Calculus II: For Biology and Medicine

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Abstract. This is the lecture note on Calculus for biology and medicine. The content contains additional topics beyond the textbook and lots of interesting problems.

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CHAPTER 1

Numbers and functions

This chapter deals with the preliminaries used in the following chapters, including the systematical description of real and complex numbers, and elementary functions. An important class of elementary functions is the class of polynomial functions, since we will see that any continuous function on a compact set (e.g., on a closed interval \([0, 1]\)) can be approximated by polynomials.

1.1. Real and complex numbers

Let \( \mathbb{N} \) be the set of all natural numbers, i.e.,
\[
\mathbb{N} := \{ n : n = 1, 2, \ldots \},
\]
and \( \mathbb{N}_{\geq 0} := \mathbb{N} \cup \{ 0 \} \) the set of whole numbers. Let us denote \( \mathbb{Z} \) and \( \mathbb{Q} \) the set of all integers and rational numbers, respectively. The relations of \( \mathbb{N}, \mathbb{Z}, \mathbb{Q} \) are related in Figure 1.1.

1.1.1. The real numbers. Let \( \mathbb{R} \) be the set of all real numbers. For any real numbers \( a, b \in \mathbb{R} \) with \( a < b \), define
\[
\text{open interval } (a, b) := \{ x \in \mathbb{R} : a < x < b \},
\]
\[
\text{closed interval } [a, b] := \{ x \in \mathbb{R} : a \leq x \leq b \}.
\]
We also use half-open intervals:
\[
[a, b) := \{ x \in \mathbb{R} : a \leq x < b \}, \quad (a, b] := \{ x \in \mathbb{R} : a < x \leq b \}.
\]

Figure 1.1. Numbers

\[1\]March 25, 2013
The above three types of intervals are clearly bounded; unbounded intervals are sets of the forms \( \{x \in \mathbb{R} : x > a\} \). Here are the possible cases:

\[
(a, \infty) := \{x \in \mathbb{R} : x > a\}, \\
(-\infty, a] := \{x \in \mathbb{R} : x \leq a\}, \\
(a, \infty) := \{x \in \mathbb{R} : x > a\}, \\
(-\infty, a) := \{x \in \mathbb{R} : x < a\}.
\]

The symbols “\( \infty \)” and “\(-\infty\)” “plus infinity” and “minus infinity”, respectively. In particular,

\[ \mathbb{R} = (-\infty, \infty). \]

**Definition 1.1.1.** The **absolute value** of a real number \( a \), denoted by \(|a|\), is

\[
|a| := \begin{cases} 
 a, & a \geq 0, \\
 -a, & a < 0.
\end{cases}
\]

Looking at Figure 1.3, we see the \( x = 0 \) is the “singular point” of the blue line; a geometric interpretation says that the function \( y = x \) has no tangent line at \( x = 0 \). The second picture in Figure 1.3 locally looks like the first one, at points \( x = b, c, d \).

**1.1.2. Lines in the plane.** The **linear equation** in the plane is given by

\[
Ax + By + C = 0,
\]
where \( A, B, \) and \( C \) are some constants. When \( A = B = 0 \) in (1.1.2), we must have \( C = 0 \). Hence, we may assume that \( A \) and \( B \) are not both equal to 0. Geometrically, the graph of a linear equation is a straight line.

If the two points \((x_1, y_1)\) and \((x_2, y_2)\) lie on a straight line (see Figure 1.4), then the slope of the line is

\[
m := \frac{y_2 - y_1}{x_2 - x_1}.
\]

(1.1.3)

Two points (or one point and the slope) are sufficient to determine the equation of a straight line.

(i) If we are given one point \((x_0, y_0)\) and the slope \( m \), we can use the point-slope form of a straight line to write its equation:

\[
y - y_0 = m(x - x_0).
\]

(1.1.4)

(ii) The equation (1.1.4) can be equivalently written as the slope-intercept form

\[
y = mx + b,
\]

(1.1.5)

where \( b \) is the \( y \)-intercept, which is the point of intersection of the line with the \( y \)-axis.

We say \( y \) is proportional to \( x \), if

\[
y = mx
\]

for some \( m \in \mathbb{R} \); in this case we call \( m \) the constant of proportionality and we write

\[
y \propto x.
\]

In particular, \( y - b \propto x \) from (1.1.5).

Given two lines \( l_1 \) and \( l_2 \) in the plane.

(i) When \( l_1 \) and \( l_2 \) have no points in common or are identical, they are called parallel, denoted by \( l_1 \parallel l_2 \). Note that two noncoincident lines are parallel if and only if their slopes are identical. In particular, for two noncoincident, nonvertical lines \( l_1 \) and \( l_2 \) with slopes \( m_1 \) and \( m_2 \), respectively, we have

\[
l_1 \parallel l_2 \iff m_1 = m_2.
\]
Two \( l_1 \) and \( l_2 \) are called \textbf{perpendicular}, denoted \( l_1 \perp l_2 \), if their intersection forms an angle of \( 90^\circ \). Two nonvertical lines are perpendicular if and only if their slopes are negative reciprocals; that is, if \( l_1 \) and \( l_2 \) are nonvertical lines with slopes \( m_1 \) and \( m_2 \), then
\[
l_1 \perp l_2 \iff m_1 m_2 = -1.
\]

1.1.3. \textbf{Equation of the circle.} A \textbf{circle} is the set of all points at a given distance, called the \textbf{radius}, from a given point, called the \textbf{center}.

If \( r \) is the distance from \((x_0, y_0)\) to \((x, y)\), then we have
\[
(x - x_0)^2 + (y - y_0)^2 = r^2. \tag{1.1.6}
\]

If \( r = 1 \) and \((x_0, y_0)\) the circle is called the \textbf{unit circle}, denoted \( S^1 \) (that is, \( S^1 \) is the sphere of dimension 1).

1.1.4. \textbf{Trigonometry.} Consider a point \( B = (x, y) \) on the unit circle \( S^1 \) (see Figure 1.6). The point \( B \) and \( O = (0, 0) \) form an angle \( \theta \). Define
\[
\begin{align*}
sin \theta &= y, & \csc \theta &= \frac{1}{\sin \theta}, \\
cos \theta &= x, & \sec \theta &= \frac{1}{\cos \theta}, \\
tan \theta &= \frac{y}{x}, & \cot \theta &= \frac{1}{\tan \theta}.
\end{align*}
\]
From the definition, we have
\begin{align}
\sin^2 \theta + \cos^2 \theta &= 1, \\
\tan^2 \theta + 1 &= \sec^2 \theta.
\end{align}

Furthermore,

(i) **Symmetry:**
\begin{align*}
\sin(-\theta) &= -\sin \theta, \quad \cos(-\theta) = \cos \theta, \\
\sin \left(\frac{\pi}{2} - \theta\right) &= \cos \theta, \quad \cos \left(\frac{\pi}{2} - \theta\right) = \sin \theta, \\
\sin(\pi - \theta) &= \sin \theta, \quad \cos(\pi - \theta) = -\cos \theta.
\end{align*}

(ii) **Shifts:**
\begin{align*}
\sin \left(\theta + \frac{\pi}{2}\right) &= \cos \theta, \quad \cos \left(\theta + \frac{\pi}{2}\right) = -\sin \theta, \\
\sin(\theta + \pi) &= -\sin \theta, \quad \cos(\theta + \pi) = -\cos \theta, \\
\sin(\theta + 2\pi) &= \sin \theta, \quad \cos(\theta + 2\pi) = \cos \theta.
\end{align*}

(iii) **Angle sum and difference identities:**
\begin{align*}
\sin(\alpha \pm \beta) &= \sin \alpha \cdot \cos \beta \pm \cos \alpha \cdot \sin \beta, \\
\cos(\alpha \pm \beta) &= \cos \alpha \cdot \cos \beta \pm \sin \alpha \cdot \sin \beta.
\end{align*}

(iv) **Double-angle formula:**
\begin{align*}
\sin(2\theta) &= 2\sin \theta \cdot \cos \theta = \frac{2\tan \theta}{1 + \tan^2 \theta}, \\
\cos(2\theta) &= \cos^2 \theta - \sin^2 \theta = 2\cos^2 \theta - 1 = 1 - 2\sin^2 \theta \\
&= \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta}, \\
\tan(2\theta) &= \frac{2\tan \theta}{1 - \tan^2 \theta}.
\end{align*}

(v) **Triple-angle formula:**
\begin{align*}
\sin(3\theta) &= -4\sin^3 \theta + 3\sin \theta, \quad \cos(3\theta) = 4\cos^3 \theta - 4\cos \theta, \quad \tan(3\theta) = \frac{3\tan \theta - \tan^3 \theta}{1 - 3\tan^2 \theta}.
\end{align*}

1.1.5. **Exponentials and logarithms.** An exponential is an expression of the form
\[ a^r \]
where \( a \) is called the **base** and \( r \) the **exponent**. Unless \( r \) is an integer or unless \( r \) is a rational number of the form \( p/q \) where \( p \) is an integer and \( q \) is an odd integer, we will assume that \( a \) is positive. Some basic properties are
\begin{align*}
a^r a^s &= a^{r+s}, \quad (ab)^r = a^r b^r, \\
\frac{a^r}{a^s} &= a^{r-s}, \quad \left(\frac{a}{b}\right)^r = \frac{a^r}{b^r}, \\
a^{-r} &= \frac{1}{a^r}, \quad (a^{-r})^s = a^{-rs}.
\end{align*}
If \( y = a^x > 0 \) with \( a > 0 \) and \( a \neq 1 \), then we formally write \( x := \log_a y \).

Consequently,

\[
\begin{align*}
\log_a (xy) &= \log_a x + \log_a y, \\
\log_a \left( \frac{x}{y} \right) &= \log_a x - \log_a y, \\
\log_a (x^r) &= r \log_a x, \\
\log_a (a^x) &= x, \\
a^{\log_a x} &= x.
\end{align*}
\]

When \( a = e \), we write \( \log_a \) as \( \ln \), where \( e \) is the constant and approximated equal to 2.7182818.

1.1.6. Complex numbers and quadratic equations. Observe that the square of a real number is always nonnegative. To extend the real numbers, we introduce a symbol \( i \) satisfying

\[ i^2 = -1. \]

Formally, the equation has two solutions \( \sqrt{-1} \) and \( -\sqrt{-1} \). We usually set

\[ i := \sqrt{-1} \]

and call \( i \) the imaginary unit.

**Definition 1.1.2.** A complex number is a number of the form

\[ z = a + b\sqrt{-1}, \quad a, b \in \mathbb{R}. \]

The real number \( a =: \Re(z) \) is the real part of \( z \), and the real number \( b =: \Im(z) \) is the imaginary part. (See Figure 1.7) The set of all complex numbers is denoted \( \mathbb{C} \). When \( a = 0 \) we call \( b\sqrt{-1} \) an purely imaginary number.

A complex number \( z = a + b\sqrt{-1} \) corresponds to a point \((a, b)\) in the plane, conversely, a point \((a, b)\) in the plane corresponds to a complex number \( z = a + b\sqrt{-1} \). Consequently,

\[ a + b\sqrt{-1} = c + d\sqrt{-1} \iff a = c \text{ and } b = d. \]

\[ \text{This number is an irrational number and can be defined as (see later) } \lim_{n \to \infty} (1 + \frac{1}{n})^n. \]
Given two complex numbers \( a + b\sqrt{-1} \) and \( c + d\sqrt{-1} \), we have
\[
(a + b\sqrt{-1}) + (c + d\sqrt{-1}) = (a + c) + (b + d)\sqrt{-1},
\]
\[
(a + b\sqrt{-1})(c + d\sqrt{-1}) = ac + ad\sqrt{-1} + bc\sqrt{-1} + bd((-1)^2)
\]
\[
= (ac - bd) + (ad + bc)\sqrt{-1}.
\]

**Definition 1.1.3.** If \( z = a + b\sqrt{-1} \) is a complex number, its **conjugate**, denoted by \( \bar{z} \), is defined as
\[
\bar{z} := a - b\sqrt{-1}.
\]

**Proposition 1.1.4.** For any \( z, w \in \mathbb{C} \) we have
\[
(\bar{z}) = z, \quad \bar{z} + \bar{w} = \bar{z} + \bar{w}, \quad \bar{zw} = \bar{z}\bar{w}.
\]
Furthermore, if \( z = a + b\sqrt{-1} \) then
\[
|z|^2 := z\bar{z} = a^2 + b^2.
\]
We call \( |z| := \sqrt{|z|^2} \) the **norm** of \( z \).

If \( z = a + b\sqrt{-1} \), then
\[
|z| = r, \quad r := \sqrt{a^2 + b^2}.
\]
Thus \( z \) lies in the circle at the origin with radius \( r \).

### 1.2. Elementary functions

A **function** \( f \) is a rule that assigns each element \( x \) in the set \( A \) exactly one element \( y \) in the set \( B \). The element \( y \) is called the **image** (or **value**) of \( x \) under \( f \) and is denoted by \( f(x) \). The set \( A \) is called the **domain** of \( f \), the set \( B \) is called the **codomain** of \( f \), and the set \( f(A) = \{ y \in B : y = f(x) \text{ for some } x \in A \} \) is called the **range** of \( f \).

**Definition 1.2.1.** Two functions \( f \) and \( g \) are **equal** if and only if
1. \( f \) and \( g \) are defined on the same domain, and
2. \( f(x) = g(x) \) for all \( x \) in the domain.

By this definition, we see that two functions
\[
f : \mathbb{R} \to \mathbb{R}, \quad x \mapsto x^2, \quad g : [0,1] \to \mathbb{R}, \quad x \mapsto x^2,
\]
do not coincide with each other.

**Definition 1.2.2.** A function \( f : A \to B \) is called
1. **even** if \( f(x) = f(-x) \) for all \( x \in A \), and
2. **odd** if \( f(x) = -f(-x) \) for all \( x \in A \).
By the definition, if \( f \) is even or odd, the domain \( A \) must be symmetric.

**Definition 1.2.3.** The **composite function** \( f \circ g \) (also called the composition of \( f \) and \( g \)) is defined as

\[
(f \circ g)(x) := f[g(x)]
\]

for each \( x \) in the domain of \( g \) for which \( g(x) \) is in the domain of \( f \).

### 1.2.1. Polynomial functions.

A **polynomial function** is a function of the form

\[
f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n,
\]

where \( n \) is the nonnegative integer and \( a_0, a_1, \ldots, a_n \) are real-valued constants with \( a_n \neq 0 \). The coefficient \( a_n \) is called the **leading coefficient**, and \( n \) is called the **degree** of the polynomial function. The largest possible domain of \( f \) is \( \mathbb{R} \).

A polynomial function \( f(x) \) of degree \( n \) may not have \( n \) roots. For example, consider the polynomial function

\[
f(x) = x^2 + x + 1 = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4};
\]

this function is always strictly positive and hence has no roots. However

**Theorem 1.2.4.** Any nonconstant polynomial function with complex-valued coefficients has, counted with multiplicity, exactly \( n \) complex roots, where \( n \) is the degree of this polynomial function.

This is the **fundamental theorem of algebra**. For example, the polynomial function \( f(x) = x^2 + x + 1 \) has two complex roots

\[
-\frac{1}{2} + \sqrt{\frac{3}{4} - 1}.
\]

### 1.2.2. Rational functions.

A **rational function** is the quotient of two polynomial functions \( p(x) \) and \( q(x) \):

\[
f(x) = \frac{p(x)}{q(x)}, \quad q(x) \neq 0.
\]

### 1.2.3. Power functions.

A **power function** is of the form

\[
f(x) = x^r, \quad r \in \mathbb{R}.
\]

### 1.2.4. Exponential functions.

The function \( f \) is an **exponential function** with base \( a \) if

\[
f(x) = a^x
\]

where \( a \) is a positive constant other than 1. The largest possible domain of \( f \) is \( \mathbb{R} \).

When \( a = e \), the exponential function is called the **natural exponential function**

\[
\exp(x) := e^x.
\]
1.2.5. Inverse functions. A function \( f(x) \) is called **one-to-one** if \( f(x_1) = f(x_2) \) implies \( x_1 = x_2 \).

**Definition 1.2.5.** Let \( f : A \to B \) be a one-to-one function with range \( f(A) \). The **inverse function** \( f^{-1} \) had domain \( f(A) \) and range \( A \) and is defined by

\[
    f^{-1}(y) = x \iff y = f(x)
\]

for all \( y \in f(x) \).

Note that if \( f : A \to B \) has an inverse \( f^{-1} \), then

\[
    f^{-1}[f(x)] = x, \quad x \in A,
\]

\[
    f[f^{-1}(x)] = x, \quad x \in f(A).
\]

1.2.6. Logarithmic functions. The inverse of \( f(x) = a^x \) is called the **logarithm to base** \( a \) and is written \( f^{-1}(x) = \log_a x \).

\[
    a^{\log_a x} = x, \quad x > 0,
\]

\[
    \log_a(a^x) = x, \quad x \in \mathbb{R},
\]

\[
    a^x = \exp[x \ln a],
\]

\[
    \log_a x = \frac{\ln x}{\ln a}, \quad x > 0.
\]

1.2.7. Trigonometric functions. A function \( f(x) \) is **periodic** if there is a positive constant \( a \) such that

\[
    f(x + a) = f(x)
\]

for all \( x \) in the domain of \( f \). If \( a \) is the smallest number with this property, we call it the **period** of \( f \).

The trigonometric functions are examples of periodic functions. Observe that the period of \( \sin(x) \) or \( \cos(x) \) is \( 2\pi \).

**Example 1.2.6.** Consider the function

\[
    f(x) = a \sin(kx), \quad x \in \mathbb{R},
\]

where \( a \in \mathbb{R} \) and \( k \neq 0 \). Since \( f(x) \) takes on values between \(-a\) and \( a \), we call \( |a| \) the **amplitude**. Observe that the period of \( f(x) \) is \( 2\pi/|k| \).

**Remark 1.2.7.** (Fourier series of periodic functions) suppose that \( f(x) \) is a periodic function of period \( 2\pi \). Then we may assume that \( f(x) \) is defined on \([-\pi, \pi]\). If \( f(x) \) is integrable (for definition, see Chapter 6), we define

\[
    (1.2.1) \quad a_n := \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) \, dx, \quad b_n := \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) \, dx, \quad n \in \mathbb{N}_{\geq 0}.
\]

which are called the **Fourier coefficients** of \( f \). The formal infinite sum

\[
    (1.2.2) \quad (Sf)(x) := \frac{a_0}{2} + \sum_{n \in \mathbb{N}} a_n \cos(nx) + b_n \sin(nx)
\]
Figure 1.8. Graphs of the six basic trigonometric functions

is called the Fourier series of \( f \). If \( f \in C^1([-\pi, \pi]) \), then a theorem in Fourier theory asserts that \( Sf = f \) for each \( x \in [-\pi, \pi] \); that is,

\[
(1.2.3) \quad f(x) = (Sf)(x), \quad x \in [-\pi, \pi], \quad \text{if } f \in C^1([-\pi, \pi]).
\]

By choosing a suitable function \( f \), we can prove

\[
(1.2.4) \quad \frac{\pi^2}{6} = \sum_{n \in \mathbb{N}} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots.
\]

**Definition 1.2.8.** A number is called **algebraic** if it is the solution of a polynomial equation with **rational** coefficients. For example, \( \sqrt{2} \) is algebraic, as it satisfies the equation \( x^2 - 2 = 0 \). Numbers that are not algebraic are called **transcendental**.

**Example 1.2.9.** (\( \pi \) is transcendental) Swiss scientist Johann Heinrich Lambert (26 August 1728– 25 September 1777) in 1761 proved that \( \pi \) is irrational; French mathematician Adrien-Marie Legendre (18 September 1752– 10 January 1833) proved in 1794 that \( \pi \) is also irrational. In 1882, German mathematician Carl Louis Ferdinand von Lindemann (12 April 1852– 6 March 1893) proved that \( \pi \) is transcendental, confirming a conjecture made by both Adrien-Marie Legendre and Leonhard Euler (15 April 1707– 18 September 1783).
Remark 1.2.10. (Gauss-Legendre algorithm) The Gauss-Legendre algorithm is an algorithm to compute the digits of $\pi$. It is notable for being rapidly convergent, with only 25 iterations producing 45 million correct digits of $\pi$.

1. Choose $a_0 = 1$, $b_0 = 1/\sqrt{2}$, $t_0 = 1/4$, and $p_0 = 2$.
2. Consider
   
   $a_{n+1} = \frac{a_n + b_n}{2}$, $b_{n+1} = \sqrt{a_nb_n}$, $t_{n+1} = t_n - p_n(a_n - a_{n+1})^2$, $p_{n+1} = 2p_n$.

3. $\pi$ is then approximated as
   
   $\pi = \lim_{n \to \infty} \frac{(a_n + b_n)^2}{4t_n}$.

The first three iterations give

$$3.140\cdots, 3.14159264\cdots, 3.1415926535897932382\cdots.$$ 

We call a function $y = f(x)$ algebraic if it is the solution of an equation of the form

$$P_n(x)y^n + \cdots + P_1(x)y + p_0y = 0$$

in which the coefficients are polynomial functions in $x$ with rational coefficients. For example, the function $y = \frac{1}{1+x}$ is algebraic, as it satisfies the equation $(x+1)y - 1 = 0$. 

Figure 1.9. Lambert, Legendre, and Euler

Figure 1.10. Lindemann and Gauss
0. Functions that are not algebraic are called transcendental. All the trigonometric, exponential, and logarithmic functions are transcendental functions.

**Remark 1.2.11.** The name “transcendental” comes from Leibniz in his 1682 paper where he proved \( \sin x \) is not an algebraic function of \( x \), and Euler was probably the first person to define transcendental numbers in the modern sense.

1. Joseph Liouville first proved the existence of transcendental numbers in 1844, and in 1851 gave the first decimal examples such as the **Liouville constant**
\[
\sum_{k \in \mathbb{N}} \frac{1}{10^k}.
\]

2. Johann Heinrich Lambert conjectured that \( e \) and \( \pi \) were both transcendental numbers in his 1761 paper proving the number \( \pi \) is irrational.

3. Charles Hermite in 1873 proved that \( e \) is transcendental.

4. In 1874, Georg Cantor proved that the algebraic numbers are countable and the real numbers are uncountable. In 1878, Cantor showed that there are as many transcendental numbers as there are real numbers.

5. In 1882, Ferdinand von Lindemann published a proof that the number \( \pi \) is transcendental. He first showed that \( e \) to any nonzero algebraic power is transcendental, and since \( e^{\sqrt{-1}\pi} = -1 \) is algebraic, \( \sqrt{-1}\pi \) and therefore \( \pi \) must be transcendental.

6. In 1900, David Hilbert posed the Hilbert’s seventh problem: **if \( a \) is an algebraic number which is not zero or one, and \( b \) is an irrational algebraic number, is \( a^b \) necessarily transcendental?** The affirmative answer was provided in 1934 by the Helfond-Schneider theorem.

**1.2.8. Graphing.** The graph of 
\[
y = f(x) + a
\]
is a **vertical translation** of the graph of \( y = f(x) \). If \( a > 0 \), the graph of \( y = f(x) \) is shifted up \( a \) units; if \( a < 0 \), the graph of \( y = f(x) \) is shifted down \( |a| \) units.

The graph of 
\[
y = f(x - c)
\]
is a **horizontal translation** of the graph \( y = f(x) \). If \( c > 0 \), the graph of \( y = f(x) \) is shifted \( c \) units to the right; if \( c < 0 \), the graph of \( y = f(x) \) is shifted \( |c| \) units to the left.
CHAPTER 2

Sequences and difference equations

In his chapter, we discuss models for populations that reproduce at discrete times and we develop some of the theory needed to analyze this type of model.

2.1. Sequences

A sequence is the function $f: \mathbb{N} \rightarrow \mathbb{R}$, $n \mapsto f(n)$ as a list of numbers $a_0, a_1, a_2, \cdots$ where $a_n = f(n)$. We will write $\{a_n: n \in \mathbb{N}_{\geq 0}\}$ or $\{a_n\}_{n \in \mathbb{N}_{\geq 0}}$ if we mean the entire sequence.

Example 2.1.1. (1) $a_n = (-1)^n$, $n = 0, 1, 2, \cdots$. (2) $a_n = n^2$, $n = 0, 1, 2, \cdots$. (3) $a_n = (-1)^n/(n + 1)^2$, $n = 0, 1, 2, \cdots$.

2.1.1. $k$th order recursion. From Example 2.1.1(2), we see that

$$a_{n+1} = (n + 1)^2 = n^2 + 2n + 1 = a_n + 2\sqrt{a_n} + 1;$$

thus $a_{n+1}$ depends only on $a_n$. In general, we have the following

Definition 2.1.2. We say a sequence $\{a_n\}_{n \in \mathbb{N}_{\geq 0}}$ is $k$th order recursive if

$$a_{n+1} = g(a_n, a_{n-1}, \cdots, a_{n-k+1})$$

for some function $g$. In particular, the 1th order recursion is of the form

$$a_{n+1} = g(a_n);$$

the 2th order recursion is of the form

$$a_{n+1} = g(a_n, a_{n-1}).$$

2.1.2. Limits. Looking at Example 2.1.1 we see that, as $n$ goes to infinity,

(1) $a_n$ can not be determined;
(2) $a_n$ tends to infinity;
(3) $a_n$ tends to zero.
Definition 2.1.3. The sequence \( \{a_n\}_{n \in \mathbb{N}} \) or \( \{a_n\}_{n \geq 0} \) has limit \( a < \infty \), written as \( \lim_{n \to \infty} a_n = a \), if, for every \( \epsilon > 0 \), there exists an integer \( N \) (depending on \( \epsilon \)) such that
\[
|a_n - a| < \epsilon
\]
whenever \( n > N \). If the limit exists, the sequence is called convergent and we say that \( a_n \) converges to \( a \) as \( n \) tends to infinity. If the sequence has no limit, it is called divergent.

