Some Higher Coalgebra

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Abstract

Using the derived algebraic formalism of Lurie we describe a framework for coalgebraic structures which are neither necessarily commutative nor cocommutative. We show that Thom spectra and homotopical descent data are both natural examples of this kind of structure.

0 Introduction

Coalgebraic structures, specifically coalgebras, bialgebras and their respective categories of comodules have long played important roles in both algebraic geometry and algebraic topology. In algebraic geometry one can show that the category of descent data for a morphism of rings (not necessarily commutative) can be encoded by an equivalent category of comodules (cf. [KO74], [BCJ10] or [BW03]). Additionally, one can develop a theory of formal algebraic geometry from the point of view of coalgebras, as described in [Dem72].

In algebraic topology, the theories of Hopf-algebras and Hopf-algebroids play an important role in chromatic homotopy theory. This is described in great detail for instance in [Rav92] and [Rav86]. In that context these structures are, from a certain vantage point, making it possible for the topologist to study the descent theory inherent in the chromatic point of view.

More recently, authors like Hess, Shipley, Lurie and Rognes have described this sort of structure in homotopy theory. In [Rog08], Galois extensions in the category of commutative ring spectra are defined along with a far-reaching generalization thereof in the notion of a Hopf-Galois extension of $\mathbb{E}_\infty$-rings (originally introduced for discrete commutative rings in [KT81]). From the results and discussion of both [Rog08] and [Rot09] it becomes clear that this generalization finds many examples in stable homotopy theory, and likely elsewhere (e.g. [Kar14]). In [Hes09], [Hes10], [HS14a] and [HS14b], more precise tools are provided for working with Hopf-Galois extensions in homotopy theory, as well as coalgebraic structures in general. Lurie, in both [Lur14] and [Lur09], discusses descent theory (and in particular proves an quasicategorical version of the Barr-Beck theorem) and lays a great deal of groundwork for working with coalgebras and Hopf-algebras in quasicategories.

With the exception of Lurie, the above cited authors all use as their foundation the theory of model categories, due to Quillen. Hess in particular takes
great care to ensure that the all of the desired (co)algebraic structure interacts suitably with simplicial model category structure. In contrast, here we work with quasicategories, first described by Boardman and Vogt [BV73] and later developed in great detail by Joyal and Lurie. In particular, Lurie’s books [Lur09] and [Lur14] serve as the point of departure for this work. While we find that the formalism of quasicategories seems to allow one to describe homotopical categories of highly structured coalgebras and comodules efficiently, there are non-trivial obstructions to showing that naturally arising objects from elsewhere in homotopy theory fit into this formalism. Ultimately our greatest motivation for working with quasicategories is that we can define reasonable categories of $E_n$-co$E_m$-bialgebras and their associated categories of modules and comodules almost immediately if we assume the results of [Lur14].

The following note is intended to suggest a collection of definitions and constructions (essentially adapted from [Lur14] and [BGN14]) in the setting of quasicategories that will simplify discussions of homotopy coherent coalgebraic structure, two examples of which we describe. The first (Theorem 2.3.9) realizes descent data for a morphism of $E_n$-ring spectra as a quasicategory of comodules over a certain coalgebra:

**Theorem 0.0.1.** Given a morphism of $E_n$-ring spectra $\phi : A \to B$, with associated comonad $C \in \text{Fun}^L(L\text{Mod}_{E_1}^B, L\text{Mod}_{E_1}^B)$ there is an equivalence of quasicategories between $L\text{Comod}_C(\text{Fun}^L(L\text{Mod}_{E_1}^B, L\text{Mod}_{E_1}^B))$ and $L\text{Comod}_{B \otimes_A B}(L\text{Mod}_{E_1}^B)$.

And the second (Theorem 0.0.2) shows that given a morphism of Kan complexes $f : X \to BGL_1(R)$ for an $E_n$-ring spectrum $R$, the Thom diagonal of the associated Thom spectrum $Mf$ is a comodule structure map for the coalgebra $R[X] \simeq R \wedge X_+$:

**Theorem 0.0.2.** If $R$ is an $E_n$-ring spectrum we are given a morphism of Kan complexes $f : X \to BGL_1(R)$, $R[X]$ is an $E_{n-1}$-coalgebra and $Mf$ is a comodule for $R[X]$.

In further work we plan to leverage this structure to better understand Hopf-Galois extensions of $E_n$-algebra objects in arbitrary quasicategories. It is already known that, at least when working with simplicial model categories of spectra, Thom spectra give examples of such extensions (cf. [Rog08], [Rot09]).

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1 Quasicategorical Foundations

It is likely that the work in this note is essentially independent of the choice of “model” one chooses for the symmetric monoidal category of spectra. However, as discussed above, it seems convenient to work with quasicategories and $\infty$-operads as in [Lur09] and [Lur14]. Note that in those works, the objects
that we refer to as quasicategories are called $\infty$-categories. Here we prefer the terminology quasicategory only because the flurry of recent work in homotopy theory has made it clear that there exist a number of categorical structures that deserve to be settings for \textit{“$\infty$-category theory”}.

For us, $\mathcal{S}$ will always denote the quasicategory of spectra in the sense of [Lur14], $\mathcal{T}$ will denote the quasicategory of Kan complexes, from Definition 1.2.16.1 of [Lur09], and $q\mathsf{Cat}$ will denote the quasicategory of small quasicategories (for a suitable choice of inaccessible cardinal), given in Definition 3.0.0.1 of [Lur09]. Unless otherwise indicated, the term “space” will always mean a Kan complex in $\mathcal{T}$ (sometimes referred to in other work as $\infty$-groupoids). When we discuss Thom spectra, we will be using [ABG$^+$14] as our main source of definitions. However we will introduce the relevant terminology and theory of Thom spectra and parameterized homotopy theory as necessary.

1.1 Fibrations of Simplicial Sets

We recall some definitions from [Lur09] regarding (co)Cartesian fibrations of simplicial sets (in particular, quasicategories) and (co)Cartesian morphisms. The reader who is already familiar with this terminology in the context of classical algebraic geometry should feel free the skip this section. All we do is outline definitions to provide us with quasicategorical machinery that acts identically to its algebro-geometric analogue. Again, the following is taken almost verbatim from Section 2.4 of [Lur09].

Recall that the $i^{th}$ horn of the standard $n$-simplex, denoted $\Lambda^n_i$, is the simplicial set obtained by throwing away the face of $\Delta^n$ opposite the vertex $\{i\}$. For instance, the triangle $\Delta^2$ has three horns, one associated to each vertex: $\angle$, $\wedge$ and $\triangle$.

