(n) \rangle_\mathcal{H} \leq \langle K(n) \rangle_\mathcal{H} \). But since \( \langle K(n) \rangle_\mathcal{H} \) is minimal and \( \langle X \wedge K(n) \rangle_\mathcal{H} \neq 0 \), we know that \( \langle X \rangle_\mathcal{H} \wedge \langle K(n) \rangle_\mathcal{H} = \langle K(n) \rangle_\mathcal{H} \). But for an arbitrary lattice, if \( a \wedge b = b \) then \( b \leq a \). Hence \( \langle X \rangle_\mathcal{H} \geq \langle K(n) \rangle_\mathcal{H} \) for every \( n \in \text{supp}(X) \). In other words, the collection of \( X \)-acyclics is contained in the collection of \( K(n) \)-acyclics. Hence, \( Y \wedge K(n) \simeq * \) for every \( n \in \text{supp}(X) \).

Now suppose that \( Y \wedge K(n) \simeq * \) for every \( n \in \text{supp}(X) \). Then we know that \( (X \wedge Y) \wedge K(n) \simeq * \) for every \( n \in \text{supp}(X) \) and for every \( n \in \text{cosupp}(X) \), that is, for every \( n \in \mathbb{N} \). However, \( X \wedge Y \) is harmonic, so, because it is \( K(\mathbb{N}) \)-local but annihilated by every \( K(n) \), \( X \wedge Y \) is contractible.

**Corollary 1.** The Bousfield lattice of \( \mathcal{H} \) (which we will denote by \( \mathcal{BH} \)) is the lattice generated by the collection of Morava K-theories.
Proof. The proposition above shows that for $X$ an object of $\mathcal{H}$, $X$ is $\mathcal{H}$-Bousfield equivalent to the wedge of Morava K-theories which support it. Additionally, every Morava K-theory is harmonic, as well as the sphere, which is supported by all the Morava K-theories. Hence the Bousfield lattice of $\mathcal{H}$ is in fact a complete, atomic Boolean algebra on the Morava K-theories. It is isomorphic to the Boolean algebra of subsets of $\mathbb{N}$.

Corollary 2. For $K(I)$ and $K(J)$ in $\mathcal{BH}$, the meet $\langle K(I) \rangle \wedge \langle K(J) \rangle$ is equal to $\langle K(I) \cap K(J) \rangle = \langle K(I \cap J) \rangle$.

Remark. The above proposition and corollaries extend Hovey and Strickland’s proofs of the structure of the $E(n)$-local and $K(n)$-local Bousfield lattices. It is clear that it can be generalized to the stable homotopy category localized at an arbitrary wedge of Morava K-theories. That is to say, the Bousfield lattice of the stable homotopy category localized at $K(I)$ for some $I \subseteq \mathbb{N}$ is precisely the Boolean algebra of subsets of $I$.

It also follows from the above statements that the telescope conjecture holds in the harmonic category. However, as all harmonic spectra are $BP$-local, and the telescope conjecture holds $BP$-locally [6], we know that already.

References