Proposition 2.1.4. If the sequence \( \{a_n\}_{n \in \mathbb{N}} \) or \( \{a_n\}_{n \geq 0} \) is convergent, then the limit of this sequence is unique.

Proof. Suppose we have two limits \( a \) and \( a' \). By the definition, for every \( \epsilon > 0 \), there exist integers \( N \) and \( N' \) such that
\[
|a_n - a| < \epsilon \quad n > N, \quad \text{and} \quad |a_n - a'| < \epsilon \quad n > N'.
\]
Consequently, for \( n > N + N' \), we have
\[
|a - a'| = |(a_n - a) + (a_n - a')| \leq |a_n - a| + |a_n - a'| < 2\epsilon.
\]
Since the left-hand side does not depend on \( n \) and \( \epsilon \), letting \( \epsilon \to 0 \), we obtain \( a = a' \).

Exercise 2.1.5. Show that \( \lim_{n \to \infty} \frac{1}{n} = 0 \).

Exercise 2.1.6. Show that the sequence \( \{a_n\}_{n \in \mathbb{N}} \), where \( a_n = n \), is divergent.

In order to calculus, we need the following fundamental results.
Theorem 2.1.7. If \( \lim_{n \to \infty} a_n \) and \( \lim_{n \to \infty} b_n \) exist and \( c \) is a constant, then

\[
\lim_{n \to \infty} (a_n + b_n) = \lim_{n \to \infty} a_n + \lim_{n \to \infty} b_n,
\]

\[
\lim_{n \to \infty} (ca_n) = c \lim_{n \to \infty} a_n,
\]

\[
\lim_{n \to \infty} (a_n b_n) = \lim_{n \to \infty} a_n \cdot \lim_{n \to \infty} b_n,
\]

\[
\lim_{n \to \infty} \frac{a_n}{b_n} = \frac{\lim_{n \to \infty} a_n}{\lim_{n \to \infty} b_n}, \text{ provided } \lim_{n \to \infty} b_n \neq 0.
\]

For example,

\[
\lim_{n \to \infty} \frac{n + 1}{n} = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right) = 1 + \lim_{n \to \infty} \frac{1}{n} = 1 + 0 = 1
\]

by (2.1.1) and Exercise 2.1.5.

Remark 2.1.8. The assumption in Theorem 2.1.7 that \( \lim_{n \to \infty} a_n \) and \( \lim_{n \to \infty} b_n \) exists is necessary. For example, consider \( a_n = n \) and \( b_n = -n \); then (2.1.1) does not hold.

2.1.3. Recursions. We consider the sequences of the 1th order recursion

\[a_{n+1} = g(a_n), \quad a_0 \text{ is a given real number.}\]

We start from several elementary special cases.

1. \( a_{n+1} = ca_n \) for some nonzero constant \( c \). In this case, it is easily to see that

\[
a_n = ca_{n-1} = c^2a_{n-2} = \cdots = a_0c^n.
\]

2. \( a_{n+1} = a_n + c \) for some constant \( c \). In this case, we have

\[
a_n = a_0 + cn.
\]

3. \( a_{n+1} = c_1a_n + c_2 \) for some constants \( c_1 \neq 1 \) and \( c_2 \). When \( c_1 = 1 \), the result follows from 2. Consider the constant \( b \) such that

\[
a_{n+1} - b \quad a_n - b = c_1;
\]

then

\[a_{n+1} = c_1a_n + b(1 - c_1), \quad c_2 = b(1 - c_1).
\]

So \( b = \frac{c_2}{1 - c_1} \) and

\[
a_{n+1} - \frac{c_2}{1 - c_1} = c_1 \left(a_n - \frac{c_2}{1 - c_1}\right);\]

Applying (2.1.5) to this case, we arrive at

\[
a_n = \frac{c_2}{1 - c_1} + \left(a_0 - \frac{c_2}{1 - c_1}\right) c_1^n.
\]
Example 2.1.9. Let \( a_{n+1} = 4 - 2a_n \) with \( a_0 = \frac{4}{3} \). By (2.1.7) we have:

\[
a_n = \frac{4}{1 - (-2)} + \left( \frac{4}{3} - \frac{4}{1 - (-2)} \right) (-2)^n = \frac{4}{3}.
\]

In this case \( \lim_{n \to \infty} a_n = \frac{4}{3} \).

In Example 2.1.9 we can give another short proof. Suppose the limit \( \lim_{n \to \infty} a_n = a \) exists, by taking the limit of \( n \), we have

\[
a = \lim_{n \to \infty} a_{n+1} = 4 - 2 \lim_{n \to \infty} a_n = 4 - 2a
\]

which gives us \( a = \frac{4}{3} \). Thus approach shows that if we know the limit exists, then we can take the limits on both-sides of \( a_{n+1} = g(a_n) \) to get the desired limit. However, in order to applying this argument, we should know the existence at first. This can be seen from the following fundamental theorem.

Theorem 2.1.10. Any bounded increasing or decreasing sequence has finite limit.

We say that a sequence \( \{a_n\}_{n \in \mathbb{N} \geq 0} \) is bounded if \( |a_n| \leq M \) for all \( n \), where \( M \) is some positive constant. A sequence \( \{a_n\}_{n \in \mathbb{N} \geq 0} \) is increasing (or decreasing) if \( a_n \leq a_{n+1} \) (or \( a_n \geq a_{n+1} \)).

Example 2.1.11. Consider \( a_{n+1} = \sqrt{3}a_n \) with \( a_0 = 2 \). If \( \lim_{n \to \infty} a_n = a \) exists, then

\[
a = \sqrt{3}a \implies a = 0 \text{ or } a = 3.
\]

We now prove that the limit of \( a_n \) exists. Since

\[
a_1 = \sqrt{3} < 3, \quad a_2 = \sqrt{3}a_1 < \sqrt{3} \times 3 = 3,
\]

we claim that \( a_n \in (0, 3) \). If this claim holds for \( n \), then

\[
a_{n+1} = \sqrt{3}a_n < \sqrt{3} \times 3 = 3, \quad a_{n+1} > 0,
\]

by inductive hypothesis. Hence \( a_n \in (0, 3) \) for all \( n \). From

\[
\frac{a_{n+1}}{a_n} = \sqrt{\frac{3}{a_n}} > 1,
\]

it follows that the sequence \( \{a_n\}_{n \in \mathbb{N} \geq 0} \) is bounded and increasing, and therefore the limit \( \lim_{n \to \infty} a_n \) exists.

Returning to the 1th order recursion

\[
a_{n+1} = g(a_n).
\]

We say that \( a \) is a fixed point of the sequence \( \{a_n\}_{n \in \mathbb{N} \geq 0} \) if

\[
a_0 = a \implies a_n = a \text{ for all } n.
\]

For example, \( 4/3 \) is a fixed point of the sequence \( \{a_n\}_{n \in \mathbb{N} \geq 0} \). If \( a \) is a fixed point, then

\[
a = a_{n+1} = g(a_n) = g(a).
\]
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Definition 2.1.12. Given a function \( f : A \to A \). A number \( a \in A \) is called a **fixed point** of \( f \) if \( a = f(a) \). Thus \( a \) is an intersection point of the functions \( y = x \) and \( y = f(x) \). (see Figure 2.2)

Observe that if \( a \) is a fixed point of the sequence \( a_{n+1} = g(a_n) \), then \( a \) is a fixed point of the function \( g : \mathbb{R} \to \mathbb{R} \). Conversely, if \( a \) is a fixed point of the function \( f : \mathbb{R} \to \mathbb{R} \), then we can construct a sequence \( \{a_n\}_{n \in \mathbb{N}_0} \) of which \( a \) is a fixed point.

**Definition 2.1.13.** A function \( f : A \to B \) is called **Lipschitz continuous** if there exists a nonnegative constant \( L \) such that

\[
|f(x) - f(y)| \leq L|x - y|
\]

whenever \( x, y \in A \).

For example \( f(x) = \sqrt{x^2 + 5}/2, \ x \in \mathbb{R} \), is a Lipschitz continuous function with \( L \leq 1/2 \).
Proposition 2.1.14. Let $f : \mathbb{R} \to \mathbb{R}$ be a Lipschitz continuous function with constant $L \in (0, 1)$. If $a$ is a fixed point of $f$, then there exists a sequence $\{x_n\}_{n \in \mathbb{N}_0}$ of which $a$ is a fixed point and $\lim_{n \to \infty} a_n = a$.

Proof. Given a point $a_0 \in \mathbb{R}$, consider
\[
a_1 := f(a_0), \quad a_2 := f(a_1) = f(f(a_0)) := f^2(a_0), \quad a_n := f^n(a_0).
\]
Then
\[
|a_1 - a| = |f(a_0) - f(a)| \leq L|a_0 - a|;
\]
in general, we have
\[
|a_n - a| = |f(a_{n-1}) - f(a)| \leq L|a_{n-1} - a| \leq \cdots \leq L^n|a_0 - a|
\]
for all $n$. Letting $n \to \infty$ and noting that $L \in (0, 1)$ imply $\lim_{n \to \infty} a_n = a$. In particular, when $a_0 = a$, we have $|a_n - a| \leq L^n|a - a| = 0$ so that $a_n = a$. \[\square\]

2.2. Population models

We denote by the population size at time $t$ by $N_t$, $t = 0, 1, 2, \cdots$, and consider the model
\[
N_{t+1} = f(N_t),
\]
where the function $f$ describes the density dependence of the population dynamics.

A simple model is
\[
N_{t+1} = RN_t \quad \text{or} \quad \frac{N_t}{N_{t+1}} = \frac{1}{R},
\]
when $R > 1$, the population size will grow indefinitely, provided that $N_0 > 0$.

2.2.1. Beverton-Holt model. The Beverton-Holt model is a classic discrete-time population model which is a nonlinear version of (2.2.2)
\[
\frac{N_t}{N_{t+1}} = \frac{1}{R + 1 - \frac{1}{K} N_t} \quad \text{or} \quad N_{t+1} = \frac{RN_t}{1 + \frac{R-1}{K} N_t},
\]
where $R > 1$ and $K > 0$ is the carrying capacity.

If $N_0 > 0$, then $N_t > 0$ for all $n$; moreover
\[
\frac{N_{t+1}}{K R} = \frac{N_t}{K + (R - 1)N_t} = \frac{1}{R - 1} \left( 1 - \frac{K}{K + (R - 1)N_t} \right),
\]
\[
N_{t+1} - N_t = N_t \frac{R - 1 + \frac{R-1}{K} N_t}{1 + \frac{R-1}{K} N_t} = (R - 1)N_t \frac{K + N_t}{K + (R - 1)N_t},
\]
which imply $0 < N_{t+1} < \frac{R}{R-1}K$ and $N_{t+1} \geq N_t$. Thus the sequence $\{N_t\}_{t \in \mathbb{N}_0}$ is bounded and increasing. By Theorem 2.1.10, the limit
\[
N := \lim_{t \to \infty} N_t
\]
exists and is nonzero. Taking $t \to \infty$ in (2.2.2) yields
\[
1 = \frac{1}{R} + 1 - \frac{1}{K} N \implies N = K.
\]
2.2. Discrete logistic equation. The most popular discrete-time single-species model is the **discrete logistic equation**

\[
N_{t+1} = N_t \left[ 1 + R \left( 1 - \frac{N_t}{K} \right) \right],
\]

where \( R \) and \( K \) are positive constants. \( R \) is called the *growth parameter* and \( K \) is called the *carrying capacity*.

Defining

\[
x_t := \frac{R}{K(1+R)} N_t,
\]

we then obtain

\[
x_{t+1} = (1 + R)x_t(1 - x_t).
\]

**Exercise 2.2.1.** Prove (2.2.6).

The long-term behavior of the discrete logistic equation is very complicated, but, we can compute its fixed points. If \( x \) is a nonzero fixed point of (2.2.6), then

\[
x = (1 + R)x(1 - x) \implies x = \frac{R}{1 + R};
\]

If \( N \) is the fixed point corresponding to \( x \), then

\[
N = \frac{K(1 + R)}{R} x = \frac{K(1 + R)}{R} \frac{R}{1 + R} = K.
\]

2.2.3. Ricker’s curve. The **Ricker’s curve** is given by

\[
N_{t+1} = N_t \exp \left[ R \left( 1 - \frac{N_t}{K} \right) \right]
\]

where \( R \) and \( K \) are positive parameters. As in the discrete logistic model, \( R \) is called growth parameter and \( K \) is the carrying capacity. If \( N \) is a nonzero fixed point then

\[
N = N \exp \left[ R \left( 1 - \frac{N}{K} \right) \right] \implies N = K.
\]

2.2.4. Harvesting model. The **harvesting model** is given by

\[
N_{t+1} = (1 - c)N_t \exp \left[ R \left( 1 - \frac{(1 - c)N_t}{K} \right) \right]
\]

where \( R \) and \( K \) are positive constant, and \( c \in (0, 1) \). If \( N \) is a nonzero fixed point then

\[
N = \frac{K}{1 - c} \left[ 1 + \frac{1}{R} \ln(1 - c) \right].
\]
2.2.5. Fibonacci sequences. A famous example of the 2th order recursions is the Fibonacci sequence:

\[ N_{t+1} = N_t + N_{t-1}, \quad N_0 = 1 \text{ and } N_1 = 1. \]  

The equation comes from the following problem posed in 1202 of his Liber Abaci (Book of Calculation) by Leonardo Pisano Bigollo (also known as Leonardo of Pisa or most commonly Fibonacci):

*How many pairs of rabbits are produced if each pair reproduces one pair of rabbits at age one month and another pair of rabbits at age two months and initially there is one pair of newborn rabbits?*

The Fibonacci sequence (2.2.9) has a closed-form solution

\[ N_t = \frac{\varphi^t - \psi^t}{\varphi - \psi}, \]
where
\begin{equation}
\varphi := \frac{1 + \sqrt{5}}{2} \approx 1.6180339887\cdots
\end{equation}
is the golden ratio or golden mean, and
\begin{equation}
\psi := \frac{1 - \sqrt{5}}{2} = 1 - \varphi = -\frac{1}{\varphi} \approx -0.6180339887\cdots.
\end{equation}

**Exercise 2.2.2.** Using (2.2.10) to show
\begin{equation}
\lim_{t \to \infty} \frac{N_t}{N_{t+1}} = \varphi.
\end{equation}
CHAPTER 3

Limits and continuity

In this chapter we give the formal definition of limits and then continuities. The key description is the “ε-δ” technique.

3.1. Limits

The statement

\[ \lim_{x \to c} f(x) = L \]

where \( c, L \) are finite, means that, for every \( \epsilon > 0 \), there exists a number \( \delta > 0 \) such that

\[ |f(x) - L| < \epsilon \]

whenever \( 0 < |x - c| < \delta \). In this case, we say that \( f(x) \) converges to \( L \). If this limit does not exist, we say that \( f(x) \) diverges as \( x \) tends to \( c \).

Remark 3.1.1. In the above definition, we exclude the value \( x = c \) from the statement. (This is done in the inequality \( 0 < |x - c| \).) A reason for that is the function \( f(x) \) may not be defined at \( x = c \). For example,

\[ f(x) = \begin{cases} x, & x \neq 0, \\ 1, & x = 0. \end{cases} \]

In this case, we have \( \lim_{x \to 0} f(x) = 0 \neq f(0) \).

According to our definition of limits, we may consider one-sided limits:

1. \( \lim_{x \to c^+} f(x) = L \) means that for every \( \epsilon > 0 \) there exists a number \( \delta > 0 \) such that

\[ |f(x) - L| < \epsilon \]

whenever \( 0 < x - c < \delta \).

2. \( \lim_{x \to c^-} f(x) = L \) means that for every \( \epsilon > 0 \) there exists a number \( \delta > 0 \) such that

\[ |f(x) - L| < \epsilon \]

whenever \( 0 < c - x < \delta \).

Observe that

\[ \lim_{x \to c} f(x) = L \iff \lim_{x \to c^+} f(x) = L \text{ and } \lim_{x \to c^-} f(x) = L. \]
Example 3.1.2. Consider

\[ f(x) = |x|, \quad x \in \mathbb{R}. \]

Then

\[ \lim_{x \to 0^+} f(x) = 1, \quad \lim_{x \to 0^-} f(x) = -1, \]

however, the limit \( \lim_{x \to 0} f(x) \) does not exist.

The statement

(3.1.2) \[ \lim_{x \to c} f(x) = \infty \text{ (or } -\infty) \]

means that, for every \( M > 0 \), there exists a \( \delta > 0 \) such that

\[ f(x) > M \text{ (or } < -M) \]

whenever \( 0 < |x - c| < \delta \). Similarly, we can define

\[ \lim_{x \to c^+} f(x) = \infty, \quad \lim_{x \to c^+} f(x) = -\infty, \quad \lim_{x \to c^-} f(x) = \infty, \quad \lim_{x \to c^-} f(x) = -\infty. \]

Example 3.1.3. \( \lim_{x \to 0^+} \frac{1}{x} = \infty \) and \( \lim_{x \to 0^-} \frac{1}{x} = -\infty \).

The statement

(3.1.3) \[ \lim_{x \to \infty} f(x) = L \]

means that, for every \( \epsilon > 0 \), there exists an \( x_0 > 0 \) such that

\[ |f(x) - L| < \epsilon \]

whenever \( x > x_0 \). Similarly, we can define \( \lim_{x \to -\infty} f(x) = L \).
3.1. Limit laws. As Theorem \[2.1.7\] we have

**Theorem 3.1.4.** If \(\lim_{x \to c} f(x)\) (\(c\) can be infinity) and \(\lim_{x \to c} g(x)\) exist and \(a\) is a constant, then

\[
\begin{align*}
(3.1.4) & \quad \lim_{x \to c} [af(x)] = a \lim_{x \to c} f(x), \\
(3.1.5) & \quad \lim_{x \to c} [f(x) + g(x)] = \lim_{x \to c} f(x) + \lim_{x \to c} g(x), \\
(3.1.6) & \quad \lim_{x \to c} [f(x)g(x)] = \lim_{x \to c} f(x) \cdot \lim_{x \to c} g(x), \\
(3.1.7) & \quad \lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}, \quad \text{provided } \lim_{x \to c} g(x) \neq 0.
\end{align*}
\]

**Example 3.1.5.** Using \[3.1.7\] we have

\[
\lim_{x \to \infty} \frac{x^n}{x^m} = \begin{cases} 
\infty, & n > m, \\
1, & n = m, \\
0, & n < m.
\end{cases}
\]

**Exercise 3.1.6.** Show that

\[
(3.1.8) \quad \lim_{x \to \infty} \frac{p(x)}{q(x)} = \begin{cases} 
\infty, & \deg(p) > \deg(q), \\
L \neq 0, & \deg(p) = \deg(q), \\
0, & \deg(p) < \deg(q).
\end{cases}
\]

Here \(p(x)\) and \(q(x)\) are two polynomials of degrees \(\deg(p)\) and \(\deg(q)\), respectively, and \(L\) is the ratio of the coefficients of the leading terms in the numerator and denominator.

Another useful and simple fact is

\[
(3.1.9) \quad \lim_{x \to \infty} \frac{1}{e^x} = 0.
\]

3.1.2. Sandwich theorem. To calculate limits, we need the following

**Theorem 3.1.7.** If \(f(x) \leq g(x) \leq h(x)\) for all \(x\) in an open interval that contains \(c\) (except possibly at \(c\)) and

\[
\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = L,
\]

then

\[
\lim_{x \to c} g(x) = L.
\]

**Proof.** Given an \(\epsilon > 0\). By the definition, we can find a positive number \(\delta\) (depending on \(\epsilon\)) such that

\[
|f(x) - L| < \epsilon, \quad |h(x) - L| < \epsilon
\]
Figure 3.2. Sandwich theorem

Figure 3.3. $\frac{\sin x}{x}$ and the inequality $\cos x \leq \frac{\sin x}{x} \leq 1$

whenever $0 < |x - c| < \delta$. Hence

$$g(x) - L \leq h(x) - L < \epsilon, \quad g(x) - L > f(x) - L > -\epsilon;$$

thus $|g(x) - L| < \epsilon$. Consequently, $\lim_{x \to c} g(x) = L$. □

**Remark 3.1.8.** If $f(x) \leq g(x) \leq h(x)$ for all $x > x_0$, where $x_0$ is a positive fixed number, and

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} h(x) = L,$$

then

$$\lim_{x \to \infty} g(x) = L.$$

Looking at the Figure 3.3, we have

area of the triangle $OAP \leq$ area of the sector $OBP \leq$ area of the triangle $OBC$;

therefore $\frac{1}{2} \sin x \leq \frac{1}{2} x \leq \frac{1}{2} \frac{\sin x}{\cos x}$. Thus

(3.1.10) $\cos x \leq \frac{\sin x}{x} \leq 1, \quad 0 < x < \frac{\pi}{2}$.

**Theorem 3.1.9.** We have

(3.1.11) $\lim_{x \to 0} \frac{\sin x}{x} = 1$. 

Since \(1 - \cos x = 2\sin^2 x\), it follows that
\[
\frac{1 - \cos x}{x} = 2\sin^2 \frac{x}{2} = \sin \frac{x}{2} \cdot \sin \frac{x}{2}.
\]
The first factor tends to 1 while the second factor tends to 0. So the limit of \((1 - \cos x)/x\) should be 0. In general,

**Proposition 3.1.10.** If \(f(x)\) is bounded (i.e., \(|f(x)| \leq M\) for some positive constant \(M\)) and \(\lim_{x \to c} g(x) = 0\), then
\[
\lim_{x \to c} f(x)g(x) = 0.
\]

**Proof.** Given \(\epsilon > 0\). There exists a positive number \(\delta\) such that
\[
|g(x)| < \frac{\epsilon}{M}
\]
whenever \(0 < |x - c| < \delta\). Hence
\[
|f(x)g(x)| \leq M|g(x)| < M \cdot \frac{\epsilon}{M} = \epsilon
\]
whenever \(0 < |x - c| < \delta\). Thus \(\lim_{x \to c} f(x)g(x) = 0\). \(\Box\)

**Exercise 3.1.11.** Prove
\[
\lim_{x \to 0} \frac{\sin x(1 - \cos x)}{x^2} = 0, \quad \lim_{x \to 0} \frac{\sin(\pi x)}{x} = \pi.
\]

### 3.2. Continuity

A function \(f\) is said to be **continuous** at \(x = c\) if

\[(3.2.1) \quad \lim_{x \to c} f(x) = f(c).\]

To check whether a function is continuous at \(x = c\), we need to check the following three conditions:

1. \(f(x)\) is defined at \(x = c\),
2. \(\lim_{x \to c} f(x)\) exists,
3. \(\lim_{x \to c} f(x) = f(c)\).

If any of these three conditions fails, the function is **discontinuous** at \(x = c\).

**Example 3.2.1.** The function defined in Remark 3.1.1 is discontinuous.

A function \(f\) is said to be **continuous from the right** at \(x = c\) if

\[(3.2.2) \quad \lim_{x \to c^+} f(x) = f(c)\]

and **continuous from the left** at \(x = c\) if

\[(3.2.3) \quad \lim_{x \to c^-} f(x) = f(c).\]
Example 3.2.2. Consider the function

\[ f(x) = \begin{cases} 
1, & x > 0, \\
0, & x = 0, \\
-1, & x < 0. 
\end{cases} \]

Then \( \lim_{x \to 0^+} f(x) = 1 \) and \( \lim_{x \to 0^-} f(x) = -1 \).

3.2.1. Properties of continuous functions. Using the limit law, Theorem 3.1.4, we can prove

Theorem 3.2.3. Suppose that \( a \) is a constant and the functions \( f \) and \( g \) are continuous at \( x = c \). Then the following functions are continuous at \( x = c \):

1. \( a \cdot f \),
2. \( f + g \),
3. \( f \cdot g \),
4. \( f/g \) provided that \( g(c) \neq 0 \).

The following functions are continuous wherever they are defined:

1. polynomial functions
2. rational functions
3. power functions
4. trigonometric functions
5. exponential functions of the form \( a^x, a > 0 \) and \( a \neq 1 \)
6. logarithmic functions of the form \( \log_a x, a > 0 \) and \( a \neq 1 \)

Theorem 3.2.4. If \( g(x) \) is continuous at \( x = c \) with \( g(c) = L \) and \( f(x) \) is continuous at \( x = L \), then \( (f \circ g)(x) \) is continuous at \( x = c \). In particular

\[ \lim_{x \to c} (f \circ g)(x) = \lim_{x \to c} f[g(x)] = f \left[ \lim_{x \to c} g(x) \right] = f[g(c)] = (f \circ g)(c). \]  

Proof. Note that \( (f \circ g)(c) = f[g(c)] \) exists. Given an \( \epsilon > 0 \). There exists a positive number \( \delta \) such that

\[ |f(y) - f(L)| < \epsilon \]

whenever \( |y - L| < \delta \). For such \( \delta \), we can find another positive number \( \delta' \) such that

\[ |g(x) - g(c)| < \delta \]

whenever \( |x - c| < \delta' \). In particular,

\[ |(f \circ g)(x) - (f \circ g)(c)| < \epsilon \]

whenever \( |x - c| < \delta' \). Thus \( \lim_{x \to c} (f \circ g)(x) = (f \circ g)(c) \). \qed
### 3.2.2. Cauchy’s functional equations

Cauchy’s functional equation is given by

\[(3.2.5) \quad f(x + y) = f(x) + f(y), \quad x, y \in \mathbb{R}.
\]

Observe that \(f(x) = cx\) satisfies this functional equation \((3.2.5)\). Conversely, any continuous function \(f(x)\) satisfying \((3.2.5)\) is of the form \(f(x) = cx\) for some constant \(c\).