**Definition 1.1.1 (Inner Fibration).** A morphism of simplicial sets $p : C \to D$ is an inner fibration if for all $n \geq 0$, every inner horn inclusion $\Lambda^n_i \hookrightarrow \Delta^n$ for $0 < i < n$ and every commuting square like the following, there is a dotted arrow making the diagram commute:

\[
\begin{array}{ccc}
\Lambda^n_i & \longrightarrow & C \\
\downarrow & & \downarrow p \\
\Delta^n & \longrightarrow & D
\end{array}
\]

**Definition 1.1.2 (Cartesian Morphism).** Let $f : x \to y$ be an edge of a simplicial set $C$ and $p : C \to D$ an inner fibration of simplicial sets. Then we say that $f$ is $p$-Cartesian if the induced functor

\[
C/f \to C/y \times_{D/p(y)} D/p(f)
\]

is a trivial Kan fibration. Say that $f$ is $p$-coCartesian if $f$ is $p^{\text{op}}$-Cartesian for $p^{\text{op}} : C^{\text{op}} \to D^{\text{op}}$. 

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Remark 1.1.3. The above definition may take some unraveling. Firstly, notice that there is a functor $C_y \rightarrow D_{p(y)}$ which takes objects in $C_y$, i.e. morphisms $z \rightarrow y$, to $p(z) \rightarrow p(y)$ in $D_{p(y)}$. There is also a functor $D_{p(f)} \rightarrow D_{p(y)}$ which simply takes an object of $D_{p(f)}$, which can be represented by a commutative diagram like the following:

\[
\begin{array}{c}
\text{w} \\
\downarrow \\
\text{p(x)} \\
\downarrow \\
\text{p(f)} \\
\downarrow \\
\text{p(y)} \\
\end{array}
\]

to the morphism $w \rightarrow p(y)$ in $D$. As such, we can form the pullback of simplicial sets $C_y \times_{D_{p(y)}} D_{p(f)}$. Intuitively, this is the category of objects in $C$ over $y$ and cones in $D$ over $p(f)$ which coincide when applying $p$ and restricting to the top right edge, respectively. Now, note that $C_f$ maps to this pullback by taking a cone over $f$:

\[
\begin{array}{c}
\text{z} \\
\downarrow \\
\text{x} \\
\text{f} \\
\downarrow \\
\text{y} \\
\end{array}
\]

to the pair comprising the object $z \rightarrow y$ in $C_y$ and the image of the cone under $p$. Saying that this is an equivalence is saying two things at once: firstly, it’s saying that given an object $z$ of $C$ such that $p(z)$ is the top point of a morphism lying over $p(f)$ in $D$, there is a morphism $g : w \rightarrow z$ in $C$ such that $p(g)$ produces that morphism over $p(f)$. However, it’s also saying that, up to homotopy, this choice is unique. Another way to say that $f$ is Cartesian is to say that in the following diagram:

\[
\begin{array}{c}
\text{x} \\
\text{f} \\
\downarrow \\
\text{y} \\
\downarrow \\
\text{p} \\
\downarrow \\
\text{g} \\
\downarrow \\
\text{w} \\
\text{z} \\
\end{array}
\]

if $g$ is a fixed morphism equivalent to $p(f)$ and $y$ is a chosen element of $C$ over $z$, then $f$ and $x$ are uniquely determined up to homotopy and universal with respect to morphisms that project down to $p(f)$. Notice that these considerations make the above diagram (which doesn’t actually exist in a single category) look remarkably like a Cartesian square (i.e. one which has top left corner determined by the other corners and morphisms).

Definition 1.1.4 (Cartesian Fibration). Let $p : C \rightarrow D$ be a morphism of simplicial sets. Then $p$ is a Cartesian fibration if the following conditions are satisfied:

1. The morphism $p$ is an inner fibration of simplicial sets.
2. For every edge \( f : x \to y \) of \( D \) and every vertex \( \tilde{y} \) such that \( p(\tilde{y}) = y \) there exists a \( p \)-Cartesian edge \( \tilde{f} : \tilde{x} \to \tilde{y} \) such that \( p(\tilde{f}) = f \).

A morphism \( p : C \to D \) of simplicial sets is a coCartesian fibration if \( p^{\text{op}} : C^{\text{op}} \to D^{\text{op}} \) is a Cartesian fibration.

**Remark 1.1.5.** In other words, \( p \) is Cartesian if every time we can lift the target of a morphism in \( D \), we can uniquely lift the entire morphism. Dually, \( p \) is coCartesian if every time we can lift the source of a morphism in \( D \) we can uniquely lift the entire morphism.

### 1.2 \( \infty \)-Operads, Algebras and Modules

We briefly review some of the constructions and definitions of [Lur14] regarding multiplicative structures in quasicategories. For a more detailed discussion of \( \infty \)-operads and an extensive investigation of their properties, see Chapter 2 of [Lur14].

**Definition 1.2.1.** Define the quasicategory \( \mathcal{F}in_* \) to be the nerve of the category of linearly ordered finite pointed sets with order preserving functions between them. We will denote the finite set \( \{*,1,\ldots,n\} \) by \( \langle n \rangle \). We will say that a morphism \( f : \langle n \rangle \to \langle m \rangle \) of \( \mathcal{F}in_* \) in \( \mathcal{F}in_* \) is inert if for each element \( k \in \{1,\ldots,m\} \subseteq \langle m \rangle \), \( f^{-1}(k) \) has exactly one element.

**Definition 1.2.2 (\( \infty \)-Operads).** An \( \infty \)-operad is a functor between quasicategories \( p : O^{\otimes} \to \mathcal{F}in_* \) satisfying the following properties (where we write \( O^{\otimes}_{(n)} \) to denote the fiber of \( p \) over \( \langle n \rangle \) and \( \rho_i \) to denote the unique inert morphism \( \langle n \rangle \to \langle 1 \rangle \) taking \( i \) to 1):

1. For every inert morphism \( f : \langle n \rangle \to \langle m \rangle \) in \( \mathcal{F}in_* \) and every object \( C \) in \( O^{\otimes}_{(n)} \) there is a \( p \)-coCartesian morphism \( \tilde{f} : C \to C' \) lifting \( f \). In particular, \( f \) induces a functor \( f_1 : O^{\otimes}_{(n)} \to O^{\otimes}_{(m)} \).