Suppose \(f(x)\) is a function (not necessarily continuous) satisfying \((3.2.5)\).

1. Letting \(x = y = 0\), we have \(f(0) = 0\).
2. Letting \(y = -x\) we have \(f(-x) = -f(x)\) for all \(x \in \mathbb{R}\).
3. For any integer \(n\) we have \(f(n) = f(1)n\).
4. For any nonzero integer \(n\) we have \(f(1/n) = f(1)/n\).
5. If \(p/q \in \mathbb{Q}\) then \(f(p/q) = pf(1/q) = f(1) \cdot \frac{p}{q}\).

Hence \(f(x) = f(1)x\) for any \(x \in \mathbb{Q}\).

Now we in addition assume \(f\) is continuous. Recall that any irrational number is the limit of a sequence of rational numbers. For example, for \(\sqrt{2} = 1.414\ldots\), we can take \(a_0 = 1, a_1 = 1.4, a_2 = 1.41, \ldots\) and then \(\lim_{n \to \infty} a_n = \sqrt{2}\). If \(x \in \mathbb{R} \setminus \mathbb{Q}\), then \(x = \lim_{n \to \infty} x_n\) for a sequence of rational numbers \(x_n\). Since \(f\) is continuous, we have

\[
f(x) = f \left( \lim_{n \to \infty} x_n \right) = \lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} f(1)x_n = f(1)x;
\]

thus \(f(x) = f(1)x\) still holds for \(x \in \mathbb{R} \setminus \mathbb{Q}\). Consequently, \(f(x) = cx\) where \(c := f(1)\).

**Proposition 3.2.5.** A continuous function \(f\) satisfying \((3.2.5)\) must be of the form \(f(x) = f(1)x\).

**Exercise 3.2.6.** Any continuous function \(f\) satisfying

\[(3.2.6) \quad f(x + y) = f(x)f(y), \quad x, y \in \mathbb{R},
\]

\(f(0) \neq 0\) and \(f(1) > 0\), must be of the form \(f(x) = a^x\) where \(a = f(1)\).

### 3.2.3. Intermediate-value theorem

An important property of continuous functions is the intermediate-value theorem. (see Figure 3.4)

**Theorem 3.2.7.** Suppose that \(f\) is continuous on the closed interval \([a, b]\). If \(M\) is any real number with \(f(a) < M < f(b)\) or \(f(b) < M < f(a)\), then there exists at least one number \(c\) on the open interval \((a, b)\) such that \(f(c) = M\).
Remark 3.2.8. Consider the function $f(x) = x$. Then for each $M \in (a, b)$, we have a unique $M \in (a, b)$ with $f(M) = M$. Hence we cannot weaken the open interval $(a, b)$ in Theorem 3.2.7 to the closed interval $[a, b]$.

Remark 3.2.9. Suppose that $f$ is continuous function defined on $[a, b]$. If $f(a) < 0 < f(b)$, then Theorem 3.2.7 shows that $f(x) = 0$ has at least one solution in $(a, b)$.
CHAPTER 4

Differentiation

In this chapter, we introduce derivatives and chain rule.

4.1. Derivatives

The derivative of a function \( f \) at \( x \), denoted \( f'(x) \), is

\[
 f'(x) := \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}
\]

provided that the limit exists. If the limit exists, then we say that \( f \) is differentiable at \( x \). (see Figure 4.1) The quotient

\[
 \frac{f(x+h) - f(x)}{h}
\]

is called the difference quotient, and we denote it by \( \frac{\Delta f}{\Delta x} \). We use the Leibniz notation

\[
 y' = \frac{dy}{dx} = f'(x) = \frac{df}{dx} = \frac{d}{dx}f(x).
\]

For particular value \( c \), we can write

\[
 \left. \frac{df}{dx} \right|_{x=c} = f'(c).
\]

4.1.1. Geometric interpretation. If the derivative of a function \( f \) exists at \( x = c \), then the tangent line at \( x = c \) is the line going through the point \( (c, f(c)) \) with slope \( f'(c) \). The equation of the tangent line is given by

\[
 y - f(c) = f'(c)(x - c).
\]

Figure 4.1. Derivative and tangent line
Remark 4.1.1. If the derivative of a function $f$ exists at $x = c$, then

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}.$$ 

In particular,

$$\lim_{h \to 0} \frac{f(c+h) - f(c-h)}{2h} = \frac{1}{2} \lim_{h \to 0} \left[ \frac{f(c+h) - f(c)}{h} + \frac{f(c-h) - f(c)}{-h} \right] = f'(c).$$

However, the converse is not true. Consider

$$f(x) = |x|, \quad c = 0.$$ 

Then

$$f'(x) = \begin{cases} 
1, & x > 0, \\
-1, & x < 0.
\end{cases}$$

Hence the derivative of $f(x)$ does not exist. But

$$\frac{f(c+h) - f(c-h)}{2h} = 0.$$

4.1.2. Differentiability and continuity. A continuous function may not be differentiable, see $f(x) = |x|$. Conversely, we have

**Theorem 4.1.2.** If $f$ is differentiable at $x = c$, then $f$ is also continuous at $x = c$.

**Proof.** From

$$f(x) - f(c) = \frac{f(x) - f(c)}{x-c} \cdot (x-c),$$

we have

$$\lim_{x \to c} [f(x) - f(c)] = \lim_{x \to c} \frac{f(x) - f(c)}{x-c} \cdot (x-c) = f'(c) \cdot 0 = 0$$

by Proposition [3.1.10].

4.2. Basic rules and chain rule

If $f(x)$ is the constant function $f(x) = a$, then

$$\frac{d}{dx} f(x) = 0.$$

4.2.1. Power rule. In general, we have

$$\frac{d}{dx} (x^n) = nx^{n-1}.$$
4.2. Basic Rules and Chain Rule

Theorem 4.2.1. Suppose $a$ is a constant and $f(x)$ and $g(x)$ are differentiable at $x$. Then

\begin{align*}
\frac{d}{dx}[af(x)] &= a \frac{d}{dx}f(x), \\
\frac{d}{dx}[f(x) + g(x)] &= \frac{d}{dx}f(x) + \frac{d}{dx}g(x).
\end{align*}

4.2.2. Product and Quotient Rules. From

\[ \frac{d}{dx}x^5 = 5x^4 \neq 6x^3 = \left( \frac{d}{dx}x^3 \right) \left( \frac{d}{dx}x^2 \right), \]

we see that $(f(x)g(x))' \neq f'(x) \cdot g'(x)$.

Theorem 4.2.2. If $f(x)$ and $g(x)$ are differentiable at $x$, then

\[ \frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x). \]

Theorem 4.2.3. If $f(x)$ and $g(x)$ are differentiable at $x$, and $g(x) \neq 0$, then

\[ \frac{d}{dx} \left[ \frac{f(x)}{g(x)} \right] = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}. \]

By Theorem 4.2.3, we have

\[ \frac{d}{dx}(x^{-n}) = -nx^{-n-1}. \]

In general,

\[ \frac{d}{dx}(x^r) = rx^{r-1}. \]

4.2.3. Chain Rule. To find the derivative of composite functions, we need the chain rule.

Theorem 4.2.4. (Chain rule) If $g$ is differentiable at $x$ and $f$ is differentiable at $y = g(x)$, then the composite function $(f \circ g)(x) = f[g(x)]$ is differentiable at $x$, and

\[ (f \circ g)'(x) = f'[g(x)]g'(x). \]
Example 4.2.5. We have

\[ \frac{d}{dx} \sin x = \cos x, \quad \frac{d}{dx} \cos x = -\sin x, \]
\[ \frac{d}{dx} \tan x = \sec^2 x, \quad \frac{d}{dx} \cot x = -\csc^2 x, \]
\[ \frac{d}{dx} \sec x = \sec x \cdot \tan x, \quad \frac{d}{dx} \csc x = -\csc x \cdot \cot x, \]
\[ \frac{d}{dx} e^x = e^x, \]
\[ \frac{d}{dx} a^x = a^x \cdot \ln a, \]
\[ \frac{d}{dx} \ln x = \frac{1}{x}, \]
\[ \frac{d}{dx} \log_a x = \frac{1}{(\ln a)x}. \]

Theorem 4.2.6. (Derivative of the inverse function) If \( f(x) \) is one-to-one and differentiable with inverse function \( f^{-1}(x) \) and \( f'(f^{-1}(x)) \neq 0 \), then

\[ \frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}. \]

For example,

\[ \frac{d}{dx} \arcsin x = \frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}, \]
\[ \frac{d}{dx} \arctan x = \frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}. \]

4.2.4. Linear approximation and error propagation. Assume that \( y = f(x) \) is differentiable at \( x = a \); then

\[ L(x) := f(a) + f'(a)(x - a) \]

is the tangent line approximation or the linearization of \( f \) at \( x = a \). If \( |x - a| \) is sufficiently small, we have

\[ f(x) \approx L(x) = f(a) + f'(a)(x - a). \]
CHAPTER 5

Applications of differentiation

In this chapter, we discuss the behavior of functions by using differentiations.

5.1. Extrema and the mean-value theorem

This section defines a extrema, gives conditions that guarantee extrema (via the extreme-value theorem), and provides a characterization of extrema (Fermat’s theorem).

5.1.1. Extreme-value theorem. Let \( f \) be a function defined on the set \( D \) that contains the number \( c \). Then \( f \) has a global (or absolute) maximum at \( x = c \) if

\[
    f(c) \geq f(x)
\]

for all \( x \in D \), and \( f \) has a global (or absolute) minimum at \( x = c \) if

\[
    f(c) \leq f(x)
\]

for all \( x \in D \). (see Figure 5.1)

**Theorem 5.1.1.** (Extreme-value theorem) If \( f \) is continuous on a closed interval \([a, b]\), \( -\infty < a < b < \infty \), then \( f \) has a global maximum and a global minimum in \([a, b]\).

The proof of Theorem 5.1.1 is beyond the scope of this course and will be omitted.

![Figure 5.1. Absolute maximum and minimum](image)
5. APPLICATIONS OF DIFFERENTIATION

Remark 5.1.2. (1) Theorem 5.1.1 only tells the existence of global (or absolute) extrema (global maximum or global minimum). We can find functions that have more than one global extrema; for example,

\[ f(x) = \begin{cases} 
  1, & 0 \leq x \leq 1, \\
  2 - x, & 1 \leq x \leq 2. 
\end{cases} \]

(2) If \( f \) is continuous on an open interval \((a, b)\), then \( f \) may not have a global extrema. For example, \( f(x) = \frac{1}{x} \) with \( x \in (0, 1) \).

5.1.2. Local extrema. A function \( f \) defined on a set \( D \) has a local (or relative) maximum at a point \( c \) if there exists a \( \delta > 0 \) such that

\[ f(c) \geq f(x) \]

for all \( x \in (c - \delta, c + \delta) \cap D \). A function \( f \) defined on a set \( D \) has a local (or relative) minimum at a point \( c \) if there exists a \( \delta > 0 \) such that

\[ f(c) \leq f(x) \]

for all \( x \in (c - \delta, c + \delta) \cap D \). Local maxima and local minima are collectively called local (or relative) extrema.

Theorem 5.1.3. (Fermat’s theorem) If \( f \) has a local extremum at an interior point \( c \) and \( f'(c) \) exists, then \( f'(c) = 0 \).

Proof. We only prove the case that \( f \) has a local minimum at \( c \). Then there exists a \( \delta > 0 \) such that

\[ f(c) \leq f(x) \]

for all \( x \in (c - \delta, c + \delta) \cap D \). For \( x \in (c, c + \delta) \cap D \), we have

\[ \frac{f(c) - f(x)}{x - c} \leq 0 \quad \text{or} \quad \frac{f(x) - f(c)}{x - c} \geq 0; \]

thus

\[ \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} \geq 0. \]

Similarly, we can show that

\[ \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \leq 0. \]
5.1. EXTREMA AND THE MEAN-VALUE THEOREM

Figure 5.3. Rolle’s theorem

Since \( f'(c) \) exists, we must have \( f'(c) = 0 \).

However, the converse of Theorem 5.1.3 is not true. For example, consider the function \( f(x) = x, \ x \in [-1, 1] \); we see that \( f'(0) = 0 \) but \( f \) has no local extremums in \((-1, 1)\).

5.1.3. Mean-value theorem. For the function \( f(x) = x^2, \ x \in [-1, 1] \), we easily see that \( f'(0) = 0 \). This result also holds for general differentiable functions.

**Theorem 5.1.4.** (Rolle’s theorem) If \( f \) is continuous on the closed interval \([a, b]\) and differentiable on the open interval \((a, b)\), and if \( f(a) = f(b) \), then there exists a number \( c \in (a, b) \) such that \( f'(c) = 0 \).

**Proof.** If \( f \) is a constant function, then the assertion is obvious. Now we suppose that \( f \) is nonconstant. Since \( f \) is continuous on the closed interval \([a, b]\), according to Theorem 5.1.1 \( f \) has a global maximum and a global minimum in \([a, b]\). For example, we assume that \( f \) has a global maximum at \( x_0 \in [a, b] \). If \( x_0 = a \), then

\[
f(a) = f(x_0) \geq f(a) \implies f(x_0) = f(a) = f(b).
\]

Hence \( f \) has a global maximum at \( a \) and \( b \). By Theorem 5.1.1 again, \( f \) has a global minimum at \( c \in [a, b] \). Since \( f \) is nonconstant, we must have \( c \in (a, b) \). By Theorem 5.1.3 we get \( f'(c) = 0 \).

**Theorem 5.1.5.** (Mean-value theorem (MVT)) If \( f \) is continuous on the closed interval \([a, b]\) and differentiable on the open interval \((a, b)\), then there exists at least one number \( c \in (a, b) \) such that

\[
\frac{f(b) - f(a)}{b - a} = f'(c).
\]
PROOF. Define
\[ F(x) := f(x) - \frac{f(b) - f(a)}{b-a}(x-a), \quad x \in [a,b]. \]
Then \( F \) is continuous on \([a,b]\) and differentiable on \((a,b)\); furthermore \( F(a) = f(a) = F(b) \). By Theorem 5.1.4, \( F'(c) = 0 \) for some \( c \in (a,b) \). Since
\[ F'(x) = f'(x) - \frac{f(b) - f(a)}{b-a}, \]
it follows that \( 0 = F'(c) = f'(c) - \frac{f(b) - f(a)}{b-a} \).
}\]}

**Corollary 5.1.6.** If \( f \) is continuous on the closed interval \([a,b]\) and differentiable on the open interval \((a,b)\) such that
\[ m \leq f'(x) \leq M \]
for all \( x \in (a,b) \), then
\[ m(b-a) \leq f(b) - f(a) \leq M(b-a). \]

**Exercise 5.1.7.** Prove Corollary 5.1.6.

If \( f \) is a constant, then \( f'(x) = 0 \) for all \( x \in \mathbb{R} \). Conversely we have

**Corollary 5.1.8.** If \( f \) is continuous on the closed interval \([a,b]\) and differentiable on the open interval \((a,b)\), with \( f'(x) = 0 \) for all \( x \in (a,b) \), then \( f \) is constant on \([a,b]\).

**Exercise 5.1.9.** Prove Corollary 5.1.8.
Exercise 5.1.10. If $f$ is continuous on the closed interval $[a,b]$ and differentiable on the open interval $(a,b)$, then there exists a number $c \in (a,b)$ such that
\[
f'(c) = 0 \quad \text{or} \quad f(c) = \frac{f(a) + f(b)}{2}.
\]
(Hint: Consider the function $F(x) = [f(x) - f(a)][f(x) - f(b)]$.)

5.2. Monotonicity, concavity, and inflection points

This section discusses the important concepts of monotonicity—whether a function is decreasing or increasing—and concavity—whether a function bends upwards or downward.

5.2.1. Monotonicity. A function $f$ defined on an interval $I$ is called (strictly) increasing on $I$ if
\[f(x_1) < f(x_2) \quad \text{whenever} \quad x_1 < x_2 \in I,
\]
and is called (strictly) decreasing on $I$ if
\[f(x_1) > f(x_2) \quad \text{whenever} \quad x_1 < x_2 \in I.
\]
An increasing or decreasing function is called monotonic.

Remark 5.2.1. The monotonicity of a function $f$ defined on the interval $I$. For example, consider the function $f(x) = |x|$, $x \in [-1,1]$. Then $f$ is increasing on $[0,1]$ while is decreasing on $[-1,0]$.

Proposition 5.2.2. (First-derivative test for monotonicity) Suppose $f$ is continuous on $[a,b]$ and differentiable on $(a,b)$.

(a) If $f'(x) > 0$ for all $x \in (a,b)$, then $f$ is increasing on $[a,b]$.
(b) If $f'(x) < 0$ for all $x \in (a,b)$, then $f$ is decreasing on $[a,b]$.

Proof. We only prove part (a). Fixed $x_1 < x_2$ in $(a,b)$. By Theorem 5.1.5 we have
\[
\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c)
\]
for some $c \in (x_1, x_2)$. Since $f'(x) > 0$ for all $x \in (a,b)$, we get $f(x_2) > f(x_1)$.

5.2.2. Concavity. A differentiable function $f(x)$ is concave up on an interval $I$ if the first derivative $f'(x)$ is an increasing function on $I$. $f(x)$ is concave down on an interval $I$ if the first derivative $f'(x)$ is a decreasing function on $I$.

Theorem 5.2.3. (Second-derivative test for concavity) Suppose $f$ is twice differentiable on an open interval $I$.

(a) If $f''(x) > 0$ for all $x \in I$, then $f$ is concave up on $I$.
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Figure 5.5. Inflection points

(b) If \( f''(x) < 0 \) for all \( x \in I \), then \( f \) is concave down on \( I \).

Exercise 5.2.4. Prove Theorem 5.2.3.

5.2.3. Extrema. A continuous function has a local minimum at \( c \) if the function is decreasing to the left of \( c \) and increasing to the right of \( c \). A continuous function has a local maximum at \( c \) if the function is increasing to the left of \( c \) and decreasing to the right of \( c \).

Theorem 5.2.5. (Second-derivative test for local extrema) Suppose \( f \) is twice differentiable on an open interval containing \( c \).

(a) If \( f'(c) = 0 \) and \( f''(c) < 0 \), then \( f \) has a local maximum at \( x = c \).
(b) If \( f'(c) = 0 \) and \( f''(c) > 0 \), then \( f \) has a local minimum at \( x = c \).

5.2.4. Inflection points. Inflection points are points where the concavity of a function changes—that is, where the function changes from concave up to concave down or from concave down to concave up.

Example 5.2.6. \( 0 \) is the only one inflection point of \( f(x) = x^3, \ x \in \mathbb{R} \).

If \( f(x) \) is twice differentiable and has an inflection point at \( x = c \), then \( f''(c) = 0 \). (see Figure 5.5)

5.2.5. Asymptotes. A line \( y = b \) is a horizontal asymptote if either

\[
\lim_{x \to -\infty} f(x) = b \quad \text{or} \quad \lim_{x \to \infty} f(x) = b.
\]
A line $x = c$ is a **vertical asymptote** if

$$
\lim_{x \to c^+} f(x) = +\infty \quad \text{or} \quad \lim_{x \to c^+} f(x) = -\infty
$$

or

$$
\lim_{x \to c^-} f(x) = +\infty \quad \text{or} \quad \lim_{x \to c^-} f(x) = -\infty.
$$

### 5.3. L'Hospital’s rule

The rule is named after 17th-century French mathematician Guillaume de l'Hôpital, who published the rule in his book *Analyse des Infiniment Petits pour l’Intelligence des Lignes Courbes* (literal translation: Analysis of the Infinitely Small for the Understanding of Curved Lines) (1696), the first textbook on differential calculus. However, it is believed that the rule was discovered by the Swiss mathematician Johann Bernoulli.

**Theorem 5.3.1.** (L’Hôpital’s rule) Suppose that $f$ and $g$ are differentiable functions and that

$$
\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0
$$

or

$$
\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = \infty.
$$

If

$$
\lim_{x \to a} \frac{f'(x)}{g'(x)} = L
$$

then

$$
\lim_{x \to a} \frac{f(x)}{g(x)} = L.
$$
5. APPLICATIONS OF DIFFERENTIATION

Proof. We treat with the first case. Define

\[ f(a) := \lim_{x \to a} f(x), \quad g(a) := \lim_{x \to a} g(x). \]

Then \( f(x) \) and \( g(x) \) are continuous at \( a \). Fixed \( x > a \). From Theorem 5.1.5 there exist \( x_1, x_2 \in (a, x) \) such that

\[ \frac{f(x) - f(a)}{x - a} = f'(x_1), \quad \frac{g(x) - g(a)}{x - a} = g'(x_2). \]

Observe that \( x_1, x_2 \to a^+ \) as \( x \to a^+ \). Hence

\[ \lim_{x \to a^+} \frac{f(x)}{g(x)} = \lim_{x \to a^+} \frac{f'(x_1)(x - a)}{g'(x_2)(x - a)} = \lim_{x \to a^+} \frac{f'(x_1)}{g'(x_2)} = L. \]

Similarly, we can show \( \lim_{x \to a^-} \frac{f(x)}{g(x)} = L \). \( \square \)

Example 5.3.2. (1) \((\infty/\infty)\) Evaluate

\[ \lim_{x \to \infty} \frac{\ln x}{x}. \]

(2) \((0 \cdot \infty)\) Evaluate

\[ \lim_{x \to 0^+} x \ln x. \]

(3) \((0/0)\) Evaluate

\[ \lim_{x \to 0} \frac{\sin x}{x}. \]

(4) \((\infty - \infty)\) Evaluate

\[ \lim_{x \to \infty} (x - \sqrt{x^2 + x}). \]

(5) \((0^0)\) Evaluate

\[ \lim_{x \to 0^+} x^x. \]

Proof. For (1),

\[ \lim_{x \to \infty} \frac{\ln x}{x} = \lim_{x \to \infty} \frac{1/x}{1} = 0. \]

For (2), where \( y = 1/x \),

\[ \lim_{x \to 0^+} x \ln x = \lim_{y \to \infty} \frac{-\ln y}{y} = 0. \]

For (3),

\[ \lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x}{1} = 1. \]

For (4),

\[ \lim_{x \to \infty} (x - \sqrt{x^2 + x}) = \lim_{x \to \infty} \frac{-x}{x + \sqrt{x^2 + x}} = \lim_{x \to \infty} \frac{-1}{1 + \sqrt{1 + \frac{1}{x}}} = \frac{-1}{2}. \]

For (5),

\[ \lim_{x \to 0^+} x^x = \lim_{x \to 0^+} e^{\ln x} = \lim_{x \to 0^+} e^0 = 1 \]

by (2). \( \square \)
5.4. Antiderivative

A function $F$ is called an antiderivative of $f$ on an interval $I$ if $F'(x) = f(x)$ for all $x \in I$. (see Figure 5.7)

**Corollary 5.4.1.** If $F(x)$ and $G(x)$ are antiderivatives of the continuous function $f(x)$ on an interval $I$, then there exists a constant $C$ such that

$$G(x) = F(x) + C$$

for all $x \in I$.

**Proof.** By definition, we have $(G - F)'(x) = f(x) - f(x) = 0$ for all $x \in I$. According to Corollary 5.1.8, $G(x) - F(x) = C$ is constant. □
CHAPTER 6

Integrations

In this chapter, we give the definition on integrals and introduce the fundamental theorem of calculus.

6.1. Definite integrals

Let $f$ be a function on the interval $[a, b]$. We partition $[a, b]$ into $n$ subintervals by choosing $n - 1$ numbers $x_1, \ldots, x_{n-1}$ in $(a, b)$ such that

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$  

The $n$ subintervals $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$ form a partition of $[a, b]$, which we denote by $\mathcal{P} = [x_0, x_1, x_2, \ldots, x_n]$. The partition $\mathcal{P}$ depends on the number of subintervals $n$ and on the choice of points $x_0, x_1, \ldots, x_n$. The length of the $k$th subinterval $[x_{k-1}, x_k]$ is denoted by $\Delta x_k$. The length of the longest subinterval is called the norm of $\mathcal{P}$ and is denoted by $||\mathcal{P}||$; thus

$$||\mathcal{P}|| = \max\{\Delta x_1, \Delta x_2, \ldots, \Delta x_n\}.$$  

In each subinterval $[x_{k-1}, x_k]$, we choose a point $c_k$ and construct a rectangle with base $\Delta x_k$ and height $|f(c_k)|$. (see Figure 6.1) The sum of these products is denoted by $S_P$; that is,

$$(6.1.1) \quad S_P := \sum_{k=1}^{n} f(c_k) \Delta x_k.$$  

The value of the sum depends on the choice of the partition $\mathcal{P}$ and the choice of the points $c_k \in [x_{k-1}, x_k]$ and is called a Riemann sum for $f$ on $[a, b]$.

![Figure 6.1. Riemann sum](image-url)
### 6.1.1. Riemann integrals

Let $f$ be a function on $[a, b]$.