2. Let \( C \) and \( C' \) be objects of \( O^{\otimes}_{(n)} \) and \( O^{\otimes}_{(m)} \) respectively and let \( f : \langle n \rangle \to \langle m \rangle \) be a morphism of \( \mathcal{F}in_* \). Let \( Map^f_{\otimes}(C,C') \) be the union of those connected components of \( Map_{\otimes}(C,C') \) which lie over \( f \). Choose \( p \)-coCartesian morphisms \( C' \to C'_i \) lying over the inert morphisms \( \rho^i : \langle n \rangle \to \langle 1 \rangle \) for each \( 1 \leq i \leq n \). Then the induced map

\[
Map^f_{\otimes}(C,C') \to \prod_{1 \leq i \leq n} Map^{\rho_{(n)}^i}_{\otimes}(C,C'_i)
\]

is a homotopy equivalence.

3. For every finite collection of objects \( C_1,\ldots,C_n \) in \( O^{\otimes}_{(1)} \) there exists an object \( C \) in \( O^{\otimes}_{(n)} \) and a collection of \( p \)-coCartesian morphisms \( C \to C_i \) covering \( \rho^i : \langle n \rangle \to \langle 1 \rangle \).
**Definition 1.2.3** (Inert Morphisms). Let \( p : \mathcal{O}^\otimes \to \text{Fin}_* \) be an \( \infty \)-operad. Then a morphism \( f \) in \( \mathcal{O}^\otimes \) is inert if \( p(f) \) is inert in \( \text{Fin}_* \) and \( f \) is \( p \)-coCartesian.

**Remark 1.2.4.** Note that the data of an \( \infty \)-operad is not the same as the data of a coCartesian fibration of simplicial sets \( p : \mathcal{O}^\otimes \to \text{Fin}_* \). That will, as we will define in Definition 1.2.6, be the data of a symmetric monoidal structure on \( \mathcal{O}^\otimes_{\langle 1 \rangle} \). In particular, a symmetric monoidal structure always defines an \( \infty \)-operad, but the converse is not true.

**Definition 1.2.5** (Maps of \( \infty \)-operads). Given two \( \infty \)-operads \( p : \mathcal{O}^\otimes \to \text{Fin}_* \) and \( p' : \mathcal{O}'^\otimes \to \text{Fin}_* \), a map of \( \infty \)-operads is a functor of quasicategories \( \mathcal{O}^\otimes \to \mathcal{O}'^\otimes \) which preserves inert morphisms and causes the evident triangle of categories involving \( p \) and \( p' \) to commute.

**Definition 1.2.6.** Let \( C^\otimes \) be a quasicategory, \( p : \mathcal{O}^\otimes \to \text{Fin}_* \) an \( \infty \)-operad and \( f : C^\otimes \to \mathcal{O}^\otimes \) a coCartesian fibration of simplicial sets. If the composition \( p \circ f : C^\otimes \to \text{Fin}_* \) exhibits \( C^\otimes \) as an \( \infty \)-operad then we say that \( C = C^\otimes_{\langle 1 \rangle} \) is \( \mathcal{O} \)-monoidal and that \( f \) is a coCartesian fibration of \( \infty \)-operads. In particular, to say that \( C \) is a symmetric monoidal (or \( \text{Fin}_* \)-monoidal) quasicategory is to say that there is a coCartesian fibration of simplicial sets \( C^\otimes \to \text{Fin}_* \) such that \( C = C^\otimes_{\langle 1 \rangle} \).

**Remark 1.2.7.** Note that being a coCartesian fibration of \( \infty \)-operads is strictly stronger than being a coCartesian fibration of simplicial sets.

**Definition 1.2.8** (Algebras for \( \infty \)-operads). For a coCartesian fibration of \( \infty \)-operads \( p : \mathcal{O}^\otimes \to \text{Fin}_* \), we define the category of \( \mathcal{O} \)-algebras in \( C^\otimes \) to be the quasicategory of \( \infty \)-operad maps (see Definition 2.1.2.7 of [Lur14]) from \( \mathcal{O}^\otimes \) to \( C^\otimes \) which cause the following triangle to commute:

\[
\begin{array}{ccc}
\mathcal{O}^\otimes & \to & C^\otimes \\
\downarrow \scriptstyle{id} & & \downarrow \scriptstyle{p} \\
\mathcal{O}^\otimes & \to & \mathcal{O}^\otimes 
\end{array}
\]

This quasicategory will be denoted by \( \text{Alg}_{\mathcal{O}}(C) \).

**Definition 1.2.9.** We recall some important \( \infty \)-operads that will be used throughout the paper:

1. The quasicategory \( \text{Fin}_* \) itself is an \( \infty \)-operad with underlying quasicategory \( \Delta^0 \). Let \( \mathcal{C} \) be a quasicategory. We will refer to the objects of \( \text{Alg}_{\text{Fin}_*}(\mathcal{C}) \) as commutative algebra objects of \( \mathcal{C} \) and often denote them by \( \text{CAlg}(\mathcal{C}) \).

2. Let \( p : C^\otimes \to \text{Fin}_* \) determine a symmetric monoidal structure on a quasicategory \( \mathcal{C} \). Recall that there is an \( \infty \)-operad \( \mathcal{A}s^\otimes \to \text{Fin}_* \) which characterizes the structure of associative algebras. Then recall from Notation 4.1.1.9 of [Lur14] that the fiber product \( \mathcal{C}^\otimes \times_{\text{Fin}_*} \mathcal{A}s^\otimes \) admits a
coCartesian fibration of \(\infty\)-operads over \(\text{Ass}^\otimes\) and as such it is a so-called planar operad (cf. Definition 4.1.1.6 of [Lur14]). We will refer to \(\infty\)-operad morphisms \(\text{Ass}^\otimes \to C^\otimes \times_{\text{Fin}_*} \text{Ass}^\otimes\) as associative algebra objects of \(C\) and often denote them by \(\text{Alg}(C)\).

3. Recall from Definition 5.1.0.2 of [Lur14] that we have the \(\infty\)-operads \(E_k\) of “little \(k\)-cubes” which interpolate between \(\text{Ass}\) and \(\text{Fin}_*\). Indeed, \(E_1 \simeq \text{Ass}^\otimes\), and there are canonical morphisms of \(\infty\)-operads \(E_k \to E_{k+1}\) such that \(\text{Fin}_* \simeq \text{colim}_k(E_k)\). For a symmetric monoidal quasicategory \(C^\otimes \to \text{Fin}_*\), we define \(E_k\)-algebras similarly to associative algebras, as sections of the pullback fibration \(C^\otimes \times_{\text{Fin}_*} E_k \to E_k\) that preserve inert morphisms.