**Definition 6.1.1.** Let $\mathcal{P} = [x_0, x_1, x_2, \ldots, x_n]$, $n = 1, 2, \ldots$, be a sequence of partition of $[a, b]$ with $||\mathcal{P}|| \to 0$. Set $\Delta x_k = x_k - x_{k-1}$ and $c_k \in [x_{k-1}, x_k]$. The **definite integral** of $f$ from $a$ to $b$ is

$$
\int_a^b f(x) \, dx := \lim_{||\mathcal{P}|| \to 0} \sum_{k=1}^n f(c_k) \Delta x_k
$$

if the limit exists, in which case $f$ is said to be (Riemann) **integrable** on the interval $[a, b]$. Moreover, $f(x)$ is called the **integrand**.

The phase “if the limit exists” means, in particular, that the value of $\lim_{||\mathcal{P}|| \to 0} S_{\mathcal{P}}$ does not depend on how we choose the partitions and the points $c_k \in [x_{k-1}, x_k]$ as we take the limit.

**Theorem 6.1.2.** All continuous functions are Riemann integrals; that is, if $f(x)$ is continuous on $[a, b]$ then

$$
\int_a^b f(x) \, dx
$$

exists.

**Remark 6.1.3.** (1) We can find a function that is Riemann integral but not continuous. For example,

$$
f(x) = \begin{cases} 
|x|, & x \in [-1, 0) \cup (0, 1], \\
1, & x = 0.
\end{cases}
$$

(2) Consider the function

$$
f(x) = \begin{cases} 
x, & x \in \mathbb{Q} \cap [0, 1], \\
0, & x \in (\mathbb{R}/\mathbb{Q}) \cap [0, 1].
\end{cases}
$$

It can be showed that the function $f(x)$ is not Riemann integrable.

(3) Another important concept is **Lebesgue integrals**, which can be think of the Riemann integral in terms of the partition of $y$-axis. A Lebesgue integrable function is Riemann integrable, but the converse is not true. For example, the Lebesgue integral of $f(x)$ defined in (2) equals 0.

### 6.1.2. Properties of the Riemann integrals

In this subsection, we collect important properties that will help us to evaluate definite integrals.

**Proposition 6.1.4.** If $f$ is integrable over $[a, b]$, then

$$
\int_a^a f(x) \, dx = 0, \quad \int_b^a f(x) \, dx = -\int_a^b f(x) \, dx.
$$
6.2. THE FUNDAMENTAL THEOREM OF CALCULUS

Figure 6.2. Riemann integrals and Lebesgue integrals

Proposition 6.1.5. Assume that \( f \) and \( g \) are integrable over \([a, b]\).

\[
\int_a^b kf(x) \, dx = k \int_a^b f(x) \, dx, \quad k \text{ is constant},
\]

\[
\int_a^b [f(x) + g(x)] \, dx = \int_a^b f(x) \, dx + \int_a^b g(x) \, dx,
\]

\[
\int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx,
\]

where \( c \) is any interior point of \([a, b]\). Moreover

(i) If \( f(x) \geq 0 \) on \([a, b]\), then

\[
\int_a^b f(x) \, dx \geq 0.
\]

(ii) If \( f(x) \leq g(x) \) on \([a, b]\), then

\[
\int_a^b f(x) \, dx \leq \int_a^b g(x) \, dx.
\]

(iii) If \( m \leq f(x) \leq M \) on \([a, b]\), then

\[
m(b - a) \leq \int_a^b f(x) \, dx \leq M(b - a).
\]

6.2. The fundamental theorem of calculus

Let \( f(x) \) be a continuous function on \([a, b]\) and let

\[
F(x) := \int_a^x f(u) \, du.
\]

Then

\[
F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h \to 0} \frac{1}{h} \left[ \int_a^{x+h} f(u) \, du - \int_a^x f(u) \, du \right]
\]

\[
= \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(u) \, du.
\]
6.2.1. **The fundamental theorem of calculus I.** The fundamental theorem for calculus tells that $F'(x) = f(x)$.

**Theorem 6.2.1.** (The fundamental theorem of calculus I) If $f$ is continuous on $[a,b]$, then the function $F$ defined by

\[
(6.2.1) \quad F(x) = \int_a^x f(u) \, du, \quad a \leq x \leq b
\]

is continuous on $[a,b]$ and differentiable on $(a,b)$ with

\[
(6.2.2) \quad F'(x) = f(x).
\]

**Proof.** Since $f$ is continuous on $[a,b]$, by Theorem 5.1.1, $m \leq f(x) \leq M$ on $[a,b]$ for some constants $m$ and $M$. Consequently, by (6.1.9),

\[
m \leq \frac{1}{h} \int_x^{x+h} f(u) \, du \leq M.
\]

Define

\[
I := \frac{1}{h} \int_x^{x+h} f(u) \, du, \quad h \in [0, b-a].
\]

Then $m \leq I \leq M$. By Theorem 3.2.7, we have

\[
f(c_h) = I = \frac{1}{h} \int_h^{x+h} f(u) \, du
\]

for some $c_h \in [x, x+h]$. Hence

\[
f(x) = f \left( \lim_{h \to 0} c_h \right) = \lim_{h \to 0} f(c_h) = \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(u) \, du.
\]

Thus $F'(x) = \lim_{h \to 0} I = f(x)$. \[\square\]

**Proposition 6.2.2.** If $g(x)$ and $h(x)$ are differentiable functions and $f(u)$ is continuous for $u$ between $g(x)$ and $h(x)$, then

\[
(6.2.3) \quad \frac{d}{dx} \int_{g(x)}^{h(x)} f(u) \, du = f[h(x)]h'(x) - f[g(x)]g'(x).
\]

**Proof.** By Propositions 6.1.4 and 6.1.5 it suffices to prove

\[
\frac{d}{dx} \int_{g(x)}^{h(x)} f(u) \, du = f[h(x)]h'(x).
\]

Introduce a new variable

\[
v := h(x).
\]

Then

\[
\frac{d}{dx} \int_0^{h(x)} f(u) \, du = \frac{d}{dv} \int_0^v f(u) \, du \cdot \frac{dv}{dx} = h'(x)f[h(x)]
\]

by Theorem 6.2.1. \[\square\]
6.2. The Fundamental Theorem of Calculus

6.2.2. Antiderivatives and indefinite integrals. Let \( f \) be a continuous function on an interval \( I \) and choose two fixed numbers \( a, b \in I \). Consider

\[
F(x) := \int_a^x f(u) \, du, \quad G(x) := \int_b^x f(u) \, du.
\]

Then \( F'(x) = f(x) = G'(x) \) by Theorem \[6.2.1\] According to Corollary \[5.4\], we have

\[
F(x) = G(x) + C
\]

for some constant \( C \); in particular,

\[
C = F(x) - G(x) = \int_a^x f(u) \, du - \int_b^x f(u) \, du = \int_a^b f(u) \, du.
\]

The above discussion tells us that any antiderivative can be obtained from a special one by adding a constant. We denote by

\[
\int f(x) \, dx
\]

any antiderivative of \( f \), called an \textbf{indefinite integral}. Note that

\[
\int f(x) \, dx = C + \int_a^x f(u) \, du
\]

for some constant \( C \).

For example

\[
\int x^3 \, dx = \frac{1}{4} x^4 + C.
\]

A collection of indefinite integrals:

\begin{align*}
(6.2.4) \quad & \int x^a \, dx = \frac{x^{a+1}}{a+1} + C, \quad a \neq -1, \\
(6.2.5) \quad & \int \frac{1}{x} \, dx = \ln |x| + C, \\
(6.2.6) \quad & \int e^x \, dx = e^x + C, \quad \int a^x \, dx = \frac{a^x}{\ln a} + C, \\
(6.2.7) \quad & \int \cos x \, dx = \sin x + C, \quad \int \sin x \, dx = -\cos x + C, \\
(6.2.8) \quad & \int \sec^2 x \, dx = \tan x + C, \quad \int \csc^2 x \, dx = -\cot x + C, \\
(6.2.9) \quad & \int \sec x \tan x \, dx = \sec x + C, \quad \int \csc x \cot x \, dx = -\csc x + C, \\
(6.2.10) \quad & \int \tan x \, dx = \ln |\sec x| + C, \quad \int \cot x \, dx = -\ln |\csc x| + C, \\
(6.2.11) \quad & \int \frac{1}{1 + x^2} \, dx = \tan^{-1} x + C, \quad \int \frac{1}{\sqrt{1-x^2}} \, dx = \sin^{-1} x + C.
\end{align*}
6. integrations

6.2.3. The fundamental theorem of calculus II. Let \( f \) be a continuous on \([a, b]\) and set

\[
G(x) = \int_{a}^{x} f(u) \, du, \quad x \in [a, b].
\]

Then \( G'(x) = f(x) \) and \( G(x) \) is an antiderivative of \( f(x) \). If \( F(x) \) is any antiderivative of \( f(x) \), then

\[
G(x) = F(x) + C, \quad x \in [a, b],
\]

by Theorem \(6.2.1\). In particular,

\[
\int_{a}^{b} f(u) \, du = G(b) = F(b) + C = F(b) + [G(a) - F(a)] = F(b) - F(a).
\]

Theorem 6.2.3. (The fundamental theorem of calculus II) If \( f \) is continuous on \([a, b]\), then

\[
(6.2.12) \quad \int_{a}^{b} f(x) \, dx = F(b) - F(a),
\]

where \( F(x) \) is an antiderivative of \( f(x) \).

6.3. Applications of integration

If \( f \) is a nonnegative, continuous function on \([a, b]\), then

\[
(6.3.1) \quad A := \int_{a}^{b} f(x) \, dx
\]

represents the area of the region bounded by the graph of \( f(x) \) between \( a \) and \( b \), the vertical lines \( x = a \) and \( x = b \), and the \( x \)-axis between \( a \) and \( b \). (see Figure 6.3)

6.3.1. Areas. From Figure 6.4 we have
If \( f \) and \( g \) are continuous on \([ a, b]\), with \( f(x) \geq g(x) \) for all \( x \in [a, b] \), then the area of the region between the curves \( y = f(x) \) and \( y = g(x) \) from \( a \) to \( b \) is equal to

\[
\text{(6.3.2) Area} = \int_a^b [f(x) - g(x)] \, dx.
\]

Suppose that a region is bounded by \( x = f(y) \) and \( x = g(y) \), with \( g(y) \leq f(y) \) for \( x \leq y \leq d \); that is, \( f(y) \) is to the right of \( g(y) \) for all \( y \in [c, d] \). Then the area of the shaded region is given by

\[
\text{(6.3.3) Area} = \int_c^d [f(y) - g(y)] \, dy.
\]

### 6.3.2. Average values.

If \( f(x) \) is a continuous function on \([a, b] \). The average value of \( f \) on \([a, b] \) is

\[
\text{(6.3.4) } f_{\text{avg}} := \frac{1}{b-a} \int_a^b f(x) \, dx.
\]

**Theorem 6.3.1.** (Mean-value theorem for definite integrals) If \( f(x) \) is a continuous function on \([a, b]\), then

\[
\text{(6.3.5) } f(c) = f_{\text{avg}}
\]

for some number \( c \in (a, b) \).

**Proof.** Since \( f \) is continuous on \([a, b]\), by Theorem 5.1.1, \( m \leq f(x) \leq M \) on \([a, b]\) for some constants \( m \) and \( M \). Consequently, by (6.1.9),

\[
m \leq \frac{1}{b-a} \int_a^b f(x) \, dx \leq M.
\]

By theorem 3.2.7 the number \( c \in (a, b) \) is obtained. \( \square \)
6.3.3. Rectification of curves. Let \( y = f(x), a \leq x \leq b \), be a differentiable function whose derivative is continuous on \([a, b]\). (see Figure 6.5) We partition the interval \([a, b]\) into subintervals by using the partition \( \mathcal{P} = [x_0, x_1, \ldots, x_n] \), where \( a = x_0 < x_1 < x_2 < \cdots < x_n = b \), and approximate the curve by a polygon that consists of the straight-line segments connecting neighboring points on the curve.

Set
\[
\Delta x_k := x_k - x_{k-1}, \quad \Delta y_k := y_k - y_{k-1}.
\]

Then the length of the line segment is given by
\[
\sqrt{\Delta x_k^2 + \Delta y_k^2}.
\]

Using the partition \( \mathcal{P} \), we find that the length of the polygon is
\[
L_P = \sum_{i=1}^{n} \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}.
\]

By Theorem 5.1.5
\[
f'(c_k) = \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} = \frac{\Delta y_k}{\Delta x_k}
\]
for some number \( c_k \in (x_{k-1}, x_k) \). Consequently,
\[
L_P = \sum_{i=1}^{n} \sqrt{1 + [f'(c_k)]^2} \Delta x_k
\]

and the length of the curve \( y = f(x) \) is
\[
L := \lim_{||\mathcal{P}|| \to 0} L_P = \int_{a}^{b} \sqrt{1 + [f'(x)]^2} \, dx.
\]

If \( f(x) \) is differentiable on \((a, b)\) and \( f'(x) \) is continuous on \([a, b]\), then the length of the curve \( y = f(x) \) from \( a \) to \( b \) is given by
\[
(6.3.6) \quad L = \int_{a}^{b} \sqrt{1 + [f'(x)]^2} \, dx.
\]
6.3. APPLICATIONS OF INTEGRATION

Since \( \frac{dy}{dx} = f'(x) \), it follows from (6.3.6) that

(6.3.7) \[ L = \int_a^b \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx = \int_a^b \sqrt{(dx)^2 + (dy)^2}. \]

We call the expression \( \sqrt{(dx)^2 + (dy)^2} \) the **arc length differential** and denote it by \( ds \).

6.3.4. **Integral inequalities.** For any real numbers \( a_1, \cdots, a_n \) and \( b_1, \cdots, b_n \), a basic inequality is

(6.3.8) \[ \left( \sum_{i=1}^n a_i b_i \right)^2 \leq \left( \sum_{i=1}^n a_i^2 \right) \left( \sum_{i=1}^n b_i^2 \right) \]

called the **Cauchy-Schwarz inequality**.

**Exercise 6.3.2.** Verify (6.3.8) for \( n = 2 \).

Another basic inequality is the **arithmetic-geometric inequality**

(6.3.9) \[ \frac{1}{\frac{a}{b}} \leq \sqrt{ab} \leq \frac{a+b}{2} \leq \sqrt{\frac{a^2 + b^2}{2}}, \quad a, b > 0. \]

**Exercise 6.3.3.** Use (6.3.9) to prove

(6.3.10) \[ ab \leq |ab| \leq \varepsilon a^2 + \frac{1}{4\varepsilon} b^2 \]

for any \( a, b \geq 0 \) and \( \varepsilon > 0 \).

An integral version of (6.3.8) is the following

**Theorem 6.3.4.** (Hölder’s inequality) Suppose that \( p \) and \( q \) are two real numbers, \( 1 \leq p, q \leq \infty \) with \( \frac{1}{p} + \frac{1}{q} = 1 \), and \( f \) and \( g \) are two Riemann integrable functions on \([a, b]\). Then the functions \(|f(x)|^p \) and \(|g(x)|^q \) are also Riemann integrable and

(6.3.11) \[ \left| \int_a^b f(x)g(x) \, dx \right| \leq \left( \int_a^b |f(x)|^p \, dx \right)^{1/p} \left( \int_a^b |g(x)|^q \, dx \right)^{1/q}. \]

An integral version of (6.3.9) is the following
Theorem 6.3.5. (Minkowski inequality) Suppose that $p \geq 1$ is a real number and $f(x), g(x)$ are Riemann integrable functions on $[a, b]$. Then $|f(x)|^p, |g(x)|^p$ and $|f + g|^p$ are also Riemann integrable and

\[
\left( \int_a^b |f(x) + g(x)|^p \, dx \right)^{\frac{1}{p}} \leq \left( \int_a^b |f(x)|^p \, dx \right)^{\frac{1}{p}} + \left( \int_a^b |g(x)|^p \, dx \right)^{\frac{1}{p}}.
\]

The third basic inequality is Young’s inequality

\[
ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \quad a, b \geq 0, \quad \frac{1}{p} + \frac{1}{q} = 1, \quad p, q > 1.
\]

**Proof.** We may assume that $b \leq a^{p-1}$. Consider the function $f(x) := x^{p-1}$ with $x \in [0, a]$. Then the inverse function $g(x)$ of $f(x)$ is equal to $g(x) = x^{1/(p-1)}$. By Figure 6.6, we see that the area of the red region is given by

\[
\int_0^a f(x) \, dx = \int_0^a x^{p-1} \, dx = \frac{a^p}{p},
\]

while the area of the yellow region is given by

\[
\int_0^b f^{-1}(y) \, dy = \int_0^b g(x) \, dx = \int_0^b x^{\frac{1}{p-1}} \, dx = \frac{p-1}{p} b^{\frac{p}{p-1}} = \frac{b^q}{q}.
\]

Since we assume that $b \leq a^{p-1}$, from Figure 6.6 we have

\[
ab \leq \text{area of the red region} + \text{area of the yellow region} = \frac{a^p}{p} + \frac{b^q}{q}.
\]

Similarly, we can deal with the case that $b \geq a^{p-1}$.

**Exercise 6.3.6.** Verify (6.3.13) when $b \geq a^{p-1}$.
CHAPTER 7

Integration techniques

In this chapter we introduce and discuss several techniques on computing (indefinite, definite, improper) integrals. Most methods are presented in the textbook, however, other approaches will be illustrated by considering practical examples which mainly come from mathematics research.

An elementary rule or axiom is that an indefinite integral does not depend on the variables. For example, the following two indefinite integrals

\[ \int u \, du, \quad \int y \, dy \]

are the same. The above rule is the original stage of the following several methods.

7.1. Basic rules

If we consider the function \( y = x^2 \), then

\[ \frac{dy}{dx} = 2x, \quad dy = 2x \, dx, \]

and hence

\[ \int 2x^3 \, dx = \int x^2 \, 2x \, dx = \int y \, dy. \]

If we consider another function \( u = x^2 \), then the same argument shows that

\[ \int u \, du = \int x^2 \, 2x \, dx = \int 2x^3 \, dx. \]

Hence

\[ \int u \, du = \int 2x^3 \, dx = \int y \, dy. \]

7.1.1. Substitution rule for indefinite integrals. In general, we can prove

**Proposition 7.1.1.** If \( u = g(x) \) for some \( C^1 \)-function \( g \), then

\[ f[g(x)]g'(x) \, dx = \int f(u) \, du. \]

We say a function \( g(x) \) is \( C^1 \), if the derivative of \( g(x) \) exists and is continuous.
PROOF. Let \( F(x) \) be an anti-derivative of \( f(x) \), i.e., \( F'(x) = f(x) \). Then

\[
\frac{d}{dx} F[g(x)] = F'[g(x)] g'(x) = f[g(x)] g'(x)
\]

by the chain rule. Thus \( F[g(x)] \) is an anti-derivative of \( f[g(x)] g'(x) \) and hence

\[
\int f[g(x)] g'(x) \, dx = F[g(x)] + C = F(u) + C = \int f(u) \, du
\]

showing the formula (7.1.1).

□

Example 7.1.2. Evaluate

\[
\int 4x \sqrt{x^2 + 1} \, dx, \quad \int 4(x^3 + x) \sqrt{x^2 + 1} \, dx, \quad \int x \sqrt{2x - 1} \, dx.
\]

PROOF. For the first indefinite integral, we set \( u = x^2 + 1 \) and hence

\[
\int 2 \sqrt{x^2 + 1} \, dx = \int 2u^{1/2} \, du = \frac{4}{3} u^{3/2} + C = \frac{4}{3} (x^2 + 1)^{3/2} + C.
\]

Similarly,

\[
\int 4(x^3 + x) \sqrt{x^2 + 1} \, dx = \int 2(x^2 + 1) \sqrt{x^2 + 1} \, dx = \int 2u^{3/2} \, du = \frac{4}{5} u^{5/2} + C = \frac{4}{5} (x^2 + 1)^{5/2} + C.
\]

To evaluate the third one, we let \( u := 2x - 1 \). Then

\[
\int x \sqrt{2x - 1} \, dx = \int \frac{u + 1}{2} \sqrt{u} \, du = \frac{1}{4} \int \left( u^{3/2} + u^{1/2} \right) \, du = \frac{1}{10} u^{5/2} + \frac{1}{6} u^{3/2} + C = \frac{1}{10} (2x - 1)^{5/2} + \frac{1}{6} (2x - 1)^{3/2} + C.
\]

□

7.1.2. Substitution rule for definite integrals. Proposition 7.1.1 can be transformed to definite integrals.

Proposition 7.1.3. If \( u = g(x) \) for some \( C^1 \)-function \( g \), then

\[
(7.1.2) \quad \int_a^b f[g(x)] g'(x) \, dx = \int_{g(a)}^{g(b)} f(u) \, du.
\]

PROOF. The proof is essentially similar as that of Proposition 7.1.1. Let \( F(x) \) be an anti-derivative of \( f(x) \). Then \( F[g(x)] \) is an anti-derivative of \( f[g(x)] g'(x) \) and hence, by the fundamental theorem of calculus, we have

\[
\int_a^b f[g(x)] g'(x) \, dx = F[g(b)] - F[g(a)] = \int_{g(a)}^{g(b)} f(u) \, du,
\]

where we use the definition of \( F \).

□
Example 7.1.4. Evaluate \( \lim_{k \to \infty} I_k \), where

\[
I_k := \int_0^{\pi/2} \sin^k x \cos x \, dx, \quad k \in \mathbb{N}.
\]

**Proof.** Letting \( u = \sin x \) we have

\[
I_k = \int_0^1 u^k \, du = \left. \frac{1}{k+1} u^{k+1} \right|_0^1 = \frac{1}{k+1}.
\]

Hence \( \lim_{k \to \infty} I_k = 0 \). \( \square \)

### 7.1.3. Trigonometric functions method.

Recall trigonometric functions formulas

\[
\sin x = \frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}, \quad \cos x = \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}, \quad \tan x = \frac{2 \tan \frac{x}{2}}{1 - \tan^2 \frac{x}{2}}.
\]

Since

\[
\frac{d}{dx} \tan x = \frac{d}{dx} \frac{\sin x}{\cos x} = \frac{1}{\cos^2 x} \frac{\sin^2 x + \cos^2 x}{\cos^2 x} = 1 + \tan^2 x,
\]

it follows that

\[
2d \tan \frac{x}{2} = \left( 1 + \tan^2 \frac{x}{2} \right) dx
\]

and hence

\[
\frac{dx}{1 + \epsilon \cos x} = \frac{dx}{1 + \epsilon \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}} = \frac{1 + \tan^2 \frac{x}{2}}{1 + \epsilon + (1 - \epsilon) \tan^2 \frac{x}{2}} dx
\]

Example 7.1.5. Evaluate

\[
\int_0^\pi \frac{1}{1 + \epsilon \cos x} \, dx = \frac{\pi}{\sqrt{1 - \epsilon^2}}, \quad \epsilon \in (-1, 1).
\]

**Proof.** Letting \( u = \tan \frac{x}{2} \) we have

\[
\int \frac{1}{1 + \epsilon \cos x} \, dx = \frac{2}{1 - \epsilon} \int \frac{du}{1 + \frac{1 - \epsilon}{1 + \epsilon} u^2} = \frac{2}{1 - \epsilon^2} \int \frac{1}{1 + \left( \frac{1 - \epsilon}{1 + \epsilon} u \right)^2} \, d\left( \frac{1 - \epsilon}{1 + \epsilon} u \right).
\]

Evaluating at \( \pi \) and 0 implies

\[
\int_0^\pi \frac{dx}{1 + \epsilon \cos x} = \frac{2}{\sqrt{1 - \epsilon^2}} \tan^{-1} \left( \frac{\sqrt{1 - \epsilon}}{1 + \epsilon} \frac{x}{2} \right) \bigg|_0^\pi = \frac{\pi}{\sqrt{1 - \epsilon^2}},
\]

because \( \tan^{-1}(\infty) = \pi/2 \). \( \square \)
7.2. Integration by parts

Let \( u = u(x) \) and \( v = v(x) \) be differentiable functions. Then

\[
(uv)' = u'v + uv'
\]

or

\[
(7.2.1) \quad uv' = (uv)' - u'v.
\]

Integrating both sides with respect to \( x \), we obtain

\[
(7.2.2) \quad \int u(x)v'(x)\,dx = u(x)v(x) - \int u'(x)v(x)\,dx
\]

or, in short form,

\[
(7.2.3) \quad \int udv = uv - \int vdu.
\]

7.2.1. Recursion method. This method reduces the “power” of functions by 1 each time.

Example 7.2.2. Evaluate

\[
I_k = \int \sin^k x\,dx, \quad J_k = \int \cos^k x\,dx.
\]

Proof. By Proposition 7.2.1 for any \( k \geq 2 \), we have

\[
I_k = -\int \sin^{k-1} x\,d\cos x = -\sin^{k-1} x \cos x + (k-1) \int \cos^2 x \sin^{k-2} x\,dx
\]

\[
= -\cos x \sin^{k-1} x + (k-1) \int \sin^{k-2} x(1-\sin^2 x)\,dx
\]

\[
= -\cos x \sin^{k-1} x + (k-1)I_{k-2} - (k-1)I_k;
\]

thus

\[
I_k = \frac{k-1}{k}I_{k-2} - \frac{1}{k} \cos x \sin^{k-1} x.
\]

Example 7.2.3. Evaluate

\[
I_k = \int_0^{\pi/2} \sin^k x\,dx.
\]

Proof. From Example 7.2.2 we have

\[
I_k = \frac{k-1}{k}I_{k-2}, \quad k \geq 2.
\]
On the other hand,

\[ I_0 = \int_0^{\pi/2} dx = \frac{\pi}{2}, \quad I_1 = \int_0^{\pi/2} \sin x \, dx = -\cos x \Big|_0^{\pi/2} = 1. \]