We will refer to the objects of \(\text{Alg}_{E_k}(C)\) as \(E_k\)-algebra objects of \(C\).

\textbf{Remark 1.2.10.} One should think of the above structure as yielding multiplications on \(C\) by giving ways of going between the fibers of \(p\) over \(\langle n \rangle\) and \(\langle m \rangle\), which are \(C^n\) and \(C^m\) respectively. Moreover, one should interpret the fact that the fibration is coCartesian as being a suitable quasicategorical generalization of the notion from classical category theory of a Grothendieck opfibration. That is, it provides a mechanism for functorially pushing forward along paths in the base. An example of such a structure is for instance the categorical fibration over the category of affine varieties whose fiber over a variety \(\text{Spec}(R)\) is the category of quasi-coherent sheaves on \(\text{Spec}(R)\). This is an opfibration because one can take the direct image sheaf along a map of varieties \(\text{Spec}(R) \to \text{Spec}(S)\).

Recall from Definition 3.3.3.8 of [Lur14] that we can also define modules over algebras with the language of \(\infty\)-operads.

\textbf{Definition 1.2.11 (Operadic Modules).} From Definition 3.3.3.8 and Theorem 3.3.3.9 of [Lur14], we have that for a unital, coherent \(\infty\)-operad \(O^\otimes\), an \(O\)-monoidal quasicategory \(C^\otimes \to O^\otimes\) and an \(O\)-algebra \(A : O^\otimes \to C^\otimes\), we have an \(\infty\)-operad given by a coCartesian fibration \(\text{Mod}_A^O(C) \otimes \to O^\otimes\). Objects of the fiber of this fibration over \(O\) can be thought of as \(A\)-modules in \(C\) with a prescribed \(A\)-action. If the operad \(O^\otimes\) is clear, we will write simply \(\text{Mod}_A(C)\) or \(\text{Mod}_A\).

\textbf{Remark 1.2.12.} In the case that \(n = 1\), \(\text{Mod}_{E_1}A\) should be thought of as the quasicategory of bimodules over an associative algebra \(A\). Moreover, for \(n \geq 1\) and an \(E_n\)-algebra \(R\) in a quasicategory \(C\), there is a forgetful functor \(\text{Mod}_{E_n}^R(C) \to \text{Mod}_{E_1}^R(C)\) associated to thinking of \(R\) as an \(E_1\)-algebra (see Theorem 5.1.3.2 of [Lur14]). In this note we will primarily work with categories of left (or right) \(R\)-modules, typically denoted \(L\text{Mod}_R\). It is not clear that this choice is necessary to our stated goals but it is certainly more in line with the recent literature (e.g. [ABG+14], [ABG11] and [ACB14]).

\textbf{Proposition 1.2.13 (Left Modules).} Given an \(E_n\)-monoidal quasicategory \(C\) and an \(E_n\)-algebra \(R\) in \(C\), there is a category of left modules over \(R\) regarded as an \(E_1\)-algebra, \(L\text{Mod}_R(C)\). This category is \(E_{n-1}\)-monoidal.

\textit{Proof.} See Sections 4.2.1 and 5.1.4 of [Lur14].
Remark 1.2.14. Recall from Section 5.1.4 of [Lur14] that if \( \mathcal{C} \) is symmetric monoidal and \( R \) is a \( \mathcal{F}in_\ast \)-algebra (i.e. an \( \mathbb{E}_\infty \)-algebra), then \( \text{LMod}_R(\mathcal{C}) \simeq \text{Mod}_{\mathbb{F}in_\ast}(\mathcal{C}) \). This equivalence is canonical.

2 Coalgebraic Structures in Quasicategories

2.1 Coalgebras, Bialgebras and Comodules

In this section we provide definitions of coalgebras, bialgebras and comodules in an arbitrary \( \mathbb{E}_n \)-monoidal quasicategory as well as \( \mathcal{S} \), the quasicategory of spectra. We show in this section (Proposition 2.1.15) that all spaces are \( \mathbb{E}_\infty \)-coalgebras and that \( n \)-fold loop spaces stabilize to cocommutative \( \mathbb{E}_n \)-bialgebras, as one would expect.

Remark 2.1.1. Recall from Remark 2.4.2.7 of [Lur14] that an \( \mathcal{O} \)-monoidal structure on a quasicategory \( \mathcal{C} \) induces an \( \mathcal{O} \)-monoidal structure on \( \mathcal{C}^{\text{op}} \) which is unique up to a contractible space of choices. In particular, in the language of [BGN14], we have a coCartesian fibration of \( \infty \)-operads \( (p^\vee)^{\text{op}} : (\mathcal{C}^{\text{op}})^{\text{op}} \to \mathcal{O}^{\text{op}} \) defining an \( \mathcal{O} \)-monoidal structure on \( \mathcal{C}^{\text{op}} \). Note that there is a quasicategory of \( \mathcal{O} \)-algebras in \( \mathcal{C}^{\text{op}} \) associated to this \( \mathcal{O} \)-monoidal structure, \( \text{Alg}_\mathcal{O}(\mathcal{C}^{\text{op}}) \).

Definition 2.1.2 (Coalgebras). Let \( \mathcal{C} \) be an \( \mathcal{O} \)-monoidal quasicategory for \( \mathcal{O} \) an \( \infty \)-operad. Then define the quasicategory of \( \mathcal{O} \)-coalgebras in \( \mathcal{C} \) to be \( \text{Alg}_\mathcal{O}(\mathcal{C}^{\text{op}})^{\text{op}} \), which we will usually denote by \( \text{CoAlg}_\mathcal{O}(\mathcal{C}) \). If \( \mathcal{O} = \mathbb{F}in_\ast \) we will write \( \text{CCoAlg}(\mathcal{C}) \) for the quasicategory of cocommutative \( \mathbb{F}in_\ast \)-coalgebras in \( \mathcal{C} \).

Remark 2.1.3. Note that if \( \mathcal{O} \) in the above definition is not equivalent to \( \mathbb{F}in_\ast \) then it may not be the case that \( \text{Alg}_\mathcal{O}(\mathcal{C}^{\text{op}}) \) admits an \( \mathcal{O} \)-monoidal structure. This sometimes restricts the sorts of object we can discuss, hence we have the cocommutative objects of Definition 2.1.4 and the \( \mathbb{E}_n \)-objects of 2.1.6.