When \( k = 2m \) is even, we have

\[ I_{2m} = \frac{2m - 1}{2m} I_{2m-2} = \frac{2m - 1}{2m} \frac{2m - 2}{2m - 2} I_{2m-4} = \cdots = \frac{(2m-1)!!}{(2m)!!} \frac{\pi}{2}. \]

When \( k = 2m + 1 \) is odd, we have

\[ I_{2m+1} = \frac{2m}{2m + 1} I_{2m-1} = \frac{2m}{2m + 1} \frac{2m - 2}{2m - 1} I_{2m-3} = \cdots = \frac{(2m)!!}{(2m+1)!!}. \]

\[ \square \]

**Example 7.2.4.** Evaluate, for \( k \in \mathbb{N} \),

\[ I_k = \int x^k e^x \, dx, \quad I_k(a) = \int x^k e^{ax} \, dx \quad (a \neq 0). \]

**Proof.** The basic idea to deal with \( I_k \) and \( I_k(a) \) is an observation that \( \frac{d}{dx} e^{ax} = ae^{ax} \). In other words,

\[ de^{ax} = ae^{ax} \, dx. \]

Since \( I_k = I_k(1) \), we suffice to consider the second integral \( I_k(a) \). Letting \( u = x^k/a \) and \( v = e^{ax} \) in (7.2.3), we have, for \( k \geq 1 \),

\[ I_k(a) = \int x^k e^{ax} \, dx = \frac{1}{a} \int x^k e^{ax} \, dx = \int \frac{1}{a} x^k \, de^{ax} \]

\[ = \frac{1}{a} x^k e^{ax} - \frac{1}{a} \int x^k e^{ax} \, dx = \frac{1}{a} x^k e^{ax} - \frac{1}{a} \int k x^{k-1} e^{ax} \, dx \]

\[ = \frac{1}{a} x^k e^{ax} - \frac{k}{a} I_{k-1}(a). \]

Note that

\[ I_0(a) = \int e^{ax} \, dx = \frac{1}{a} e^{ax} + C. \]

for example,

\[ I_1(a) = \frac{1}{a} x e^{ax} - \frac{1}{a} I_0(a) = \frac{x}{a} e^{ax} - \frac{1}{a^2} e^{ax} + C, \]

\[ I_2(a) = \frac{1}{a} x^2 e^{ax} - \frac{2}{a} I_1(a) = \left( \frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3} \right) e^{ax} + C. \]

It is a good exercise for you to find the general formula for \( I_k(a) \). \[ \square \]

**Example 7.2.5.** Evaluate, for \( k \in \mathbb{N} \),

\[ I_k = \int (\ln x)^k \, dx. \]
PROOF. Taking $u = (\ln x)^k$ and $v = x$ in (7.2.3) implies
\[ I_k = x(\ln x)^k - \int k(\ln x)^{k-1} dx = x(\ln x)^k - kI_{k-1}. \]
Since
\[ I_1 = \int \ln x \, dx = x \ln x - \int dx = x \ln x - x + C, \]
we can calculate all $I_k$’s by the above recursion formula and the initial value. □

7.2.2. Gamma function and Beta function. The gamma function $\Gamma(x)$ is defined by
\[
(7.2.4) \quad \Gamma(x) := \int_0^\infty t^{x-1}e^{-t} \, dt, \quad x > 0.
\]
This is an improper integral which is well-defined (see later). By (7.2.3), we have
\[
\int t^{x-1}e^{-t} \, dt = \frac{1}{x} \int_0^\infty e^{-t} \, dt = \frac{t^x}{xe^t} + \frac{1}{x} \int t^x e^{-t} \, dt.
\]
Since
\[
\lim_{t \to \infty} \frac{t^x}{xe^t} = 0 = \lim_{t \to 0} \frac{t^x}{xe^t},
\]
it follows that
\[
(7.2.5) \quad \Gamma(1 + x) = x\Gamma(x).
\]
Using the recursion formula (7.2.5), we have
\[
(7.2.6) \quad \Gamma(n + 1) = n!, \quad n \in \mathbb{N}.
\]
There are some fundamental properties of $\Gamma(x)$:

(1) **Euler’s reflection formula**: for any $0 < x < 1$, we have
\[
(7.2.7) \quad \Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}.
\]
In particular,
\[
(7.2.8) \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\frac{\pi}{\sin(\pi/2)}} = \sqrt{\pi}.
\]
(2) **Duplication formula**: for any $x > 0$, we have
\[
(7.2.9) \quad \Gamma(x)\Gamma\left(x + \frac{1}{2}\right) = 2^{1-2x}\sqrt{\pi}\Gamma(2x).
\]
Another important and useful integral is the **Beta function** defined by
\[
(7.2.10) \quad B(x, y) := \int_0^1 t^{x-1}(1 - t)^{y-1} \, dt, \quad x, y > 0.
\]
This is also an improper integral. The beta function was studied by Euler and Legendre and was given its name by Jacques Binet.

(3) $B(x, y) = B(y, x)$.

(4) For any $x, y > 0$ we have
\[
(7.2.11) \quad B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.
\]
Example 7.2.6. (Example 7.2.3 revisited) We now use gamma/beta functions to give another way to evaluate $I_k$ defined in Example 7.2.3. Writing $t = \sin^2 u$ in (7.2.10) with $u \in [0, \pi/2]$, we arrive at

\[
B(x, y) = \int_0^{\pi/2} (\sin u)^{2x-2} (\cos u)^{2y-2} \sin u \cdot \cos u \, du
\]

in particular,

\[
B\left(\frac{k+1}{2}, \frac{1}{2}\right) = 2 \int_0^{\pi/2} (\sin u)^k \, du = 2I_k.
\]

Using (7.2.11), we have

\[
I_k = \frac{1}{2} \frac{\Gamma\left(\frac{k+1}{2}\right) \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{k+2}{2}\right)} = \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k+2}{2}\right)}.
\]

If $k = 2m$ is even, then

\[
I_{2m} = \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(m + \frac{1}{2}\right)}{\Gamma(m + 1)} = \frac{\sqrt{\pi}}{2} \frac{1}{m!} \Gamma\left(m + \frac{1}{2}\right)
\]

by (7.2.6); from (7.2.5), we see that

\[
\Gamma\left(m + \frac{1}{2}\right) = \Gamma\left(1 + m - 1 + \frac{1}{2}\right) = \left(m - 1 + \frac{1}{2}\right) \Gamma\left(m - 1 + \frac{1}{2}\right) = \frac{2m - 1}{2} \Gamma\left(m - 1 + \frac{1}{2}\right) = \frac{2m - 1}{2} \cdot \frac{2m - 2}{2} \Gamma\left(m - 2 + \frac{1}{2}\right) = \cdots = \frac{(2m - 1)!!}{2^m} \Gamma\left(\frac{1}{2}\right) = \frac{(2m - 1)!!}{2^m \sqrt{\pi}}.
\]

Hence

\[
I_{2m} = \frac{\pi (2m - 1)!!}{2^{2m} m!} = \frac{\pi (2m - 1)!!}{2^m (2m)!!}.
\]

The odd case is left to readers.

7.3. Rational functions and partial fractions

A rational function $f$ is the quotient of two polynomials. That is

\[
f(x) = \frac{P(x)}{Q(x)},
\]

where $P(x)$ and $Q(x)$ are polynomials. A basic idea to treat with rational functions is to write $f(x)$ as a sum of a polynomial and simpler rational functions. Such a sum is called a partial-fraction decomposition. These simpler rational functions are of the form

\[
\frac{A}{(ax + b)^n}, \quad \frac{Bx + C}{(ax^2 + bx + c)^n},
\]

where $A, B, C, a, b,$ and $c$ are constants and $n$ is a positive integer.
If the degree of \( P(x) \) in (7.3.1) is greater than or equal to the degree of \( Q(x) \), then the first step in the partial-fraction decomposition is to use long division to write \( f(x) \) as a sum of a polynomial and a rational function, where the rational function is such that the degree of the polynomial in the numerator is less than the degree of the polynomial in the denominator. (Such rational functions are called proper.)

7.3.1. Partial-fraction decomposition I. We discuss the cases of distinct and repeated linear factors.

If the linear factor \( ax + b \) is contained \( n \) times in the factorization of the denominator of a proper rational function, then the partial-fraction decomposition contains terms of the form

\[
\frac{A_1}{ax + b} + \frac{A_2}{(ax + b)^2} + \cdots + \frac{A_n}{(ax + b)^n},
\]

where \( A_1, \ldots, A_n \) are constants.

Case 1a: \( Q(x) \) is a product of \( m \) distinct linear factors. \( Q(x) \) is thus of the form

\[
Q(x) = a(x - x_1) \cdots (x - x_m)
\]

where \( x_1, \ldots, x_m \) are the \( m \) distinct roots of \( Q(x) \). The rational function can then be written as

\[
P(x) Q(x) = \frac{1}{a} \left[ \frac{A_1}{x - x_1} + \cdots + \frac{A_m}{x - x_m} \right].
\]

Case 1b: \( Q(x) \) is a product of repeated linear factors. \( Q(x) \) is thus of the form

\[
Q(x) = a(x - x_1)^{n_1} \cdots (x - x_m)^{n_m}
\]

where \( x_1, \ldots, x_m \) are the \( m \) distinct roots of \( Q(x) \), and \( n_1, \cdots, n_m \) are positive integers. Then

\[
P(x) Q(x) = \frac{1}{a} \left[ \frac{A_{1,1}}{x - x_1} + \cdots + \frac{A_{1,n_1}}{(x - x_1)^{n_1}} + \cdots + \frac{A_{m,1}}{x - x_m} + \cdots + \frac{A_{m,n_m}}{(x - x_1)^{n_m}} \right].
\]

For example,

\[
\int \frac{1}{x(x-1)} \, dx = \int \left( -\frac{1}{x} + \frac{1}{x-1} \right) \, dx = \ln |x-1| - \ln |x| + C,
\]

and

\[
\int \frac{x}{(x+1)^2} \, dx = \int \left[ \frac{1}{x+1} + \frac{-1}{(x+1)^2} \right] \, dx = \ln |x+1| + \frac{1}{x+1} + C.
\]

7.3.2. Irreducible quadratic factors. We discuss the cases of distinct and repeated irreducible quadratic factors.

If the irreducible quadratic factor \( ax^2 + bx + c \) is contained \( n \) times in the factorization denominator of a proper rational function, then the partial-fraction decomposition contains terms of the form

\[
\frac{B_1 x + C_1}{ax^2 + bx + c} + \frac{B_2 x + C_2}{(ax^2 + bx + c)^2} + \cdots + \frac{B_n x + C_n}{(ax^2 + bx + c)^n},
\]
where $B_1, C_1, \ldots, B_n, C_n$ are constants.

**Case 1a:** $Q(x)$ is a product of $m$ distinct irreducible quadratic factors. $Q(x)$ is thus of the form

$$Q(x) = (a_1 x^2 + b_1 x + c_1) \cdots (a_m x^2 + b_m x + c_m)$$

where $a_1, b_1, c_1, \ldots, a_m, b_m, c_m$ are constants. Then

$$P(x) = \frac{B_1 x + C_1}{a_1 x^2 + b_1 x + c_1} + \cdots + \frac{B_m x + C_m}{a_m x^2 + b_m x + c_m}.$$ 

**Case 2a:** $Q(x)$ is a product of $m$ repeated irreducible quadratic factors. $Q(x)$ is thus of the form

$$Q(x) = (a_1 x^2 + b_1 x + c_1)^{n_1} \cdots (a_m x^2 + b_m x + c_m)^{n_m},$$

where $a_1, b_1, c_1, \ldots, a_m, b_m, c_m$ are constants, and $n_1, \ldots, n_m$ are positive integers. Then

$$P(x) = \frac{B_{1,i} x + C_{1,i}}{(a_1 x^2 + b_1 x + c_1)^i} + \cdots + \frac{B_{m,i} x + C_{m,i}}{(a_m x^2 + b_m x + c_m)^{n_m}}.$$ 

For example,

$$\int \frac{2x^3 - x^2 + 2x - 2}{(x^2 + 2)(x^2 + 1)} \, dx = \int \left[ \frac{2x}{x^2 + 2} + \frac{-1}{x^2 + 1} \right] \, dx = \ln(x^2 + 2) - \tan^{-1} x + C,$$

and

$$\int \frac{x^2 + x + 1}{(x^2 + 1)^2} \, dx = \int \left[ \frac{1}{x^2 + 1} + \frac{x}{(x^2 + 1)^2} \right] \, dx = \tan^{-1} x - \frac{1}{2(x^2 + 1)} + C.$$

**Example 7.3.1.** Evaluate

$$\int_0^1 \frac{x^4(1-x)^4}{1 + x^2} \, dx = \frac{22}{7} - \pi.$$ 

**Proof.** Since

$$\frac{x^4(1-x)^4}{1 + x^2} = x^6 - 4x^5 + 5x^4 - 4x^2 + 4 - \frac{4}{1 + x^2},$$

it follows that

$$\int_0^1 \frac{x^4(1-x)^4}{1 + x^2} \, dx = \frac{22}{7} - \pi.$$ 

Since $1 \leq 1 + x^2 \leq 2$ for any $x \in [0, 1]$, we have

$$\int_0^1 \frac{1}{2} x^4(1-x)^4 \, dx \leq \int_0^1 \frac{x^4(1-x)^4}{1 + x^2} \, dx \leq \int_0^1 x^4(1-x)^4 \, dx.$$ 

Recall the Beta function (7.2.10). Hence

$$\int_0^1 x^4(1-x)^4 \, dx = B(5, 5) = \frac{\Gamma(5)\Gamma(5)}{\Gamma(10)} = \frac{4!4!}{9!} = \frac{1}{630}.$$
by (7.2.11) and (7.2.6). Consequently
\[
\frac{1}{1260} < \frac{22}{7} - \pi \leq \frac{1}{630}
\]
which gives $3.140 \leq \pi \leq 3.142$. \qed

### 7.4. Improper integrals

Look at the gamma function defined in (7.2.4). At first glance, the integration interval is unbounded and the function $t^{x-1}e^{-t}$ is infinity as $t \to 0^+$. 

#### Exercise 7.4.1

Show that
\[
\lim_{t \to 0^+} t^{x-1}e^{-t} = \begin{cases} 
0, & x > 1, \\
1, & x = 1, \\
\infty, & x < 1.
\end{cases}
\]

We call an integral is improper if

1. one or both limits of integration are infinite (that is, the integration interval is unbounded), or
2. the integrand becomes infinite at one or more points of the interval of integration.

#### 7.4.1. Unbounded intervals

By cutting off the interval, we can define

**Definition 7.4.2.** If $f(x)$ is a continuous function on $[a, \infty)$ or $(-\infty, a]$, then we define

\[
\int_a^\infty f(x) \, dx := \lim_{z \to \infty} \int_z^a f(x) \, dx,
\]

and

\[
\int_{-\infty}^a f(x) \, dx := \lim_{z \to -\infty} \int_z^a f(x) \, dx.
\]
From the definition above, we know that for each \( z \), the integrals
\[
\int_{a}^{z} f(x) \, dx, \quad \int_{z}^{a} f(x) \, dx
\]
are finite, however, after taking the limits, the improper integrals may be infinity.

**Example 7.4.3.** A typical example is
\[
(7.4.3) \quad \int_{1}^{\infty} \frac{1}{x^p} \, dx, \quad p > 0.
\]
For example,
\[
\int_{1}^{\infty} \frac{1}{x^2} \, dx = \lim_{z \to \infty} \int_{1}^{z} \frac{1}{x^p} \, dx = \lim_{z \to \infty} \left( 1 - \frac{1}{z} \right) = 1,
\]
\[
\int_{1}^{\infty} \frac{1}{\sqrt{x}} \, dx = \lim_{z \to \infty} \int_{1}^{z} \frac{1}{\sqrt{x}} \, dx = \lim_{z \to \infty} (2\sqrt{z} - 2) = \infty.
\]
In general, since
\[
\int_{1}^{\infty} \frac{1}{x^p} \, dx = \begin{cases} \frac{1}{1-p} (z^{1-p} - 1), & p \neq 1, \\ \ln z, & p = 1, \end{cases}
\]
we arrive at
\[
(7.4.4) \quad \int_{1}^{\infty} \frac{1}{x^p} \, dx = \begin{cases} \frac{1}{p-1}, & p > 1, \\ \infty, & 0 < p \leq 1. \end{cases}
\]

As in the case of sequences, we can define

**Definition 7.4.4.** Let \( f(x) \) be continuous on the interval \([a, \infty)\). If
\[
\lim_{z \to \infty} \int_{a}^{z} f(x) \, dx
\]
exists and has a finite value, we say that the improper integral
\[
\int_{a}^{\infty} f(x) \, dx
\]
converges and define
\[
\int_{0}^{\infty} f(x) \, dx := \lim_{z \to \infty} \int_{a}^{z} f(x) \, dx.
\]
Otherwise we say that the improper integral **diverge**.

From Example 7.4.3 and Definition 7.4.4, we see that the integral
\[
\int_{1}^{\infty} \frac{1}{x^p} \, dx
\]
is convergent if \( p > 1 \) while is divergent if \( 0 < p \leq 1 \).
Example 7.4.5. Consider the integral
\[ \int_{0}^{\infty} \frac{1}{1 + x^2} \, dx. \]
According to (6.2.11), we have
\[ \int_{0}^{z} \frac{1}{1 + x^2} \, dx = \tan^{-1} x \bigg|_{0}^{z} = \tan^{-1} z, \]
and then
\[ \int_{0}^{\infty} \frac{1}{1 + x^2} \, dx = \lim_{z \to \infty} \tan^{-1} z = \frac{\pi}{2}. \]
Definition 7.4.4 can be extended to the integrals
\[ \int_{-\infty}^{a} f(x) \, dx. \]
For example
\[ \int_{-\infty}^{0} \frac{1}{1 + x^2} \, dx = \lim_{z \to -\infty} (-\tan^{-1} z) = \frac{\pi}{2}. \]
If \( f(x) \) is continuous on \( (-\infty, \infty) \), we define
\[ (7.4.5) \quad \int_{-\infty}^{\infty} f(x) \, dx := \int_{-\infty}^{a} f(x) \, dx + \int_{a}^{\infty} f(x) \, dx, \]
where \( a \) is a real number. If both improper integrals on the right-hand side of (7.4.5) are convergent, then the value of the improper integral on the left-hand side of (7.4.5) is the sum of the two limiting values on the right-hand side.

I give some remarks on the above definition.

Remark 7.4.6. (1) More precisely, if for any given \( a \in \mathbb{R} \), the integrals
\[ \int_{-\infty}^{a} f(x) \, dx, \quad \int_{a}^{\infty} f(x) \, dx \]
are convergent, then we define the integral \( \int_{-\infty}^{\infty} f(x) \, dx \) via (7.4.5). In this case, we can show that the definition (7.4.5) does not depend on the particular choice of \( a \). Indeed, for any given two real numbers \( a \) and \( b \), say \( a < b \), we have
\[ \int_{a}^{\infty} f(x) \, dx = \lim_{z \to \infty} \int_{a}^{z} f(x) \, dx = \lim_{z \to \infty} \left[ \int_{a}^{z} f(x) \, dx + \int_{a}^{b} f(x) \, dx \right] = \int_{b}^{\infty} f(x) \, dx + \int_{a}^{b} f(x) \, dx. \]
Similarly, we can show
\[ \int_{-\infty}^{b} f(x) \, dx = \int_{a}^{b} f(x) \, dx + \int_{-\infty}^{a} f(x) \, dx. \]
Since all integrals are finite, it follows that
\[ \int_a^\infty f(x) \, dx + \int_{-\infty}^a f(x) \, dx = \int_b^\infty f(x) \, dx + \int_{-\infty}^b f(x) \, dx. \]
Thus the definition (7.4.5) does not depend on the choice of \( a \).

(2) Another important way to define the integral \( \int_{-\infty}^\infty f(x) \, dx \) is the **Cauchy’s principal-value integral**

\[ (7.4.6) \quad \text{p.v.} \int_{-\infty}^\infty f(x) \, dx := \lim_{z \to \infty} \int_{-z}^z f(x) \, dx. \]

(3) If \( \int_{-\infty}^\infty f(x) \, dx \) exists in the sense of (7.4.5), then
\[ \int_{-\infty}^\infty f(x) \, dx = \lim_{z \to \infty} \left[ \int_{-z}^a f(x) \, dx + \int_a^z f(x) \, dx \right] = \text{p.v.} \int_{-\infty}^\infty f(x) \, dx. \]

However, the converse is not true. Thus, we can find a continuous function \( f(x) \) defined on \((-\infty, \infty)\), which has finite integral in the sense of (7.4.6) but not of (7.4.5). For example,
\[ f(x) := \sin x, \quad x \in (-\infty, \infty). \]

Then
\[ \text{p.v.} \int_{-\infty}^\infty \sin x \, dx = \lim_{z \to \infty} \int_{-z}^z \sin x \, dx = \lim_{z \to \infty} 0 = 0, \]
\[ \int_{-\infty}^\infty \sin x \, dx = \infty. \]

Let \( f(x) := e^x - x, \quad x \in [0, \infty). \)

Since \( f'(x) = e^x - 1 \geq e^0 - 1 = 0 \), it follows from Proposition 5.2.2 that \( f(x) \) is increasing on \([0, b]\) for each fixed \( b \) and then on \([0, \infty)\). Hence
\[ (7.4.7) \quad e^x - x = f(x) \geq f(0) = e^0 - 0 = 1 \implies e^x \geq 1 + x, \quad x \in [0, \infty). \]

**Exercise 7.4.7.** Show that
\[ e^x \geq 1 + x + \frac{x^2}{2}, \quad x \in [0, \infty). \]

(Hint: Consider the function \( f(x) = e^x - 1 - x - \frac{x^2}{2} \) and use (7.4.7).)

In general, we have, for any given positive integer \( n \),
\[ (7.4.8) \quad e^x \geq \sum_{i=0}^{n} \frac{x^i}{i!} = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} \geq \frac{1}{n!} x^n, \quad x \in [0, \infty). \]

Actually, we can show that (beyond the scope of this course)
\[ (7.4.9) \quad e^x = \lim_{n \to \infty} \sum_{i=0}^{n} \frac{x^i}{i!} = \sum_{i=0}^{\infty} \frac{x^i}{i!}, \quad x \in (-\infty, \infty). \]

We call (7.4.9) the **Taylor series** of \( e^x \).
**Exercise 7.4.8.** Verify (7.4.8) for \( n = 3, 4 \).

We now use the inequality (7.4.8) to show that the improper integral (7.2.4) is convergent at least for \( x \geq 1 \).

**Example 7.4.9.** Recall the gamma function

\[
\Gamma(x) := \int_0^\infty t^{x-1}e^{-t} \, dt.
\]

We now show that \( \Gamma(x) \) is convergent whenever \( x \geq 1 \).

**Case 1:** \( x = 1 \). Then

\[
\Gamma(1) = \int_0^\infty e^{-t} \, dt = \left. -e^{-t} \right|_0^\infty = 1.
\]

**Case 2:** \( x > 1 \). Divide \( \Gamma(x) \) into two terms

\[
\Gamma(x) = \int_0^1 t^{x-1}e^{-t} \, dt + \int_1^\infty t^{x-1}e^{-t} \, dt;
\]

since \( x - 1 > 0 \) and \( 0 \leq t \leq 1 \), we have \( t^{x-1} \leq 1 \) and

\[
\int_0^1 t^{x-1}e^{-t} \, dt \leq \int_0^1 e^{-t} \, dt = \left. -e^{-t} \right|_0^1 = 1 - \frac{1}{e}.
\]

Choose an integer \( n \) such that \( x - 1 \leq n \); for \( t \geq 1 \), we then get \( t^{x-1} \leq t^n \). Using (7.4.8) implies

\[
\frac{1}{n!} \left( \frac{t}{2} \right)^n \leq e^{t/2} \implies t^{x-1} \leq 2^n n! e^{t/2}
\]

and hence

\[
\int_1^\infty t^{x-1}e^{-t} \, dt \leq \int_1^\infty 2^n n! e^{t/2} \, dt = 2^n n! \int_1^\infty e^{-t/2} \, dt = \frac{2^n n!}{\sqrt{e}}.
\]

Therefore \( \Gamma(x) \) is convergent for \( x \geq 1 \).

**7.4.2. Unbounded integrands.** A typical example is the integral

\[
\int_0^1 \frac{1}{\sqrt{x}} \, dx.
\]

The integrand \( f(x) = 1/\sqrt{x} \) is unbounded as \( x \to 0^+ \). By cutting off the point 0, we can study

\[
I(c) := \int_c^1 \frac{1}{\sqrt{x}} \, dx, \quad 0 < c < 1.
\]

Now the function \( f(x) \) is well-defined and continuous on \([c, 1]\); moreover

\[
I_c = 2\sqrt{x} \bigg|_c^1 = 2(1 - \sqrt{c})
\]

and

\[
\lim_{c \to 0^+} I_c = \lim_{c \to 0^+} (2 - 2\sqrt{c}) = 2.
\]
This example suggests us the following

**Definition 7.4.10.** If \( f \) is continuous on \((a, b]\) and \( \lim_{x \to a^+} f(x) = \pm \infty \), we define

\[
\int_a^b f(x) \, dx := \lim_{c \to a^+} \int_c^b f(x) \, dx,
\]

provided that this limit exists. If the limit exists, we say that the improper integral on the left-hand side of (7.4.10) **converges**; if the limit does not exist, we say that the improper integral **diverges**.