Definition 2.1.4 (Cocommutative \( \mathbb{E}_n \)-Bialgebras). Let \( \mathcal{C} \) be a symmetric monoidal quasicategory. Note that \( \text{Alg}(\mathcal{C}^{\text{op}})^{\text{op}} \) admits a symmetric monoidal structure (see Remark 3.2.4.4 of [Lur14]). Hence its opposite, \( \text{CCoAlg}(\mathcal{C}) \), also admits a symmetric monoidal structure. In other words, there is a coCartesian fibration \( q : \text{CCoAlg}(\mathcal{C})^{\text{op}} \to \mathcal{F}in_\ast \) such that the fiber of (1) is \( \text{CCoAlg}(\mathcal{C}) \). As such, we can pull back \( q \) along the inclusions \( \mathbb{E}_n \hookrightarrow \mathcal{F}in_\ast \) for any \( n \) to obtain an \( \mathbb{E}_n \)-monoidal structure on \( \text{CCoAlg}(\mathcal{C}) \). Define a cocommutative \( \mathbb{E}_n \)-bialgebra of \( \mathcal{C} \) to be an \( \mathbb{E}_n \)-algebra object of \( \text{CCoAlg}(\mathcal{C}) \). Denote the quasicategory of such objects by \( \mathbb{E}_n \text{-BiAlg}(\mathcal{C}) \).

Warning 2.1.5. Beginning with an \( \mathcal{O} \)-monoidal quasicategory \( \mathcal{C} \) with associated coCartesian fibration \( \mathcal{C}^{\text{op}} \to \mathcal{O}^{\text{op}} \) we are implicitly using the results of [BGN14] to produce a coCartesian fibration over \( \mathcal{O}^{\text{op}} \) describing the \( \mathcal{O} \)-monoidal
structure on $C^{op}$. However, the reader should be aware that the actual construction of this coCartesian fibration is highly non-trivial. As a whole, $(C^\otimes, \star)^{op}$ looks very different from $(C^\otimes)^{op}$.

For the next definition, recall from Proposition 3.2.4.3 and Variant 5.1.2.8 of [Lur14] (and subsequent discussion) that the quasicategory of $E_k$-algebras in an $E_{k+j}$-monoidal quasicategory is generally only $E_j$-monoidal. As a result, for an $E_n$-monoidal quasicategory, our constructions only allow us to work with $coE_j$-$E_k$-bialgebras for $j, k \geq 0$ and $j + k = n$.

**Definition 2.1.6 (co$E_k$-$E_j$-Bialgebras).** Let $C$ be an $E_n$-monoidal quasicategory. Then for any $k \leq n$ there is a quasicategory of $E_k$-coalgebras in $C$ (see Definition 2.1.2), $CoAlg_{E_k}(C)$. As the opposite of a quasicategory of $E_{n-k}$-algebras, $CoAlg_{E_k}(C)$ is $E_{n-k}$-monoidal. As such, for each $j \leq n - k$, there are quasicategories $Alg_{E_j}(CoAlg_{E_k}(C))$. For a fixed $j, k < n$, we call $Alg_{E_j}(CoAlg_{E_k}(C))$ the category of $coE_{k}$-$E_j$-bialgebras in $C$. We will denote this category by $kBiAlg_j(C)$ where the lower right index gives the degree of commutativity, and the upper left index gives the degree of cocommutativity.

**Remark 2.1.7.** In the above definition, if $n = \infty$, then we (informally) have that $n - k = n$ for every $k$. In other words, in a symmetric monoidal quasicategory, $Alg_{E_k}$ is again symmetric monoidal (cf. Examples 3.2.4.4 of [Lur14]). As such, in a symmetric monoidal quasicategory, we can define $mBiAlg_n$ for arbitrary $m$ and $n$.

**Remark 2.1.8.** Note that for an $E_{k+j}$-monoidal category $C$, an object $H$ of $Alg_{E_k}(C^{op})$ admits a lifting of the inclusion of the base point $(0) \to (1)$, inducing an algebra unit map $1_C \to H$. Hence $H$ admits a counit $\epsilon : H \to 1_C$ in $CoAlg_{E_k}(S)$. Similarly, $H$ admits a comultiplication $\delta : H \to H \otimes H$ which is “$E_k$-cocommutative” up to coherent higher homotopy. We have ensured that the $E_j$-algebra structure on $kBiAlg_j(C)$ is compatible with this coalgebra structure by demanding that this structure pulls back the $E_j$-monoidal structure of $CoAlg_{E_k}(S)$.

**Lemma 2.1.9.** Let $C$ be a symmetric monoidal quasicategory and let $*$ denote the symmetric monoidal product on $CoAlg_{E_k}(C)$ induced by the symmetric monoidal product on $CoAlg_{E_k}(C)^{op} = Alg_{E_k}(C^{op})$ as the category of $E_k$-algebras in $C^{op}$. Then for $H, K \in CoAlg_{E_k}(C)$, the underlying $C$ object of $H \ast K$ is equivalent to $H \otimes K$, where $\otimes$ denotes the symmetric monoidal product of $C$.

**Proof.** Let $*^{op}$ be the symmetric monoidal structure on $Alg_{E_k}(C^{op})$. From Remark 3.2.4.4 of [Lur14] we recall that for each object $J$ of $Fin_*$, there is an evaluation functor of $\infty$-operads $ev_J : CAlg(C^{op})^{op} \to (C^\otimes)^{op}$. In other words, if $\otimes^{op}$ is the symmetric monoidal structure on $C^{op}$, $X \ast^{op} Y \simeq X \otimes^{op} Y$. Since the $op$-involution preserves objects, this proves the Lemma.

**Corollary 2.1.10.** Let $H$ be an object of $kBiAlg_j(C)$ for $C$ a symmetric monoidal quasicategory. Then the underlying object of $H$ admits an $E_j$-algebra structure.
Proof. This follows from the fact that the evaluation functor given above is a symmetric monoidal functor (again, see Remark 3.2.4.4 of [Lur14]).

**Proposition 2.1.11.** Let $\mathcal{C}$ and $\mathcal{D}$ be small $E_n$-monoidal quasicategories and $f : \mathcal{C} \to \mathcal{D}$ an $E_n$-monoidal functor. Then for $j + k = n$, if $H$ is an object of $kBiAlg_j(\mathcal{C})$ then $f(H)$ is an object of $kBiAlg_j(\mathcal{D})$.