Similarly, if \( f \) is continuous on \([a, b)\) and \( \lim_{x \to b^-} f(x) = \pm \infty \), we define

\[
\int_a^b f(x) \, dx := \lim_{c \to b^-} \int_a^c f(x) \, dx,
\]

provided that this limit exists.

If \( f \) is continuous on \((a, b)\) and \( \lim_{x \to a^+} f(x) = \pm \infty \) and \( \lim_{x \to b^-} f(x) = \pm \infty \), we define

\[
\int_a^b f(x) \, dx := \lim_{\epsilon, \eta \to 0} \int_{a+\epsilon}^{b-\eta} f(x) \, dx,
\]

provided that this limit exists. Note that the limit in (7.4.12) is the limit of multivariable which will be treated in later.

For example

\[
\int_0^1 \frac{dx}{(x-1)^{2/3}} = \left. 3(x-1)^{1/3} \right|_0^1 = 3,
\]

\[
\int_0^1 \ln x \, dx = (x \ln x - x) \bigg|_0^1 = -1 - \lim_{x \to 0^+} x \ln x = -1.
\]

If \( f(x) \) is discontinuous at a point \( c \in (a, b) \), then we define

\[
\int_a^b f(x) \, dx := \lim_{\epsilon \to 0^+} \int_c^{c+\epsilon} f(x) \, dx + \lim_{\eta \to 0^+} \int_{a+\eta}^b f(x) \, dx.
\]

By this definition, we have

\[
\int_{-1}^1 \frac{1}{x^3} \, dx = \infty.
\]

We can also define Cauchy’s principal-value integral

\[
p.v. \int_a^b f(x) \, dx := \lim_{\epsilon \to 0^+} \left[ \int_{c+\epsilon}^b f(x) \, dx + \int_{a+\epsilon}^{c-\epsilon} f(x) \, dx \right].
\]

For example,

\[
p.v. \int_{-1}^1 \frac{1}{x^3} \, dx = \lim_{\epsilon \to 0^+} \left[ \int_{\epsilon}^1 \frac{dx}{x^3} + \int_{-1}^{-\epsilon} \frac{dx}{x^3} \right] = 0.
\]

### 7.4.3. A comparison result for improper integrals.

The inequality (6.1.8) can be generalized to improper integrals.
Proposition 7.4.11. Suppose that \( f(x) \) and \( g(x) \) are continuous on \([a, \infty)\).

1. If \( f(x) \leq g(x) \) on \([a, \infty)\) and \( \int_a^\infty g(x) \, dx \) is convergent, then \( \int_a^\infty f(x) \, dx \) is also convergent and

\[
\int_a^\infty f(x) \, dx \leq \int_a^\infty g(x) \, dx.
\]

2. If \( f(x) \geq g(x) \) on \([a, \infty)\) and \( \int_a^\infty g(x) \, dx \) is divergent, then \( \int_a^\infty f(x) \, dx \) is also divergent.

Proof. (1) In this case, for any fixed number \( z > a \), we have

\[
\int_a^z f(x) \, dx \leq \int_a^z g(x) \, dx;
\]

letting \( z \to \infty \) implies (7.4.15).

(2) If \( \int_a^\infty f(x) \, dx \) is convergent, by (1), we must have that the improper integral \( \int_a^\infty g(x) \, dx \) is convergent, a contradiction. \( \square \)

Example 7.4.12. Evaluate

\[
I := \int_0^\infty e^{-x^2} \, dx.
\]

Note that

\[
I = \int_0^1 e^{-x^2} \, dx + \int_1^\infty e^{-x^2} \, dx.
\]

For any \( 0 \leq x \leq 1 \), we have \( 0 \leq e^{-x^2} \leq 1 \) and then

\[
\int_0^1 e^{-x^2} \, dx \leq \int_0^1 dx = 1.
\]

When \( x > 1 \), we have then \( -x^2 \leq -x \) so that

\[
\int_1^\infty e^{-x^2} \, dx \leq \int_1^\infty e^{-x} \, dx = -e^{-x}\bigg|_1^\infty = 1/\sqrt{e}.
\]

Set

\[
t = x^2, \quad x \geq 0.
\]

Then

\[
I = \int_0^\infty e^{-t} \, d\sqrt{t} = \frac{1}{2} \int_0^\infty t^{-1/2} e^{-t} \, dt = \frac{1}{2} \Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2},
\]

by (7.2.8).

7.5. Taylor approximation

The Taylor polynomial of degree \( n \) about \( x = 0 \) for the function \( f(x) \) is given by

\[
P(x) := f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n,
\]
provided all derivatives $f'(0), f''(0), \cdots, f^{(n)}(0)$ exist.

For example, $\sum_{i=0}^{n} i^i / i!$ in (7.4.8) is the Taylor polynomial of degree $n$ about $x = 0$ for $e^x$.

More general, the Taylor polynomial of degree $n$ about $x = a$ for the function $f(x)$ is given by

\[
P_n(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n.
\]

Exercise 7.5.1. Compute the Taylor polynomial of degree 3 about $x = 0$ for $\sin x$ and $\cos x$.

7.5.1. Taylor’s formula. Let $f$ be a continuous function whose continuously derivatives exist up to second order. By Theorem 6.2.3 we have

\[
f(x) = f(a) + \int_a^x f'(t) \, dt.
\]

Applying Proposition 7.2.1 to $u = -(x - t)$ and $v = f'(t)$, we get

\[
\int_a^x f'(t) \, dt = \int_a^x f'(t) d(-(x - t)) = -(x - t)f'(t) \bigg|_a^x + \int_a^x (x - t)f''(t) \, dt
\]

\[
= (x - a)f'(a) + \int_a^x (x - t)f''(t) \, dt.
\]

That is

\[
f(x) = f(a) + f'(a)(x - a) + \int_a^x (x - t)f''(t) \, dt.
\]

In general

Theorem 7.5.2. (Taylor’s formula) Suppose that $f : I \to \mathbb{R}$ where $I$ is an interval, $a \in I$, and $f$ and its first $n + 1$ derivatives are continuous at $a \in I$. Then, for $x \in I$,

\[
f(x) = f(a) + \sum_{i=1}^{n} \frac{f^{(i)}(a)}{i!} (x - a)^i + R_{n+1}(x),
\]

where

\[
R_{n+1}(x) = \frac{1}{n!} \int_a^x (x - t)^n f^{(n+1)}(t) \, dt.
\]

By Theorem 6.3.1 we have

\[
f(x) = f(a) + \int_a^x f'(t) \, dt = f(a) + (x - a)f'(c)
\]

for some $c$ between $a$ and $x$. 

Corollary 7.5.3. Suppose that \( f : I \to \mathbf{R} \) where \( I \) is an interval, \( a \in I \), and \( f \) and its first \( n + 1 \) derivatives are continuous at \( a \in I \). Then, for \( x \in I \), there exists a \( c \) between \( a \) and \( x \) such that the error term \( R_{n+1}(x) \) is of the form

\[
R_{n+1}(x) = \frac{f^{(n+1)}(c)}{(n+1)!} (x-a)^{n+1}.
\]  

Proof. Since \( f^{(n+1)}(t) \) is continuous, it follows that \( m \leq f^{(n+1)}(t) \leq M \) for all \( t \in I \). In particular,

\[
\frac{m}{n!} \int_a^x (x-t)^n \, dt \leq R_{n+1}(x) \leq \frac{M}{n!} \int_a^x (x-t)^n \, dt,
\]

which gives us \( \frac{m}{(n+1)!} (x-a)^{n+1} \leq R_{n+1}(x) \leq \frac{M}{(n+1)!} (x-a)^{n+1} \). Thus \( m \leq \frac{R_{n+1}(x)}{(x-a)^{n+1}} \leq M \). By Theorem 3.2.7, \( \frac{(n+1)!}{(x-a)^{n+1}} R_{n+1}(x)/(x-a)^{n+1} = f^{(n+1)}(c) \) for some \( c \) between \( a \) and \( x \). \( \square \)

For example,

\[
e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!} + R_{n+1}(x), \quad x \in (-\infty, \infty),
\]

\[
\sin x = \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i+1}}{(2i+1)!} + R_{n+1}(x), \quad x \in (-\infty, \infty),
\]

\[
\cos x = \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i}}{(2i)!} + R_{n+1}(x), \quad x \in (-\infty, \infty),
\]

\[
\ln(1+x) = \sum_{i=1}^{\infty} (-1)^i \frac{x^i}{i} + R_{n+1}(x), \quad x \in (-1,1],
\]

\[
\frac{1}{1-x} = \sum_{i=0}^{\infty} x^i + R_{n}(x), \quad x \in (-1,1).
\]

7.5.2. Taylor’s series. Indeed, we can show that \( \text{(7.5.6)} \)–\( \text{(7.5.10)} \) become

\[
e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!}, \quad x \in (-\infty, \infty),
\]

\[
\sin x = \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i+1}}{(2i+1)!}, \quad x \in (-\infty, \infty),
\]

\[
\cos x = \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i}}{(2i)!}, \quad x \in (-\infty, \infty),
\]

\[
\ln(1+x) = \sum_{i=1}^{\infty} (-1)^{i+1} \frac{x^i}{i}, \quad x \in (-1,1),
\]

\[
\frac{1}{1-x} = \sum_{i=0}^{\infty} x^i, \quad x \in (-1,1).
\]
A function \( f(x) \) defined on an open interval \( I \) is called **smooth** if all its higher order derivatives are continuous. For example \( e^x, \sin x, \cos x \) are smooth functions on \((-\infty, \infty)\). The set of all smooth functions defined on \( I \) is denoted by \( C^\infty(I) \). We say \( f \in C^k(I) \) if all higher order derivative up to \( k \)th order are continuous.

Observe that

\[
C^k(I) \subset C^\infty(I) \quad \text{for all } k.
\]

A function \( f(x) \) defined on an open interval \( I \) is called **analytic** if for any \( a \in I \), there exists an open interval \( I_a \) such that \( a \in I_a \subset I \) and

\[
f(x) = \sum_{n=0}^{\infty} c_n (x - a)^n, \quad x \in I_a.
\]

The set of all analytic functions defined on \( I \) is denoted by \( C^\omega(I) \).

**Example 7.5.4.** \( e^x \in C^\omega((-\infty, \infty)) \). For any \( a \in (-\infty, \infty) \), we have

\[
e^x = e^a \cdot e^{x-a} = e^a \sum_{i=0}^{\infty} \frac{(x-a)^i}{i!}
\]

by (7.5.11).

Since \( \frac{d^n}{dx^n} e^x = e^x \), it follows that \( e^x \) is a smooth function. In general, we have

**Proposition 7.5.5.** For any open interval \( I \), we have \( C^\omega(I) \subset C^\infty(I) \) and \( C^\omega(I) \neq C^\infty(I) \). If \( f \in C^\omega(I) \), then, for any \( a \in I \),

\[
f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n
\]

for any \( x \) near \( a \).

I only give a smooth function on \((-\infty, \infty)\) which is not analytic.

**Example 7.5.6.** Consider

\[
f(x) = \begin{cases} 
e^{-1/x}, & x > 0, \\
0, & x \leq 0.
\end{cases}
\]

Since

\[
\lim_{x \to 0^+} e^{-1/x} = \lim_{x \to 0^+} \frac{1}{e^{1/x}} = \lim_{y \to \infty} \frac{1}{ey} = 0,
\]

where \( y = 1/x \). This shows that \( f(x) \) is continuous on \((-\infty, \infty)\). To prove \( f \in C^\infty((-\infty, \infty)) \), we suffice to check that \( \lim_{x \to 0^+} f^{(n)}(x) = f(0) = 0 \) for all \( n \in \mathbb{N} \).
Calculate
\[ f'(x) = \frac{1}{x^2} e^{-1/x}, \]
\[ f''(x) = \frac{-2x}{x^4} e^{-1/x} + \frac{1}{x^2} = \left( \frac{1}{x^4} - \frac{2}{x^3} \right) e^{-1/x}, \]
\[ f'''(x) = \left( \frac{-4x^3}{x^8} + \frac{6x^2}{x^6} + \frac{1}{x^5} - \frac{2}{x^4} \right) e^{-1/x} = \left( \frac{1}{x^5} - \frac{6}{x^3} + \frac{6}{x^4} \right) e^{-1/x}. \]

By induction on \( n \), we can show that
\[ f^{(n)}(x) = P_{2n} \left( \frac{1}{x} \right) e^{-1/x} \]
where \( P_{2n}(t) \) denotes the polynomial (without constant term) of degree \( 2n \) in terms of \( t \). Hence
\[ \lim_{x \to 0^+} f^{(n)}(x) = \lim_{y \to \infty} P_{2n}(y); \]
by (7.4.8), we have \( e^y \geq \frac{1}{(2n+1)!} y^{2n+1} \) and then
\[ \lim_{x \to 0^+} f^{(n)}(x) \leq \lim_{y \to \infty} \frac{(2n+1)! P_{2n}(y)}{y^{2n+1}} = 0 \]
by (3.1.8). Thus \( f(x) \) is smooth.

We now show that \( f(x) \) is not analytic. Otherwise,
\[ f(x) = \sum_{n=0}^{\infty} c_n x^n \]
around 0. By (7.5.17), we must have \( f(x) \equiv 0 \) near 0. But, for any \( \epsilon > 0 \), by the definition, we get \( f(x) > 0 \), a contradiction. Therefore, \( f(x) \in C^\infty((\infty, \infty)) \) but \( f(x) \notin C^\infty((\infty, \infty)) \).

For any real number \( \alpha \) and any positive integer \( n \), define
\[ \binom{\alpha}{n} = \frac{\alpha(\alpha - 1) \cdots (\alpha - n + 1)}{n!}. \]

When \( \alpha \) is a positive integer greater than \( n \), the definition (7.5.19) coincides with the usual binomial coefficients.

For all \( |x| < 1 \) and all \( \alpha \in \mathbb{R} \), we have
\[ (1 + x)^\alpha = 1 + \sum_{n=1}^{\infty} \binom{\alpha}{n} x^n. \]

Other important Taylor’s series are
We have
\begin{align*}
\sin^{-1} x &= \sum_{n=0}^{\infty} \frac{(2n)!}{4^n n!(2n+1)} x^{2n+1}, \quad |x| \leq 1, \\
\cos^{-1} x &= \frac{\pi}{2} - \sin^{-1} x, \quad |x| \leq 1, \\
\tan^{-1} x &= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}, \quad |x| \leq 1.
\end{align*}

In particular, taking \(x = 1\) in (7.5.23) yields
\begin{equation}
\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots.
\end{equation}

**Exercise 7.5.7.** Verify
\begin{align*}
\frac{e^x}{\cos x} &= 1 + x + x^2 + \frac{2}{3} x^3 + \frac{1}{2} x^4 + \cdots, \\
(1 + x)e^x &= \sum_{n=0}^{\infty} \frac{n+1}{n!} x^n.
\end{align*}

The hyperbolic functions are
\begin{align*}
\sinh x &= \frac{e^x - e^{-x}}{2}, \quad \cosh x = \frac{e^x + e^{-x}}{2}, \\
\tanh x &= \frac{\sinh x}{\cosh x}, \quad \coth x = \frac{\cosh x}{\sinh x}.
\end{align*}

**Exercise 7.5.8.** Prove
\begin{align*}
\sinh(-x) &= -\sinh x, \quad \cosh(-x) = \cosh x, \\
\tanh(-x) &= -\tanh x, \quad \coth(-x) = -\coth x, \\
1 &= \cosh^2 x - \sinh^2 x, \\
\sinh^{-1} x &= \ln \left(x + \sqrt{1 + x^2}\right), \quad \cosh^{-1} x = \ln \left(x + \sqrt{x^2 - 1}\right) \quad (x \geq 1), \\
\tanh^{-1} x &= \frac{1}{2} \ln \left(\frac{1 + x}{1 - x}\right) \quad (|x| < 1), \quad \coth^{-1} x = \frac{1}{2} \ln \left(\frac{x + 1}{x - 1}\right) \quad (|x| > 1), \\
\frac{d}{dx} \sinh x &= \cosh x, \quad \frac{d}{dx} \cosh x = \sinh x, \\
\frac{d}{dx} \tanh x &= 1 - \tanh^2 x, \quad \frac{d}{dx} \coth x = 1 - \coth^2 x, \\
\frac{d}{dx} \sinh^{-1} x &= \frac{1}{\sqrt{1 + x^2}}, \quad \frac{d}{dx} \cosh^{-1} x = \frac{1}{\sqrt{x^2 - 1}}, \\
\frac{d}{dx} \tanh^{-1} x &= \frac{1}{1 - x^2}, \quad \frac{d}{dx} \coth^{-1} x = \frac{1}{1 - x^2}.
\end{align*}
The Taylor’s series of hyperbolic functions are

$$\sinh x = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1},$$

(7.5.27)

$$\cosh x = \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n}.$$  

(7.5.28)

**Exercise 7.5.9.** Evaluate the first two terms of their Taylor polynomials:

$$\tanh x = x - \frac{1}{3} x^3 + \cdots,$$

$$\coth x - \frac{1}{x} = \frac{1}{3} x - \frac{1}{45} x^3 + \cdots.$$
A differential equation about \( y(x) \) is an equation containing function \( y \) and its higher-order derivatives. For example,
\[
y'(x) = y(x).
\]

In this chapter we give methods to solve differential equations.

**8.1. Solving differential equations**

If a differential equation contains only the first derivative, it is called a **first-order** differential equation. For example,
\[
y'(x) = y(x), \quad y'(x) = \sqrt{x + y(x)}.
\]

In this chapter, we focus only on first-order differential equations of the form
\[
\frac{dy}{dx} = f(x)g(y).
\]

The right-hand side of (8.1.1) is the product of two functions, one depending only on \( x \), the other only on \( y \).

**8.1.1. Separable differential equations.** Look at the following differential equation
\[
\frac{dy}{dx} = \frac{1}{y}.
\]

Multiplying by \( ydx \) on both sides implies
\[
ydy = dx;
\]

so
\[
\int ydy = \int dx \implies \frac{1}{2}y^2 = x + C \implies y^2 = 2x + C.
\]

The above discussion suggests a way to solve (8.1.1). From (8.1.1) we have
\[
\frac{dy}{g(y)} = f(x)dx
\]

which implies
\[
\int \frac{dy}{g(y)} = \int f(x)dx.
\]
8.1.2. Pure-time differential equations. If $g(y) \equiv 1$, then we say (8.1.1) is a pure-time differential equation:

(8.1.3) \[ \frac{dy}{dx} = f(x), \quad x \in I, \]

for some interval $I$. In this case, according to (8.1.2), we have

(8.1.4) \[ y(x) = y = \int f(x) \, dx + C. \]

In particular, for any $x_0 \in I$, we have

(8.1.5) \[ y(x) = \int_{x_0}^{x} f(x) \, dx + y_0, \]

where $y_0 := y(x_0)$.

**Example 8.1.1.** Suppose that the volume $V(t)$ of a cell at time $t$ changes according to

\[ V'(t) = \cos t, \quad \text{with } V(0) = 3. \]

**Proof.** Using (8.1.5), we have

\[ V(t) = \int_{0}^{t} \cos s \, ds + V(0) = \sin s \bigg|_{0}^{t} + 3 = \sin t - \sin 0 + 3 = 3 + \sin t \]

for any $t \geq 0$. \(\square\)

8.1.3. Autonomous differential equations. Many biological models have the form

(8.1.6) \[ \frac{dy}{dx} = g(y) \]

called an autonomous differential equation. For example,

\[ N'(t) = 2N(t), \]
where \( N(t) \) denotes the size of the population at time \( t \). By (8.1.2), we have

\[(8.1.7) \quad \int \frac{dy}{g(y)} = \int dx.\]

The solutions now depend on the form of \( g(y) \).

**Case 1:** \( g(y) = k(y - a) \), where \( k \neq 0 \). In this case, (8.1.7) becomes

\[\int \frac{dy}{y - a} = \int k \, dx \implies \ln |y - a| = kx + C.\]

Consequently,

\[(8.1.8) \quad y(x) = y = C'e^{kx} + a, \quad C' := \pm e^C.\]

**Example 8.1.2.** Solve \( \frac{dy}{dx} = 2 - 3y \), where \( y(1) = 1 \).

**Proof.** Note that \( 2 - 3y = -3(y - \frac{2}{3}) \). By (8.1.8) we have

\[y(x) = C'e^{-3x} + \frac{2}{3} = y(x) = C'e^{-3x} + \frac{2}{3} \]

But \( y(1) = 1 \), we get

\[1 = y(1) = C'e^{-3} + \frac{2}{3} \implies C' = \frac{1}{3}e^3\]

and hence \( y(x) = \frac{2}{3} + \frac{1}{3}e^{-3x} \). Letting \( x \to \infty \) yields \( \lim_{x \to \infty} y(x) = 2/3 \). \( \square \)

**Case 2:** \( g(y) = k(y - a)(y - b) \), where \( k \neq 0 \). In this case, (8.1.7) becomes

\[\int \frac{dy}{(y - a)(y - b)} = \int k \, dx.\]

When \( a = b \), we immediately have

\[\int \frac{dy}{(y - a)^2} = \int k \, dx \implies -\frac{1}{y - a} = kx + C\]

and hence

\[(8.1.9) \quad y = a - \frac{1}{kx + C}, \quad \text{if } a = b.\]

When \( a \neq b \), the rational function \( 1/(y - a)(y - b) \) can be decomposed as

\[\frac{1}{(y - a)(y - b)} = \frac{A}{y - a} + \frac{B}{y - b}\]

according to (7.3.1). Since

\[\frac{A}{y - a} + \frac{B}{y - b} = \frac{(A + B)y - (Ab + Ba)}{(y - a)(y - b)}\]

we must have

\[A + B = 0 \quad \text{and} \quad Ab + Ba = -1.\]

Thus

\[A = \frac{1}{a - b}, \quad B = \frac{1}{b - a}.\]
and
\[
\frac{1}{(y-a)(y-b)} = \frac{1}{a-b} \left( \frac{1}{y-a} - \frac{1}{y-b} \right)
\]
since \(a \neq b\). Consequently,
\[
\frac{1}{a-b} \ln \left| \frac{y-a}{y-b} \right| = kx + C_1
\]
for some constant \(C_1\), and
\[
\frac{y-a}{y-b} = Ce^{(a-b)kx}, \quad C := \pm e^{C_1(a-b)}.
\]

We now arrive at
\[
(8.1.10) \quad y(x) = y = \frac{a - bCe^{(a-b)kx}}{1 - Ce^{(a-b)kx}}, \quad \text{if } a \neq b.
\]

**Example 8.1.3.** Solve
\[
\frac{dy}{dx} = (y - 2)^2, \quad y(0) = 1.
\]

**Proof.** From (8.1.9), we obtain
\[
y(x) = 2 - \frac{1}{x + C}.
\]
Since \(y(0) = 1\), it follows that
\[
1 = 2 - \frac{1}{0 + C} \implies C = 1.
\]
Hence \(y(x) = 2 - \frac{1}{x+1} = \frac{2x+1}{x+1}\). \(\square\)

**Example 8.1.4.** Solve
\[
\frac{dy}{dx} = 2(y - 1)(y + 2), \quad y(0) = 2.
\]

**Proof.** By (8.1.10) we have
\[
y(x) = \frac{1 + 2Ce^{6x}}{1 - Ce^{6x}}.
\]
Since \(y(0) = 2\), it follows that
\[
2 = \frac{1 + 2C}{1 - C} \implies C = \frac{1}{4}.
\]
Thus
\[
y(x) = \frac{4 + 2e^{6x}}{4 - e^{6x}}.
\]
\(\square\)
8.2. Equilibria and stability

We revisit Example 8.1.2:

\[
\frac{dy}{dx} = 2 - 3y, \quad y(x) = \frac{2}{3} + Ce^{-3x}.
\]

Letting \( x \to \infty \), we have

\[
\lim_{x \to \infty} y(x) = \frac{2}{3}.
\]

Observe that \( \frac{2}{3} \) is the solution of \( 2 - 3y = 0 \).

8.2.1. Equilibria. Consider the differential equations

\[
\frac{dy}{dx} = g(y).
\]

**Definition 8.2.1.** If \( \hat{y} \) satisfies \( g(\hat{y}) = 0 \), then \( \hat{y} \) is called an *equilibrium* of

\[
\frac{dy}{dx} = g(y).
\]

For example, \( \frac{2}{3} \) is an equilibrium of \( \frac{dy}{dx} = 2 - 3y \).

**Example 8.2.2.** Consider the differential equation

\[
\frac{dy}{dx} = 2 - 3y.
\]

The equilibrium \( \frac{2}{3} \) is also a solution. We now consider a new function

\[ y(x) := \frac{2}{3} + z(x) \]

for some differential function \( z(x) \); we may think of \( z(x) \) being a small perturbation of \( \frac{2}{3} \). If \( y(x) \) satisfies the above differential equation, then

\[
\frac{dz}{dx} = \frac{dy}{dx} = 2 - 3 \left( \frac{2}{3} + z \right) = -3z.
\]

Hence

\[ z(x) = z = C' e^{-3x} \]

for some constant \( C' \), and \( y(x) = \frac{2}{3} + C' e^{-3x} \). Note that

\[
\lim_{x \to \infty} y(x) = \lim_{x \to \infty} \left( \frac{2}{3} + C' e^{-3x} \right) = \frac{2}{3}.
\]

Thus, any small perturbation of \( \frac{2}{3} \) returns back to \( \frac{2}{3} \). Such an equilibrium is called *locally stable*.

**Example 8.2.3.** Look at the differential equation

\[
\frac{dy}{dx} = 1 + y.
\]
The equilibrium is \( \hat{y} = -1 \). Consider a new function
\[
y(x) := -1 + z(x)
\]
for some differential function \( z(x) \). If \( y(x) \) satisfies the above differential equation, then
\[
\frac{dz}{dx} = \frac{dy}{dx} = 1 + (-1 + z) = z \implies z(x) = z(0)e^x
\]
and
\[
y(x) = z(0)e^x - 1.
\]
However, in this case \( \lim_{z \to \infty} y(x) = \infty \) or \( -\infty \). Such an equilibrium \( -1 \) is called unstable.

We now give a precise definition of stabilities. Let \( \hat{y} \) be an equilibrium of \( \frac{dy}{dx} = g(y) \). Then
\[
(8.2.1) \quad g(\hat{y}) = 0
\]
by the definition. Consider a small perturbation
\[
(8.2.2) \quad y(x) = \hat{y} + z(x)
\]
for some differential function \( z(x) \).

**Definition 8.2.4.** We say an equilibrium \( \hat{y} \) of \( \frac{dy}{dx} = g(y) \) is **locally stable** if any small perturbation (8.2.2) returns back to \( \hat{y} \). Otherwise, we say \( \hat{y} \) is **unstable**.

If \( y(x) \) defined by (8.2.2) satisfies \( \frac{dy}{dx} = g(y) \), then
\[
(8.2.3) \quad \frac{dz}{dx} = \frac{dy}{dx} = g(\hat{y} + z).
\]
Since \( z \) is very small, by the linear approximation, we have
\[
g(\hat{y} + z) - g(\hat{y}) \approx g'(\hat{y})z;
\]
using (8.2.1), we may think
\[ g(\hat{y} + z) = g'(\hat{y})z. \]

Define
\[ \lambda := g'(\hat{y}). \]

Then (8.2.3) becomes
\[ \frac{dz}{dx} = \lambda z \implies z(x) = z = Ce^{\lambda x}; \]
thus
\[ y(x) = \hat{y} + Ce^{\lambda x}. \]

8.2.2. Stabilities. From (8.2.6), we arrive at

**Theorem 8.2.5.** Consider the differential equation
\[ \frac{dy}{dx} = g(y) \]
where \( g(y) \) is a differentiable function. Assume that \( \hat{y} \) is an equilibrium; that is, \( g(\hat{y}) = 0 \). Then
- \( \hat{y} \) is locally stable if \( g'(\hat{y}) < 0 \), and
- \( \hat{y} \) is unstable if \( g'(\hat{y}) > 0 \).

When \( g'(\hat{y}) = 0 \), we cannot make any conclusions about the behavior of \( z(x) \), since higher-order terms then become important. We will not discuss this case in the note.
CHAPTER 9

Linear algebra

In this chapter, we introduce an important concept, matrix, in solving linear equations, and also discuss its basic properties.

9.1. Linear systems

The standard form of a linear equation in two variables is

\[ \begin{align*}
Ax + By &= C, \\
Dx + Ey &= F,
\end{align*} \]

where \( A, B, C, D, E, F \) are constants and \( x, y \) are the two variables. We now consider two linear equations in two variables, i.e., the system of two linear equations in two variables,

\[ \begin{align*}
Ax + By &= C, \\
Dx + Ey &= F,
\end{align*} \]

where \( A, B, C, D, E, F \) are constants and \( x, y \) are the two variables. (We require that \( A, B \) and that \( D, E \) are not both equal to 0.) An example arising from rational functions is

\[ \frac{1}{(y - a)(y - b)} = \frac{A}{y - a} + \frac{B}{y - b}. \]

when we say that we “solve” \((9.1.2)\) for \( x \) and \( y \), we mean that we find an ordered pair \((x, y)\) that satisfies each equation of the system \((9.1.2)\). Equivalently, a solution of \((9.1.2)\) is the point of intersection of these two lines.

1. The two lines have exactly one point of intersection. In this case, the system \((9.1.2)\) has exactly one solution (Figure 9.1).

2. The two lines are parallel and do not intersect. In this case, the system \((9.1.2)\) has no solution (Figure 9.1).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{linear-systems.png}
\caption{Linear system of two linear equations in two variables. The first figure indicates the two lines have exactly one point of intersection; the second figure indicates the two lines are parallel but not intersect; the third figure indicates the two lines are identical.}
\end{figure}
The two lines are parallel and intersect. In this case, the system (9.1.2) has no solution (Figure 9.1).

**Example 9.1.1.** (Exactly one solution) Find the solution of
\[ x + y = 5, \quad -\frac{1}{2}x + y = 2. \]

**Proof.** From the first linear equation, we have \( y = 5 - x \) and then
\[ 2 = y - \frac{1}{2}x = (5 - x) - \frac{1}{2}x = 5 - \frac{3}{2}x; \]
hence \( x = 2 \) and \( y = 3 \).

**Example 9.1.2.** (No solution) Find the solution of
\[ 3x - 2y = 2, \quad 3x - 2y = -2. \]

**Proof.** From Figure 9.1, we have already known that there are no solutions. It can also be seen from the following:
\[ 2 = 3x - 2y = -2, \]
impossible!

**Example 9.1.3.** (Infinitely many solutions) Find the solution of
\[ 3x + 4y = 12, \quad 6x + 8y = 24. \]

**Proof.** If \( (x, y) \) is a solution of the first equation, then
\[ 3x + 4y = 12 \implies 2(3x + 4y) = 2 \times 12 = 24; \]
thus \( (x, y) \) is a solution of the second equation. Conversely, if \( (x, y) \) is a solution of the second equation, then
\[ 6x + 8y = 24 \implies \frac{1}{2}(6x + 8y) = \frac{1}{2} \times 24 = 12; \]
thus \( (x, y) \) is a solution of the first equation. Hence the system has infinitely many solutions \{\((t, 4 - 2t) : t \in \mathbb{R}\}\).

A basic idea to solve system of two linear equations is
\[ \ast x + \ast y = \ast \implies \ast x + \ast y = \ast \]
from which we can solve \( y \) by the second linear equation and then \( x \) by the first linear equation.

**Example 9.1.4.** Solve
\[
\begin{align*}
3x + 2y &= 8 \quad (R_1) \\
2x + 4y &= 5 \quad (R_2)
\end{align*}
\]
Proof. By the above idea we have, eliminating $x$ in the second equation,

$$(R_1) \, 3x + 2y = 8 \quad (R_3)$$
$$2(R_1) - 3(R_2) \quad \implies \quad -8y = 1 \quad (R_4)$$

Hence $y = -1/8$ and

$$3x + 2 \left( -\frac{1}{8} \right) = 8 \quad \implies \quad 3x = 8 + \frac{1}{4} = \frac{33}{4} \quad \implies \quad x = \frac{11}{4}.$$ 

Thus the solution is $(11/4, -1/8)$. \hfill \Box

A $2 \times 2$ matrix is a rectangular array of numbers

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

The elements $a_{ij}$ of the matrix $A$ are called entries. In Example 9.1.4 we can form a $2 \times 2$ matrix

$$A = \begin{pmatrix} 3 & 2 \\ 2 & 4 \end{pmatrix}$$

from $(R_1)$ and $(R_2)$, and another $2 \times 2$ matrix

$$B = \begin{pmatrix} 3 & 2 \\ 0 & -8 \end{pmatrix}.$$ 

We call $B$ an upper triangular matrix. Our basic idea in solving linear system is to transform a $2 \times 2$ matrix into an upper triangular $2 \times 2$ matrix.

9.1.1. Solving systems of linear equations. We now consider system of three linear equations

$$a_{11}x + a_{12}y + a_{13}z = b_1$$
$$a_{21}x + a_{22}y + a_{23}z = b_2$$
$$a_{31}x + a_{32}y + a_{33}z = b_3$$

A basic idea is

$$\begin{align*}
*x + *y + *z &= * \\
*x + *y + *z &= * \\
*x + *y + *z &= * \\
\Rightarrow \quad &*y + *z = * \\
&*z = *
\end{align*}$$

Example 9.1.5. Solve

$$\begin{align*}
3x + 5y - z &= 10 \quad (R_1) \\
2x - y + 3z &= 9 \quad (R_2) \\
4x + 2y - 3z &= -1 \quad (R_3)
\end{align*}$$

Proof. We first eliminate $x$

$$\begin{align*}
(R_1) \quad &3x + 5y - z = 10 \quad (R_4) \\
2(R_1) - 3(R_2) \quad &13y - 11z = -7 \quad (R_5) \\
2(R_2) - (R_3) \quad &-4y + 9z = 19 \quad (R_6)
\end{align*}$$

and next eliminate $y$

$$\begin{align*}
(R_4) \quad &3x + 5y - z = 10 \quad (R_7) \\
(R_5) \quad &13y - 11z = -7 \quad (R_8) \\
4(R_5) + 13(R_6) \quad &73z = 219 \quad (R_9)
\end{align*}$$
Solving equation \((R_0)\) for \(z\) yields \(z = 3\); solving \((R_8)\) for \(y\) and substituting the value of \(z\) gives
\[
y = \frac{1}{13}(-7 + 11z) = \frac{1}{13}(-7 + 11 \times 3) = 2.
\]
Finally, from \((R_7)\), we get \(x = 1\). \(\square\)

**Example 9.1.6.** (No solution) Solve
\[
\begin{align*}
2x - y + z &= 3 \ (R_1) \\
4x - 4y + 3z &= 2 \ (R_2) \\
2x - 3y + 2z &= 1 \ (R_3)
\end{align*}
\]
**Proof.** We first eliminate \(x\)
\[
\begin{align*}
(R_1) \quad 2x - y + z &= 3 \ (R_4) \\
2(R_1) - (R_2) \quad 2y - z &= 4 \ (R_5) \\
(R_1) - (R_3) \quad 2y - z &= 2 \ (R_6)
\end{align*}
\]
Equations \((R_5)\) and \((R_6)\) give us a contradiction. Hence the system has no solutions. \(\square\)

**Example 9.1.7.** (Infinitely many solutions) Solve
\[
\begin{align*}
x - 3y + z &= 4 \ (R_1) \\
x - 2y + 3z &= 6 \ (R_2) \\
2x - 6y + 2z &= 8 \ (R_3)
\end{align*}
\]
**Proof.** We first eliminate \(x\)
\[
\begin{align*}
(R_1) \quad x - 3y + z &= 4 \ (R_4) \\
(R_2) - (R_1) \quad y + 2z &= 2 \ (R_5) \\
(R_1) - \frac{1}{2}(R_3) \quad 0z &= 0 \ (R_6)
\end{align*}
\]
Thus we can take \(z\) to be any number \(t\). Solving \((R_5)\) for \(y\) yields
\[
y = 2 - 2z = 2 - 2t;
\]
solving \((R_4)\) gives us
\[
x = 4 + 3y - z = 4 + 3(2 - 2t) - t = 10 - 7t.
\]
Hence the solution is the set \(\{(x, y, z) : x = 10 - 7t, y = 2 - 2t, z = t, \ t \in \mathbb{R}\}\). \(\square\)

**9.1.2. Matrices.** We now introduce matrices in general forms.

**Definition 9.1.8.** A **matrix** is a rectangular array of numbers
\[
A = \begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix} = (a_{ij})_{1 \leq i \leq m, \ 1 \leq j \leq n}
\]
The elements $a_{ij}$ of the matrix $A$ are called entries. If the matrix has $m$ rows and $n$ columns, it is called $m \times n$ matrix.

If a matrix has the same number of rows as columns, it is called a square matrix. An $m \times 1$ matrix is called a column vector and a $1 \times n$ matrix is called a row vector. For example,

$$\begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & -1 \\ 5 & 4 & 3 \end{pmatrix}, \quad \begin{pmatrix} 2 \\ 7 \\ 4 \end{pmatrix}, \quad (1 \ 3 \ 0 \ 5)$$

are $3 \times 3$ square matrix, $3 \times 1$ column vector, and $1 \times 3$ row vector, respectively.

If $A$ is a square matrix, then the diagonal line of $A$ consists of the elements $a_{11}, \cdots, a_{nn}$.

The matrix $A$ in (9.1.3) can be thought of the coefficient matrix of the linear system

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

(9.1.4)

To solve systems of linear equations with the use of matrices, we introduce the augmented matrix—the coefficient matrix of the linear system (9.1.4), augmented by an additional column representing the right-hand side of (9.1.4). The augmented matrix representing the linear system (9.1.4) is therefore

$$\begin{bmatrix}
    a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\
    a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    a_{m1} & a_{m2} & \cdots & a_{mn} & b_m 
\end{bmatrix}$$

(9.1.5)

Example 9.1.9. Consider Example 9.1.5. The augmented matrix is given by

$$\begin{pmatrix}
    3 & 5 & -1 & 10 \\
    2 & -1 & 3 & 9 \\
    4 & 2 & -3 & -1
\end{pmatrix} (R_1)$$

$$\begin{pmatrix}
    3 & 5 & -1 & 10 \\
    0 & 13 & -11 & -7 \\
    0 & -4 & 9 & 19
\end{pmatrix} (R_2)$$

$$\begin{pmatrix}
    3 & 5 & -1 & 10 \\
    0 & 13 & -11 & -7 \\
    0 & 0 & 73 & 219
\end{pmatrix} (R_3)
$$

Then

$$2(R_1) - 3(R_2)$$

and

$$4(R_3) + 13(R_6)$$

When a system has fewer equations than variables, we say that it is underdetermined; when a system has more equations than variables, we say that it is overdetermined.
Example 9.1.10. Solve the underdetermined system
\[
\begin{align*}
2x + 2y - z &= 1 \ (R_1) \\
2x - y + z &= 2 \ (R_2)
\end{align*}
\]

**Proof.**
\[
\begin{bmatrix}
2 & 2 & -1 \\
2 & -1 & 1
\end{bmatrix}
\begin{bmatrix}
R_1 \\
R_2
\end{bmatrix}
\]
We first eliminate \(x\)
\[
(R_1) - (R_2)
\begin{bmatrix}
2 & 2 & -1 & 1 \\
0 & 3 & -2 & -1
\end{bmatrix}
\]
Translating this matrix block into a system of equations, we obtain
\[
\begin{align*}
2x + 2y - z &= 1 \\
3y - 2z &= -1
\end{align*}
\]
It then follows that
\[
\begin{align*}
y &= \frac{1}{3} + \frac{2}{3}z \\
x &= \frac{1}{2}(1 - 2y + z) = \frac{5}{6} - \frac{1}{6}z.
\end{align*}
\]
By a dummy variable \(t\), we get \(\{(x, y, z) : x = \frac{5}{6} - \frac{1}{6}t, \ y = -\frac{1}{3} + \frac{2}{3}t, \ z = t, \ t \in \mathbb{R}\}\). ☐

Example 9.1.11. Solve the following overdetermined system
\[
\begin{align*}
2x - y &= 1 \ (R_1) \\
x + y &= 2 \ (R_2) \\
x - y &= 3 \ (R_3)
\end{align*}
\]

**Proof.** We eliminate \(x\)
\[
\begin{align*}
(R_1) & \quad 2x - y = 1 \ (R_4) \\
(2R_2) - (R_1) & \quad 3y = 3 \ (R_5) \\
(R_2) - (R_3) & \quad 2y = -1 \ (R_6)
\end{align*}
\]
The equations \((R_5)\) and \((R_6)\) give us a contradiction. Hence this system has no solutions. ☐

9.2. Matrices

Recall that an \(m \times n\) matrix \(A\) is a rectangular array of numbers with \(m\) rows and \(n\) columns:
\[
A = \begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix} = (a_{ij})
\]
9.2. MATRICES

Definition 9.2.1. Suppose that $A = (a_{ij})$ and $B = (b_{ij})$ are two matrices. Then $A = B$ if and only if, $a_{ij} = b_{ij}$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$.

9.2.1. Matrix operations. Suppose we have two $m \times n$ matrices $A = (a_{ij})$ and $B = (b_{ij})$.

Definition 9.2.2. We define the sum of $A$ and $B$, whose entries are $c_{ij} = a_{ij} + b_{ij}$ for $1 \leq i \leq m$ and $1 \leq j \leq m$. If $c$ is a scalar, then $cA$ is an $m \times n$ matrix with entries $ca_{ij}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$.

Example 9.2.3. If

$A = \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 1 \\ -1 & -3 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$

then

$A + 2B - 3C = \begin{pmatrix} -1 & 5 \\ -1 & -15 \end{pmatrix}$

Definition 9.2.4. Suppose that $A = (a_{ij})$ is an $m \times n$ matrix. Then the transpose of $A$, denoted by $A'$, is an $n \times m$ matrix with entries $a'_{ij} := a_{ji}$.

For example, if

$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad C = \begin{pmatrix} 3 & 4 \end{pmatrix}$

then

$A' = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} \quad B' = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad C' = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

Clearly

(4) $(A + B)' = A' + B'$,
(5) \((A')' = A\),
(6) \((kA)' = kA'\).

### 9.2.2. Matrix multiplication.

We now give the definition of matrix multiplication.

**Definition 9.2.5.** Suppose that \(A = (a_{ij})\) is an \(m \times \ell\) matrix and \(B = (b_{ij})\) is an \(\ell \times n\) matrix. Then

\[ C := AB \]

is an \(m \times n\) matrix with

\[ c_{ij} := \sum_{k=1}^{\ell} a_{ik} b_{kj} = a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots + a_{i\ell} b_{\ell j} \]

for \(1 \leq i \leq m\) and \(1 \leq j \leq n\).

For example, if

\[ A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \]

then \(C = AB\) is a \(2 \times 3\) matrix and

\[ C = \begin{pmatrix} 30 & 36 & 42 \\ 66 & 81 & 96 \end{pmatrix} \]

If \(A\) and \(B\) are \(1 \times 1\) matrix, then

\(A = (a), \quad B = (b)\);

thus \(A\) and \(B\) can be viewed as two numbers. Note that

\(AB = (ab) = (ba) = BA\).

However, \(AB = BA\) is not true for general matrices.

**Example 9.2.6.** Suppose

\[ A = \begin{pmatrix} 2 & 1 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \]

Then

\[ AB = 1 \neq BA = \begin{pmatrix} 2 & 1 & -1 \\ -2 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \]

Clearly

(7) \((A + B)C = AC + BC\),
(8) \(A(B + C) = AB + AC\),
(9) \((AB)C = A(BC)\),
(10) \(A0 = 0A = 0\),
(11) \((AB)' = B'A'\).
If $A$ is a square matrix, we define

\[(9.2.1) \quad A^k := A^{k-1}A = AA^{k-1} = A \cdots A.\]

For instance, $A^2 = AA$, $A^3 = AAA$, and so on.

The **identity matrix**, denoted by $I_n$, is an $n \times n$ matrix with 1's on its diagonal line and 0's elsewhere; that is,

\[(9.2.2) \quad I_m = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \]

For example,

$I_1 = (1)$, $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Note that

12. $I_m^n = I_n$ for any positive integer $k$.
13. $AI_n = I_mA = A$ for any $m \times n$ matrix.

**9.2.3. Matrix equations.** The linear system (9.1.4) has the matrix (9.1.3). If we introduce

\[B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}\]

then (9.1.4) can be written as

\[(9.2.3) \quad AX = B.\]

If $A, X, B$ are $1 \times 1$ matrices, then $X = A^{-1}B$ by (9.2.3). In general, we can also obtain $X = A^{-1}B$ from (9.2.3), provided we can give a definition of $A^{-1}$—the inverse matrix of $A$.

**9.2.4. Inverse matrices.** Suppose that $A = (a_{ij})$ is an $n \times n$ square matrix. If there exists an $n \times n$ square matrix $B$ such that

\[(9.2.4) \quad AB = BA = I_n,\]

then $B$ is called the inverse matrix of $A$ and is denoted by $A^{-1}$. If $A$ has an inverse matrix, $A$ is called **invertible** or **nonsingular**; if $A$ does not have an inverse matrix, $A$ is called **singular**.

**Example 9.2.7.** Show that the $2 \times 2$ matrix

\[A = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}\]

is nonsingular.
Proof. We shall find a $2 \times 2$ matrix $B$ such that $AB = BA = I_2$. Write

$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

and then

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2 = AB = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} b_{11} + 2b_{21} & b_{12} + 2b_{22} \\ -b_{21} & -b_{22} \end{pmatrix}.$$ 

Consequently,

$$b_{21} = 0, \ b_{22} = -1, \ b_{11} + 2b_{21} = 1, \ b_{12} + 2b_{22} = 0,$$

and we have

$$B = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}.$$ 

Finally, we check that $BA = I_2$:

$$BA = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$ 

In this case, we have $A^{-1} = B = A$. However, $A^{-1}$ is in general not equal to $A$. □

From Example 9.2.7, we see that for any given nonsingular matrix its inverse matrix has the exact form.

Remark 9.2.8. (1) If $A$ is invertible, then its inverse matrix is unique.

Proof. Suppose $B$ and $C$ are two inverse matrices of $A$. By the definition, we have $BA = I_n$ and $AC = I_n$. Hence

$$B = BI_n = B(AC) = (BA)C = I_n C = C.$$ 

Thus $B = C$. □

(2) If $A$ is a nonsingular $2 \times 2$ matrix and $A = A^{-1}$, then

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{11}^2 + a_{12}a_{21} & (a_{11} + a_{22})a_{12} \\ a_{21}(a_{11} + a_{22}) & a_{21}a_{12} + a_{22}^2 \end{pmatrix}.$$ 

(a) If $a_{11} + a_{22} = 0$, then

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -(a_{11}a_{22} - a_{12}a_{21}) \\ 0 \end{pmatrix}$$
and $a_{11}a_{22} - a_{12}a_{21} = -1$. We call the number $a_{11}a_{22} - a_{12}a_{21}$ the determinant of $A$:

(9.2.5) $\det(A) := a_{11}a_{22} - a_{12}a_{21}$, if $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$. 

(b) If $a_{11} + a_{22} \neq 0$, then $a_{12} = a_{21} = 0$ and

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a_{11}^2 & 0 \\ 0 & a_{22}^2 \end{pmatrix}.$$
consequently, \((a_{11}, a_{22}) = (1, 1)\) or \((-1, -1)\). Hence
\[
A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2, \quad \text{or} \quad A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I_2.
\]
In summary, if \(A\) is a nonsingular \(2 \times 2\) matrix and \(A = A^{-1}\), then
\[
A = I_2 \quad \text{or} \quad -I_2 \quad \text{or} \quad \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & -a_{11} \end{pmatrix} \quad \text{with} \quad 1 = a_{11}^2 + a_{12}a_{21}.
\]

There are basic properties for inverse matrices.

**Proposition 9.2.9.** (1) If \(A\) is an inverse \(n \times n\) matrix, then \(A^{-1}\) is also nonsingular and
\[
(A^{-1})^{-1} = A.
\]
(2) If \(A\) and \(B\) are invertible \(n \times n\) matrices, then
\[
(AB)^{-1} = B^{-1}A^{-1}.
\]

**Proof.** (1) Since \(A\) is invertible, it follows that
\[
AA^{-1} = A^{-1}A = I_n.
\]
This equation also says that \(A\) is the inverse matrix of \(A^{-1}\); that is \(A = (A^{-1})^{-1}\).

(2) Let \(C = AB\). We shall check that \((B^{-1}A^{-1})C = C(B^{-1}A^{-1}) = I_n\). For example,
\[
(B^{-1}A^{-1})C = (B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}I_nB = B^{-1}B = I_n.
\]
Similarly we can check the second identity. \(\square\)

We now try to find the inverse matrix of general \(2 \times 2\) matrices. Recall that if
\[
A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}
\]
is a \(2 \times 2\) matrix, then its determinant is given by
\[
\det(A) := a_{11}a_{22} - a_{12}a_{21}.
\]
For any \(\lambda \in \mathbb{R}\) we consider the matrix
\[
A - \lambda I_2 = \begin{pmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{pmatrix};
\]
then its determinant is given by
\[
\det(A - \lambda I_2) = (a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21} = \lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21}).
\]
Define the **trace** of \(A\) by
\[
\text{tr}(A) := a_{11} + a_{22}
\]
the sum of diagonal line of \(A\). Then
\[
\det(A - \lambda I_2) = \lambda^2 - \text{tr}(A)\lambda + \det(A).
\]
Proposition 9.2.10. (1) If \( A \) and \( B \) are two \( 2 \times 2 \) matrices, then
\[
\text{tr}(A + B) = \text{tr}(A) + \text{tr}(B), \quad \det(AB) = \det(A) \cdot \det(B).
\]
(2) If \( A \) is an invertible \( 2 \times 2 \) matrix, then \( \det(A) \neq 0 \) and
\[
\det(A^{-1}) = \frac{1}{\det(A)}.
\]

Proof. (1) Write
\[
A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}.
\]
Hence
\[
A + B = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{pmatrix}, \quad AB = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix};
\]
we then get
\[
\text{tr}(A + B) = (a_{11} + b_{11}) + (a_{22} + b_{22}) = (a_{11} + a_{22}) + (b_{11} + b_{22}) = \text{tr}(A) + \text{tr}(B).
\]
On the other hand,
\[
\det(AB) = (a_{11}b_{11} + a_{12}b_{21})(a_{21}b_{12} + a_{22}b_{22})
\]
\[
- (a_{11}b_{12} + a_{12}b_{21})(a_{21}b_{11} + a_{22}b_{21}) = a_{11}a_{21}b_{11}b_{12} + a_{12}a_{21}b_{21}b_{12} + a_{11}a_{22}b_{11}b_{22} + a_{12}a_{22}b_{21}b_{22}
\]
\[
- a_{11}a_{21}b_{12}b_{11} - a_{12}a_{21}b_{22}b_{11} - a_{11}a_{22}b_{12}b_{21} - a_{12}a_{22}b_{22}b_{21}
\]
\[
= a_{12}a_{21}(b_{21}b_{12} - b_{22}b_{11}) + a_{11}a_{22}(b_{11}b_{22} - b_{12}b_{21})
\]
\[
= \det(A) \cdot \det(B).
\]
(2) By \( AA^{-1} = I_2 \) and (1) we have
\[
\det(A) \cdot \det(A^{-1}) = \det(I_2) = 1
\]
and hence \( \det(A^{-1}) = 1/\det(A) \). \( \square \)

Write
\[
A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad A^{-1} = B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}
\]
By Proposition 9.2.10 \( \det(A) \neq 0 \). From \( AB = I_2 \) we have
\[
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2 = AB = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix}.
\]
We get four equations:

\[ a_{11}b_{11} + a_{12}b_{21} = 1, \quad (R_1) \]
\[ a_{21}b_{11} + a_{22}b_{21} = 0, \quad (R_2) \]

and

\[ a_{11}b_{12} + a_{12}b_{22} = 0, \quad (R_3) \]
\[ a_{21}b_{12} + a_{22}b_{22} = 1. \quad (R_4) \]

From \( a_{11}(R_2) - a_{21}(R_1) \) we find that

\[ \det(A)b_{21} = -a_{21}; \]

since \( \det(A) \neq 0 \), it follows that

\[ b_{21} = -\frac{a_{21}}{\det(A)}. \]

Substituting \( b_{21} \) into \((R_1)\) and solving for \( b_{11} \), we obtain

\[ b_{11} = \frac{a_{22}}{\det(A)}. \]

Similarly, using \((R_3)\) and \((R_4)\), we arrive at

\[ b_{12} = -\frac{a_{12}}{\det(A)}, \quad b_{22} = \frac{a_{11}}{\det(A)}. \]

**Theorem 9.2.11.** If

\[ A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \]

is invertible, then

\[ A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix} \]

Consequently, a \( 2 \times 2 \) matrix \( A \) is invertible if and only if \( \det(A) \neq 0 \).