**Proof.** The statement that $f$ is an $E_n$-monoidal functor means in particular that $f$ corresponds to a map of $\infty$-operads over $E_n$ for which both of the vertical maps in the following diagram are coCartesian:

\[
\begin{array}{ccc}
\mathcal{C} \otimes & \xrightarrow{f \otimes} & \mathcal{D} \\
\downarrow & & \downarrow \\
E_n & & E_n
\end{array}
\]

Equivalently, by Lurie’s straightening formalism, there is a natural transformation of maps of $\infty$-operads in $Fun_{Fin_n}(E_n, qCat)$ from the functor representing the $E_n$-structure on $\mathcal{C}$ to the functor representing the $E_n$-structure on $\mathcal{D}$. This induces a functor $\tilde{f}$ in $Fun_{Fin_n}(E_n \times \Delta^1, qCat)$, which we can compose with $op: qCat \to qCat$ to produce another functor $\tilde{f}^{op}$ which is equivalent to $f$ on objects. It follows formally that $f$ preserves both monoidal and comonoidal structure.

**Definition 2.1.12 (Comodules).** Let $\mathcal{C}$ be an $E_n$-monoidal quasicategory and let $H$ be an object of $Alg_{E_k}(\mathcal{C}^{op})$ for $0 < k \leq n$. Then, using Proposition 1.2.13 we know that there is an $E_{k-1}$-monoidal quasicategory $LMod_H(\mathcal{C}^{op})$. Hence we define the category of left comodules over $H$ to be the quasicategory $LMod_H(\mathcal{C}^{op})^{op}$. We will denote this category by $LComod_H(\mathcal{C})$ or $LComod_H$.

**Lemma 2.1.13.** If $\mathcal{C}$ is an $E_n$-monoidal category and $A$ is an (at least) $E_1$-coalgebra in $\mathcal{C}$ then the category $LComod_A(\mathcal{C})$ admits $K$ indexed colimits for every small simplicial set $K$. Moreover, the forgetful functor $LComod_A(\mathcal{C}) \to \mathcal{C}$ preserves these colimits.

**Proof.** One notices that the category of comodules is the opposite of a category of modules, which is is closed under limits as demonstrated in Corollary 4.2.3.3 of [Lur14]

**Definition 2.1.14 (Cotensor Product).** Let $\mathcal{C}$ be an $E_m$-monoidal category, $H$ an object of $CoAlg_{E_n}(\mathcal{C})$ for $0 < n \leq m$, and $B$ and $C$ objects of $LComod_H$. Then using Construction 4.4.2.7 of [Lur14] we can form a simplicial object $Bar_H(B,C)_\bullet$ in $\mathcal{C}^{op}$ called the two-sided bar construction of $B$ and $C$ over $H$. If the colimit of $Bar_H(B,C)_\bullet$ exists, we call it the relative tensor product of $B$ and $C$ over $H$, and sometimes denote it by $B \otimes_H C$. Let $Cobar_H(B,C)_\bullet$ denote the cosimplicial object of $\mathcal{C}$ corresponding to $Bar_H(B,C)_\bullet$. If the limit of $Cobar_H(B,C)_\bullet$ exists in $\mathcal{C}$ then we call it the cotensor product of $B$ and $C$ over $H$ and will sometimes denote it by $B \square_H C$. 

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Proposition 2.1.15. Any Kan complex $X$ is a cocommutative coalgebra object of $\mathcal{T}$, the quasicategory of Kan complexes (our model of topological spaces) with the Cartesian symmetric monoidal structure.

Proof. Recall that $\mathcal{T}$ is equipped with a coCartesian fibration $p : \mathcal{T}^\otimes \to \text{Fin}^*$ defining the Cartesian symmetric monoidal structure thereon (the one given by taking Cartesian products of spaces). Then the monoidal structure on $\mathcal{T}^{op}$ determined by $(p^\vee)^{op} : (\mathcal{T}^{\otimes, \vee})^{op} \to \text{Fin}^*$ is the coCartesian monoidal structure given by the coproduct in $\mathcal{T}^{op}$. From Corollary 2.4.3.10 of [Lur14] we know that every object of $\mathcal{T}^{op}$ is a commutative algebra with respect to the coCartesian monoidal structure, with algebra structure coming from the universal property of coproducts. Equivalently, every space is a cocommutative coalgebra with respect to the product on $\mathcal{T}$, given explicitly by the diagonal map. □

Corollary 2.1.16. Let $f : X \to Y$ be a morphism in $\mathcal{T}$. Then $X$ is a $Y$-comodule in the Cartesian symmetric monoidal structure on $\mathcal{T}$.

Proof. We know that, using the Cartesian symmetric monoidal structure on $\mathcal{T}$, $Y$ is a commutative algebra in $\mathcal{T}^{op}$. In $\mathcal{T}^{op}$, there is a morphism $f^{op} : Y \to X$ which is a morphism of commutative algebras (again, see Corollary 2.4.3.10 of [Lur14]). Hence $X$ is a $Y$-algebra (and therefore a $Y$-module) in $\mathcal{T}^{op}$. As a result, $X$ is clearly a $Y$-comodule in $\mathcal{T}$. On the level of points, the coaction is given by $x \mapsto (x, \ast)$. □

Remark 2.1.17. Note that, as a result of Corollary 2.1.16, given any space $X$ and a pointed space $\ast \to Y$, $X$ supports a $Y$-comodule structure given by the zero map $X \to \ast \to Y$. We will call this the “trivial $Y$-comodule structure on $X$.”

Corollary 2.1.18. If $X$ is an $E_n$-algebra in $\mathcal{T}$ then $\mathbb{S}[X]$, the suspension spectrum of $X$, is an object of $\infty\text{BiAlg}_n(\mathbb{S})$.

Proof. Recall from section 4.8 of [Lur14] that $\mathcal{T}$ is a commutative algebra in $\text{Pr}_L$, the category of presentable quasicategories, with monoidal structure given by the Cartesian product of spaces. Moreover, there is a symmetric monoidal functor $\mathbb{S}[\cdot] = \Sigma^\infty_{\ast} : \mathcal{T} \to S$ presenting $S$ as a $\mathcal{T}$-algebra which takes the product of spaces to the smash product of suspension spectra. Thus there is, by virtue of the functoriality of the involution $\mathcal{C} \mapsto \mathcal{C}^{op}$ on $q\text{Cat}$, a symmetric monoidal functor $(\Sigma^\infty_{\ast})^{op} : \mathcal{T}^{op} \to S^{op}$ which takes coalgebra objects in $\mathcal{T}$ to objects in $CCoAlg(\mathbb{S})$. In particular, similarly to the proof of Proposition 2.1.11 on $n$-fold loop spaces this functor can be lifted to $\infty\text{BiAlg}_n(\mathbb{S})$, yielding the result. □

2.2 Thom Spectra and Coalgebraic Structure

Describing the coalgebraic structure inherent in Thom spectra was the primary motivation behind this note. As such we review here some of the basics of the theory of Thom spectra, drawing from [ABG+14], [ABG11] and [ACB14].

There are two important notions of Thom spectra that are equivalent:
1. From [ABG+14], for an $E_n$-ring spectrum $R$, we consider the $E_n$-monoidal quasicategory of left $R$-modules which are equivalent to $R$, and equivalences between them, denoted $BGL_1(R)$. This is in fact a Kan complex with an evident inclusion functor $BGL_1(R) \hookrightarrow LMod_R$. Given a morphism of Kan complexes $f : X \to BGL_1(R)$, we can take the composition $X \to BGL_1(R) \hookrightarrow \rightarrow LMod_R$. The colimit of this map, an $R$-module denoted $M_f$, is then called the Thom spectrum associated to $f$.

2. On the other hand from [ABG11] we have, for an $\infty$-operad $O$ admitting a map from $E_\otimes^{\otimes} 1$, an adjoint pair $Pre : Alg_{\otimes}^O(T) \rightleftarrows Alg_O(Pr^L) : Pic$ between grouplike $O$-algebras in $T$ and $O$-algebras in $Pr^L$, the quasicategory of presentable quasicategories and colimit preserving functors between them. The left adjoint $Pre$ takes a grouplike space $O$-monoidal Kan complex $X$ to the ($O$-monoidal) presentable quasicategory of presheaves on $X$, $Fun(X, T)$. The right adjoint takes a presentable $O$-monoidal quasicategory to the sub-quasicategory of invertible $O$-algebras and equivalences between them. For a quasicategory $C$, $Pic(C)$ is always a Kan complex. Moreover if $C = LMod_R$ for an $E_n$-ring spectrum, and $O = E_{n-1}$, then $BGL_1(R)$ is the base point component of $Pic(LMod_R)$. There is a comonad associated to this adjunction, and for the case of $C = LMod_R$, the counit $Pre(Pic(LMod_R)) \to LMod_R$ is called the generalized Thom spectrum functor. By inclusion of the base component, we may extend this to a functor $Pre(BGL_1(R)) \to LMod_R$.

Remark 2.2.1. Note that by Lurie’s quasicategorical Grothendieck construction (cf. 3.2 of [Lur09]), there is an equivalence between $Pre(BGL_1(R))$ and the overcategory $T/BGL_1(R)$. Thus the two Thom spectrum constructions above take equivalent types of data as input, and both have objects of $LMod_R$ as output.

Theorem 2.2.2. Constructions 1. and 2. give equivalent functors.

Proof. From Corollary 8.13 of [ABG+14] we have a unique characterization, up to equivalence of functors of quasicategories, of the Thom spectrum functor, and one checks that both functors described above satisfy that characterization. □

The first construction above has the advantage of being easy to understand and even visualize: you take a diagram in $LMod_R$ in the shape of $X$, possibly twisted by some automorphisms, and you take its colimit. This has a natural interpretation in terms of performing fiber integration on a “bundle of spectra.” The second construction is perhaps more useful for us because we know that the Thom spectrum functor preserves essentially all the operadic structure we could want, so to show, say, that a Thom spectrum is a comodule, we need only show it in $Pre(X)$, and then apply the Thom spectrum functor.

Before proving that the Thom diagonal is a structure coaction, we recall some facts regarding colored operads. We refer the reader to [Lur14] for the constructions we use, but excellent references can also be found in introductory
sections of [Her00], [BM07] and [Hor15]. Given a monoidal category $C$, the associated colored endomorphism operad will be denoted $\text{End}(C)$ (this is described by Variant 4.3.1.17 of [Lur14]). Given a colored operad $O$, the associated category of operators will be denoted by $O^\otimes$, and is given by Construction 2.1.1.7 of [Lur14]. If $C$ is simplicial then so will be $\text{End}(C)$ and $\text{End}(C)^\otimes$.

**Lemma 2.2.3.** Let $C$ be a monoidal model category with full monoidal subcategory of bifibrant objects $C^\circ$. Then the category of operators of the endomorphism operad of $(C^\circ)^{\text{op}}$, $\text{End}((C^\circ)^{\text{op}})^\otimes$, is equivalent to $(\text{End}(C^\circ)^{\text{op}})^\otimes$.

**Proof.** An investigation of the relevant constructions in [Lur14] (or the other references given) makes the result clear. □

**Theorem 2.2.4.** Let $X$ be a based Kan complex. Given a morphism $f : X \to BGL_1(R)$ for $R$ an $\mathbb{E}_n$-ring spectrum, the associated Thom spectrum $Mf$ is a comodule for the co$\mathbb{E}_n$-coalgebra $R[X]$.

**Proof.** Choose a fibration in the Quillen model structure on the ordinary category of simplicial sets, say $\tilde{f} : \tilde{X} \to BGL_1(R)$, which is equivalent to $f : X \to BGL_1(R)$ upon taking the simplicial nerve. It is not hard to check that $\tilde{f} : \tilde{X} \to BGL_1(R)$ is a strict comodule for the trivial morphism $*: \tilde{X} \to BGL_1(R)$ in $sSet/BGL_1(R)$ (with the overcategory module structure induced by the Quillen model structure on $sSet$). Moreover, both $\tilde{f} : \tilde{X} \to BGL_1(R)$ and $*: \tilde{X} \to BGL_1(R)$ are bifibrant objects in this model structure (the former since $\tilde{f}$ is a fibration and the latter since $\tilde{X}$ is a Kan complex). Hence the triangle

$$
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\Delta} & \tilde{X} \times \tilde{X} \\
\downarrow^{\tilde{f}} & & \downarrow^{f \circ \pi_1} \\
BGL_1(R) & & BGL_1(R)
\end{array}
$$

defines a coaction of $* : \tilde{X} \to BGL_1(R)$ on $\tilde{f} : \tilde{X} \to BGL_1(R)$ in $(sSet/BGL_1(R))^\circ$, the full subcategory of bifibrant objects. In other words, we have an action of $* : \tilde{X} \to BGL_1(R)$ on $\tilde{f} : \tilde{X} \to BGL_1(R)$ in $((sSet/BGL_1(R))^\circ)^{\text{op}}$. Applying Variant 4.1.3.17 of [Lur14] to $(sSet/BGL_1(R))^\circ$ we see that we have a simplicial colored endomorphism operad $\text{End}((sSet/BGL_1(R))^\circ)^{\text{op}}$ encoding its opposite monoidal structure. The action described above induces a map of colored operads $\mathbf{LM} \to \text{End}(((sSet/BGL_1(R))^\circ)^{\text{op}}) \simeq \text{End}((sSet/BGL_1(R))^\circ)^{\text{op}}$ (where $\mathbf{LM}$ is the colored operad whose algebras are monoids and modules over them, defined in Definition 4.2.1.1 of [Lur14]). Taking operadic nerves, we obtain an algebra of the $\infty$-operad $\mathcal{L}M^\otimes$ in $N^\Delta_{\text{ass}}((sSet/BGL_1(R))^{\text{op}}$.

It remains to show that this coaction is fully co$\mathbb{E}_n$-monoidal. Recall however that for an $\mathbb{E}_n$-algebra $A$ of an $\mathbb{E}_n$-monoidal quasicategory the quasicategory $LMod_A$ is naturally $\mathbb{E}_n$-monoidal. In other words, by showing that $f : X \to
$BGL_1(R)$ is an object the quasicategory of comodules for $*: X \to BGL_1(R)$ we have shown that it admits as much structure as possible.

Thus, we have shown that the diagonal map induces a $co\mathbb{E}_n$-coaction of $*: X \to BGL_1(R)$ on $f: X \to BGL_1(R)$ in $\mathcal{T}_{/BGL_1(R)}$. Since the Thom spectrum functor is strictly $\mathbb{E}_n$-monoidal, the result follows.

2.3 Coalgebras From Comonads

We now describe how to obtain coalgebras from comonads. This is essentially an application of an Eilenberg-Watts type theorem, where we recognize comonads as coalgebras in endofunctor categories and produce coalgebras in the source category by evaluating at the generating object. This procedure makes recognizing categories of descent data as equivalent to comodule categories an essentially trivial exercise.

**Definition 2.3.1.** For any quasicategory $C$ there is an $E_1$-monoidal category of functors $Fun(C, C)$, where the monoidal structure is given by composition (cf. Remark 4.7.2.31 of [Lur14]). If $F$ is an object of $Fun(C, C)$ then we say $F$ is a comonad if $F$ is equivalent to a vertex of $CoAlg(Fun(C, C))$.

**Theorem 2.3.2 (Eilenberg-Watts).** Let $B$ be an $E_1$-algebra of a symmetric monoidal quasicategory $C$, $Mod^E_1 B$ the category of $B$-bimodules and $Fun^L(LMod^E_1 B, LMod^E_1 B)$ the category of small colimit preserving endofunctors of the category of left $B$-modules. Then there is an equivalence of monoidal categories $Mod^E_1 B \sim \rightarrow Fun^L(LMod^E_1 B, LMod^E_1 B)$ given by $M \mapsto M \otimes B -$. Its inverse is given by evaluation on $B$.

**Proof.** See Proposition 7.1.2.4 of [Lur14].

**Corollary 2.3.3.** There is an equivalence of categories between the quasicategory of colimit preserving comonads $F: LMod^E_1 B \rightarrow LMod^E_1 A$ and coalgebras of $Mod^E_1 B$.

**Proof.** This follows from the theorem by restricting the monoidal equivalence to quasicategories of algebras.

**Corollary 2.3.4.** Let $\phi: A \rightarrow B$ be a morphism of (at least) $E_1$-algebras of $C$. Then the $B$-bimodule $B \otimes_A B$ is a coaogonal object of $Mod^E_1 B$.

**Proof.** Note that tensoring with $B \otimes_A B$ is equivalent to applying the forgetful functor $Mod^E_1 A \rightarrow Mod^E_1 A$ and then applying its left adjoint $- \otimes_A B: Mod^E_1 A \rightarrow Mod^E_1 B$. As the composition of a right adjoint followed by its left adjoint, this defines a comonad on $Mod^E_1 B$.

**Remark 2.3.5.** Note that one can obtain a more explicit construction of the coalgebra associated to a comonad by using the quasicategorical adjunction machinery of [RV]. In particular, there it is shown that an adjunction of quasicategories (in this case between $LMod^E_1 A$ and $LMod^E_1 B$) yields a cosimplicial object.
of \( \text{Fun}(L\text{Mod}_{E_1}^E, L\text{Mod}_{E_1}^E) \) satisfying the Segal condition, which determines, in light of Section 4.1.2 of [Lur14], a comonoid object.

**Theorem 2.3.6.** Let \( B \) be an \( E_n \)-algebra of a symmetric monoidal quasicategory \( C \), for \( n \geq 1 \). Then \( L\text{Mod}_{E_1}^E B \) is an \( E_{n-1} \)-monoidal category and the forgetful functor \( \text{Mod}_{E_1}^E B \to L\text{Mod}_{E_1}^E B \) is \( E_{n-1} \)-monoidal.

**Proof.** See Theorem 5.1.4.10 of [Lur14].

**Corollary 2.3.7.** If \( \phi : A \to B \) is a morphism of \( E_n \)-algebra objects of \( C \) then \( B \otimes_A B \) is an \( E_1 \)-coalgebra in \( L\text{Mod}_{E_1}^E B \).

**Remark 2.3.8.** Often, given a monadic adjunction of categories \( F : C \rightleftarrows D : G \), \( F \dashv G \) (i.e. one such that \( D \) is the category of algebras for the monad \( G \circ F \)), the category of comodules for the comonad \( F \circ G \) is referred to as the category of descent data for this adjunction. It is a classical theorem that the category of descent data for the extension/restriction of scalars adjunction of a morphism of commutative rings \( \phi : A \to B \) is equivalent to the category of comodules for the coring \( B \otimes_A B \). This has been proven in the homotopical setting by Hess ([Hes10]), and we reprove her result for quasicategories here.

**Theorem 2.3.9.** Given a morphism of \( E_n \)-ring spectra \( \phi : A \to B \), with associated comonad \( C \in \text{Fun}^L(L\text{Mod}_{E_1}^E, L\text{Mod}_{E_1}^E) \) there is an equivalence of quasicategories between \( L\text{Comod}_C(\text{Fun}^L(L\text{Mod}_{E_1}^E, L\text{Mod}_{E_1}^E)) \) and \( L\text{Comod}_{B \otimes_A B}(L\text{Mod}_{E_1}^E) \).

**Proof.** The monoidal equivalence of Corollary 2.3.3 is lifted to an equivalence of module categories by the functor \( \Theta_* \) of Theorem 4.8.5.5 in [Lur14].

**References**


[RV] Emily Riehl and Dominic Verity, *Homotopy coherent adjunctions and the formal theory of monads.*