**Example 9.2.12.** Find the inverse of

\[ A = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix} \]

**Proof.** By Theorem 9.2.11, since \( \det(A) = 2 \times 3 - 1 \times 5 = 1 \), \( A \) is invertible. Using (9.2.13) we have

\[ A^{-1} = \frac{1}{1} \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix} \]

□

**Example 9.2.13.** Check whether

\[ A = \begin{pmatrix} 2 & 6 \\ 1 & 3 \end{pmatrix} \]

is invertible.
Proof. Since det(A) = 2 × 3 − 1 × 6 = 0, A is not invertible.

Any system of linear equations can be written as

\[ AX = B. \]  

(9.2.14)

In a linear system of \( n \) equations in \( n \) unknowns, \( A \) is an \( n \times n \) matrix. If \( A \) is invertible, then

\[ A^{-1}B = A^{-1}(AX) = (A^{-1}A)X = I_nX = X \]

by (9.2.14).

Example 9.2.14. (Example 9.1.4 revisited) Solve

\[
\begin{align*}
3x + 2y &= 8 \\
2x + 4y &= 5
\end{align*}
\]

Proof. Let

\[
A = \begin{pmatrix} 3 & 2 \\ 2 & 4 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \end{pmatrix}, \quad B = \begin{pmatrix} 8 \\ 5 \end{pmatrix}.
\]

Then \( AX = B \). Since \( \det(A) = 12 - 4 = 8 \neq 0 \), it follows that \( A \) is invertible and

\[
A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} 4 & -2 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} 1/2 & -1/4 \\ -1/4 & 3/8 \end{pmatrix}
\]

Using (9.2.15) implies

\[
X = A^{-1}B = \begin{pmatrix} 1/2 & -1/4 \\ -1/4 & 3/8 \end{pmatrix} \begin{pmatrix} 8 \\ 5 \end{pmatrix} = \begin{pmatrix} 4 - 5 \frac{5}{8} \\ -2 + 1 \frac{5}{8} \end{pmatrix} = \begin{pmatrix} 11/8 \\ 1/8 \end{pmatrix}.
\]

Consider a special case of (9.2.14):

\[ AX = 0 \]

(9.2.16)

\[ A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \]

Clearly that \( X = 0 \) is a solution of (9.2.16), called the trivial solution. If \( AX = 0 \) has a solution \( X \neq 0 \), then \( X \) is called a nontrivial solution.

Theorem 9.2.15. Suppose that \( A \) is a \( 2 \times 2 \) matrix, and \( X \) is a \( 2 \times 1 \) matrix. Then the equation \( AX = 0 \) has a nontrivial solution if and only if \( \det(A) = 0 \).

Proof. If \( A \) is invertible, then \( \det(A) \neq 0 \) and \( X = A^{-1}0 = 0 \). Hence \( X \neq 0 \) if and only if \( \det(A) = 0 \).

Example 9.2.16. Let

\[
A = \begin{pmatrix} a & 3 \\ 2 & 1 \end{pmatrix}.
\]
Determine $a$ so that $AX = 0$ has at least one nontrivial solution and find the nontrivial solution(s).

**Proof.** Since $\det(A) = a-6$, it follows that $AX = 0$ has at least one nontrivial solution if and only if $a = 6$. In this case,

\[
\begin{pmatrix} 6 & 3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix};
\]

thus

\[
6x + 3y = 0, \quad 2x + y = 0,
\]
or $y = -2x$. Hence the system has infinitely many solutions, namely

\[
\{(x, y) : x = t, \ y = -2t, \ t \in \mathbb{R}\}.
\]

\[\square\]

**9.3. Linear maps, eigenvectors, and eigenvalues**

Let $V$ be the space of all $2 \times 1$ vectors. Then any $2 \times 2$ matrix $A$ can be viewed as a map on $V$:

\[x \mapsto Ax,\]
where $x$ is a $2 \times 1$ matrix or $2 \times 1$ column vector. We can compute $A^2x = A(Ax)$, etc. Note that

1. $A(x + y) = Ax + Ay$, and
2. $A(\lambda x) = \lambda (Ax)$ for any scalar $\lambda$.

We then call the map $x \mapsto Ax$ is **linear**.

**9.3.1. Graphical representation of vectors.** A vector $x$ is a $2 \times 1$ matrix,

\[x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.
\]

A vector can be viewed as a point $(x_1, x_2)$ in the $x_1x_2$-plane. For example the vector $x = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ is represented in Figure 9.2.
Figure 9.3. Parallelogram law

The length of $\mathbf{x}$ is defined by

\[(9.3.1) \quad r := |\mathbf{x}| := \sqrt{x_1^2 + x_2^2},\]

and the angle $\alpha$ counterclockwise formed by the vector $\mathbf{x}$ is

\[(9.3.2) \quad \tan \alpha := \frac{x_2}{x_1}.\]

We call $(r, \alpha)$ the polar coordinate system of $\mathbf{x}$:

\[(9.3.3) \quad x_1 = r \cos \alpha, \quad x_2 = r \sin \alpha.\]

**Example 9.3.1.** Consider the vector $\mathbf{x} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ in Figure 9.2. Then

\[r = |\mathbf{x}| = \sqrt{2^2 + 3^2} = \sqrt{13}, \quad \alpha = \tan^{-1} \frac{2}{3}.\]

Consider two vectors

\[a = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad b = \begin{pmatrix} 4 \\ -1 \end{pmatrix}\]

in Figure 9.3. Then

\[a + b = \begin{pmatrix} 5 \\ 1 \end{pmatrix} = b + a.\]

**Proposition 9.3.2.** Let $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$. Then

\[\mathbf{x} + \mathbf{y} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix}.\]

If $a$ is a scalar, then

\[a \mathbf{x} = \begin{pmatrix} ax_1 \\ ax_2 \end{pmatrix}.\]

The length of $a \mathbf{x}$ is equal to $|a||\mathbf{x}|$. 
9.3. Linear maps. A linear map on plane has the form
\[ x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto A x = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \]

(1) The identity map is represented by \( I_2 \):
\[ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \]
Thus the identity map leaves the vector \( x \) unchanged.

(2) Given a vector \( x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \). The vector reflecting the \( x_1 \)-axis is
\[ \begin{pmatrix} x_1 \\ -x_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \]
Hence the linear map
\[ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} ax_1 \\ bx_2 \end{pmatrix} \]
reflects vectors about \( x_1 \)-axis.

(3) Given a vector \( x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \). The vector reflecting the \( x_2 \)-axis is
\[ \begin{pmatrix} -x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \]
Hence the linear map
\[ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} ax_1 \\ bx_2 \end{pmatrix} \]
reflects vectors about \( x_2 \)-axis.

(3) A diagonal map is given by
\[ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} ax_1 \\ bx_2 \end{pmatrix}. \]

For example,
\[ \begin{pmatrix} 2 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}. \]
This map stretches the first coordinate by a factor of 2 and contracts the second coordinates by a factor of 1/2. The minus sign in front of 1/2 corresponds to reflecting the second coordinate about the \( x_1 \)-axis. This can be seen as follows:
\[ \begin{pmatrix} 2 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]

(3) Rotation. Given a vector \( x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \). Then
\[ x = \begin{pmatrix} r \cos \alpha \\ r \sin \alpha \end{pmatrix}. \]
If we rotate \( x \) by an angle \( \theta \) (\( \theta > 0 \), the rotation is counterclockwise; if \( \theta < 0 \), the rotation is clockwise by the angle \( |\theta| \)), then we get a new vector
\[ \begin{pmatrix} r \cos(\theta + \alpha) \\ r \sin(\theta + \alpha) \end{pmatrix} = \begin{pmatrix} r(\cos \theta \cos \alpha - \sin \theta \sin \alpha) \\ r(\sin \theta \cos \alpha + \cos \theta \sin \alpha) \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} r \cos \alpha \\ r \sin \alpha \end{pmatrix}. \]
thus
\[
\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} =: R_\theta \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.
\]
Clearly
\[
\det(R_\theta) = \cos^2 \theta + \sin^2 \theta = 1
\]
and
\[
R_\theta^{-1} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} = R_{-\theta}.
\]

9.3.3. Eigenvalues and eigenvectors. Recall that given a $2 \times 2$ matrix $X$, for any vector $x$, we get a new vector $Ax$. The simplest relation between those two vectors is
\[
(9.3.11) \quad Ax = \lambda x
\]
for some $\lambda \in \mathbb{R}$. For example,
\[
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x = x.
\]

**Definition 9.3.3.** Assume that $A$ is a $2 \times 2$ matrix. A nonzero vector $x$ that satisfies the equation (9.3.11),
\[
Ax = \lambda x
\]
is an eigenvector of $A$, and the number $\lambda$ is an eigenvalue of $A$.

Note that
(i) We assume that the vector $x$ in Definition 9.3.3 is different from the zero vector. The zero vector $x = 0$ always satisfies the equation $Ax = \lambda x$ and thus would not be special.
(ii) The eigenvalue $\lambda$ can be zero and however can be even be a complex number.
(iii) The action of $A$ on eigenvectors produces a particularly simple form: If we apply $A$ to an eigenvector $x$ (i.e., if we compute $Ax$), the result is a constant multiple of $x$.
(iv) If $x$ is an eigenvector of $A$ with an eigenvalue $\lambda$, then
\[
Ax = \lambda x;
\]
for any nonzero number $a$ we see that $ax$ is also an eigenvector of $A$ with the same eigenvalue $\lambda$. Hence, we have infinitely many eigenvectors to a given eigenvalue.
(v) If $x$ is an eigenvector of $A$, then from (9.3.11) we have
\[
0 = Ax - \lambda x = (A - \lambda I)x;
\]
by Theorem 9.2.15 since $x$ is nonzero, we have $\det(A - \lambda I) = 0$. By 9.2.9 we get
\[
\lambda^2 - \text{tr}(A)\lambda + \det(A) = 0.
\]
Corollary 9.3.4. If \( \mathbf{x} \) is an eigenvector of \( A \) with an eigenvalue \( \lambda \), then \( \lambda \) satisfies the quadratic equation

\[
\lambda^2 - \text{tr}(A)\lambda + \det(A) = 0.
\]

We now discuss some examples.

Example 9.3.5. Find all eigenvalues and eigenvectors of

\[
A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}
\]

Proof. Let \( \lambda \) be an eigenvector of \( A \). Then

\[
\lambda^2 - \text{tr}(A)\lambda + \det(A) = 0.
\]

Since \( \text{tr}(A) = 1 + 2 = 3 \), \( \det(A) = 1 \times 2 - 3 \times 2 = -4 \), it follows that

\[
\lambda^2 - 3\lambda - 4 = 0 \implies (\lambda + 1)(\lambda - 4) = 0.
\]

Hence \( \lambda_1 = -1 \) and \( \lambda_2 = 4 \).

(1) \( \lambda_1 = -1 \). In this case we have

\[
0 = (A + I_2)\mathbf{x} = \begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix};
\]

thus

\[
x_1 + x_2 = 0
\]

and the eigenvectors associated to \( \lambda_1 = -1 \) are of the forms

\[
\text{span} \begin{pmatrix} 1 \\ -1 \end{pmatrix} := \left\{ \begin{pmatrix} t \\ -t \end{pmatrix} : t \in \mathbb{R} \right\} = \left\{ t \begin{pmatrix} 1 \\ -1 \end{pmatrix} : t \in \mathbb{R} \right\}.
\]

Define a line \( l_1 \) represented or spanned by the vector \( \mathbf{x}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \) by

\[
l_1 = \{ (x_1, x_2) \in \mathbb{R}^2 : x_1 + x_2 = 0 \} = \text{span} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.
\]

We say a line \( l \) is invariant under the matrix \( A \), if \( A\mathbf{x} \in l \) whenever \( \mathbf{x} \in l \). Then \( l_1 \) is invariant under the matrix \( A \): for any point \( (x_1, x_2) \in l_1 \), the associated vector \( \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \) satisfies \( A\mathbf{x} = -1\mathbf{x} \), and hence \( A\mathbf{x} \in l_1 \).

(2) \( \lambda_2 = 4 \). In this case we have

\[
0 = (A - 4I_2)\mathbf{x} = \begin{pmatrix} -3 & 2 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix};
\]

thus

\[
3x_1 - 2x_2 = 0
\]

and the eigenvectors associated to \( \lambda_2 = 4 \) are of the forms

\[
\text{span} \begin{pmatrix} 2 \\ 3 \end{pmatrix} := \left\{ \begin{pmatrix} t \\ 3t/2 \end{pmatrix} : t \in \mathbb{R} \right\} = \left\{ 2s \begin{pmatrix} 2 \\ 3 \end{pmatrix} : s \in \mathbb{R} \right\} = \left\{ \begin{pmatrix} 2 \\ 3 \end{pmatrix} : s \in \mathbb{R} \right\}.
\]
Define a line \( l_2 \) represented or spanned by the vector \( x_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \) by
\[
l_2 = \{ (x_1, x_2) \in \mathbb{R}^2 : 3x_1 - 2x_2 = 0 \} = \text{span} \begin{pmatrix} 2 \\ 3 \end{pmatrix}.
\]
As in (1), \( l_2 \) is invariant under the matrix \( A \). \( \square \)

The second example is about \( \lambda_1 = \lambda_2 \).

**Example 9.3.6.** Find eigenvalues and eigenvectors of
\[
A = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}.
\]

**Proof.** Let \( \lambda \) be an eigenvalue of \( A \). Then
\[
\lambda^2 - \text{tr}(A)\lambda + \det(A) = 0.
\]
Since \( \text{tr}(A) = 1 + 3 = 4 \) and \( \det(A) = 1 \times 3 - 1 \times (-1) = 4 \), it follows that
\[
0 = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2.
\]
Hence \( \lambda = \lambda_1 = \lambda_2 = 2 \) and
\[
0 = (A - \lambda I_2)x = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.
\]
Thus
\[
x_1 + x_2 = 0;
\]
the eigenvectors associated to \( \lambda = 2 \) are of the forms
\[
\text{span} \left( \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right) = \left\{ \begin{pmatrix} t \\ -t \end{pmatrix} : t \in \mathbb{R} \right\} = \left\{ t \begin{pmatrix} 1 \\ -1 \end{pmatrix} : t \in \mathbb{R} \right\}.
\]
If we define \( l = \{ (x_1, x_2) \in \mathbb{R}^2 : x_1 + x_2 = 0 \} \), then \( l \) is invariant under the matrix \( A \). \( \square \)

The third example is about zero eigenvalue.

**Example 9.3.7.** Find the eigenvalues and eigenvectors of
\[
A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.
\]

**Proof.** If \( \lambda \) is an eigenvalue of \( A \), then
\[
\lambda^2 - 2\lambda + 0 = 0 \implies \lambda(\lambda - 2) = 0.
\]
Hence \( \lambda_1 = 0 \) and \( \lambda_2 = 2 \).
(1) \( \lambda_1 = 0 \). In this case, we have
\[
0 = (A - \lambda_1 I_2)x = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.
\]
thus $x_1 + x_2 = 0$ and the eigenvectors associated to $\lambda_1 = 0$ are of the forms
\[
\text{span} \left( \frac{1}{1} \right) = \left\{ \left( \frac{t}{t} \right) : t \in \mathbb{R} \right\} = \left\{ t \left( \frac{1}{1} \right) : t \in \mathbb{R} \right\}.
\]

(2) $\lambda_2 = 2$. In this case we have
\[
0 = (A - 2I_2)x = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix};
\]
thus $x_1 = x_2$, and the eigenvectors associated to $\lambda_2 = 2$ are of the forms
\[
\text{span} \left( \frac{1}{1} \right) = \left\{ \left( \frac{t}{t} \right) : t \in \mathbb{R} \right\} = \left\{ t \left( \frac{1}{1} \right) : t \in \mathbb{R} \right\}.
\]

□

The fourth example is about complex eigenvalues.

**Example 9.3.8.** Find eigenvalues of
\[
A = \begin{pmatrix} -2 & -1 \\ 3 & 1 \end{pmatrix}.
\]

**Proof.** Since $\text{tr}(A) = -2 + 1 = -1$ and $\text{det}(A) = -2 \times 1 - 3 \times (-1) = 1$, it follows that the eigenvalue $\lambda$ satisfies
\[
\lambda^2 + \lambda + 1 = 0.
\]
Hence
\[
\lambda_1 = \frac{-1 - i\sqrt{3}}{2}, \quad \lambda_2 = \frac{-1 + i\sqrt{3}}{2}.
\]
This example suggests
\[
\text{tr}(A) < 0 \text{ and } \text{det}(A) > 0 \iff \text{the real parts of } \lambda_1 \text{ and } \lambda_2 \text{ are negative.}
\]
Actually, this result holds in general. □

Suppose $\lambda_1 = a_1 + ib_1$ and $\lambda_2 = a_2 + ib_2$ are two complex eigenvalues of a $2 \times 2$ matrix $A$. Then
\[
\text{tr}(A) = \lambda_1 + \lambda_2 = (a_1 + a_2) + i(b_1 + b_2),
\]
\[
\text{det}(A) = \lambda_1 \lambda_2 = (a_1a_2 - b_1b_2) + i(a_1b_2 + a_2b_1).
\]
Since $\text{det}(A)$ and $\text{tr}(A)$ are real, it follows that
\[
a_1b_2 + a_2b_1 = 0, \quad b_1 + b_2 = 0.
\]
Hence
\[
b_2 = -b_1, \quad b_1(a_2 - a_1) = 0.
\]
(1) If $b_1 \neq 0$, then $a_1 = a_2$. Write
\[
b := b_1, \quad a := a_1;
\]
we obtain
\[
\lambda_1 = a + ib, \quad \lambda_2 = a - ib,
\]
so that

\[ \text{tr}(A) = 2a, \quad \text{det}(A) = a^2 + b^2 \geq a^2. \]

(2) If \( b_1 = 0 \), then \( b_2 = 0 \) and

\[ \lambda_1 = a_1, \quad \lambda_2 = a_2. \]

Hence

\[ \text{tr}(A) = a_1 + a_2, \quad \text{det}(A) = a_1a_2. \]

**Theorem 9.3.9.** Let \( A \) be a \( 2 \times 2 \) matrix with eigenvalues \( \lambda_1 \) and \( \lambda_2 \). Then the real parts of \( \lambda_1 \) and \( \lambda_2 \) are negative if and only if

\[ \text{tr}(A) < 0 \quad \text{and} \quad \text{det}(A) > 0. \]

**Example 9.3.10.** Show that both of eigenvalues of

\[ A = \begin{pmatrix} -1 & -3 \\ 1 & -2 \end{pmatrix} \]

have negative real part.

**Proof.** Since \( \text{tr}(A) = -1 - 2 = -3 \) and \( \text{det}(A) = -1 \times (-2) - 1 \times (-3) = 5 > 0 \), it follows from Theorem 9.3.9 that the real parts of \( \lambda_1 \) and \( \lambda_2 \) are negative. \( \square \)

**Example 9.3.11.** Let

\[ A = \begin{pmatrix} -2 & 5 \\ 2 & 3 \end{pmatrix} \]

Without explicitly computing the eigenvalues of \( A \), decide whether the real parts of both eigenvalues are negative.

**Proof.** Since \( \text{tr}(A) = -2 + 3 = 1 > 0 \) and \( \text{det}(A) = -2 \times 3 - 2 \times 5 = -16 < 0 \), we conclude that the real parts of both eigenvalues are not negative. Actually, \( \lambda_1 = (1 - \sqrt{65})/2 \) and \( \lambda_2 = (1 + \sqrt{65})/2. \) \( \square \)

**9.4. Analytic geometry**

The plane can be thought of the set of all points \((x_1, x_2)\) with \( x_1 \in \mathbb{R} \) and \( x_2 \in \mathbb{R} \). Define

\[ \mathbb{R}^2 := \{(x_1, x_2) : x_1 \in \mathbb{R}, \ x_2 \in \mathbb{R}\}. \]

Thus the notion \( \mathbb{R}^2 \) represents the plane. Similarly, we formally define

\[ \mathbb{R}^n := \{(x_1, x_2, \ldots, x_n) : x_1 \in \mathbb{R}, \ x_2 \in \mathbb{R}, \ldots, x_n \in \mathbb{R}\}. \]

For example, \( \mathbb{R}^3 \) is three-dimensional space, and \( \mathbb{R}^4 \) is the space and time.
Definition 9.4.1. A vector in $\mathbb{R}^n$ is an ordered $n$-tuple

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

of real numbers. The numbers $x_1, x_2, \cdots, x_n$ are called the components of $\mathbf{x}$.

The vector $\mathbf{x}$ in $\mathbb{R}^n$ can be represented by directed segments with initial point $(0, 0, \cdots, 0)$ and end point $(x_1, x_2, \cdots, x_n)$.

Definition 9.4.2. If

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix},$$

and $a \in \mathbb{R}$, we define

$$\mathbf{x} + \mathbf{y} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{pmatrix}, \quad a\mathbf{x} = \begin{pmatrix} ax_1 \\ ax_2 \\ \vdots \\ ax_n \end{pmatrix}.$$

Given two points $A = (a_1, \cdots, a_n)$ and $B = (b_1, \cdots, b_n)$ in $\mathbb{R}^n$. Then we can form two vectors

$$\overrightarrow{OA} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}, \quad \overrightarrow{OB} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$
Using parallelogram law, we have
\[ \overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}. \]
Hence

(9.4.2) \[ \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \\ \vdots \\ b_n - a_n \end{pmatrix}. \]

We call (9.4.2) the vector representation of \( \overrightarrow{AB} \).

**Example 9.4.3.** Find the vector representation of \( \overrightarrow{AB} \) when \( A = (2, -1) \) and \( B = (1, 3) \).

**Proof.** The vector representation of \( \overrightarrow{AB} \) is given by
\[ \overrightarrow{AB} = \begin{pmatrix} 1 - 2 \\ 3 - (-1) \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}. \]

If we shift the vector \( \overrightarrow{AB} \) to the origin, its tip ends at \((-1, 4)\), which confirms that the vector representation of \( \overrightarrow{AB} \) is \((-1, 4)\). \( \square \)

### 9.4.1. Length of vector.

The length of a vector
\[ \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \]
is

(9.4.3) \[ |\mathbf{x}| := \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}. \]

**Example 9.4.4.** Find the length of
\[ \mathbf{x} = \begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix}. \]

**Proof.** The length of \( \mathbf{x} \) is
\[ |\mathbf{x}| = \sqrt{1^2 + (-3)^2 + 4^2} = \sqrt{26}. \]
If we define
\[ \hat{\mathbf{x}} := \frac{1}{|\mathbf{x}|} \mathbf{x}, \]
then
\[ \hat{\mathbf{x}} = \frac{1}{\sqrt{26}} \begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{26}}{26} \\ -\frac{3\sqrt{26}}{26} \\ \frac{4\sqrt{26}}{26} \end{pmatrix}. \]
and \( |\hat{x}| = \sqrt{\frac{1}{26} + \frac{9}{26} + \frac{16}{26}} = 1. \)

If \( x \) is a nonzero vector, then we call

\( (9.4.4) \quad \hat{x} := \frac{1}{|x|} x \)

the normalization of \( x \); since \( |\hat{x}| = 1 \), we also call \( \hat{x} \) the unit vector in the direction of \( x \).

**Example 9.4.5.** Normalize the vector

\[
x = \begin{pmatrix} 3 \\ -6 \\ 6 \end{pmatrix}.
\]

**Proof.** Since \( |x| = \sqrt{3^2 + (-6)^2 + 6^2} = \sqrt{81} = 9 \), we get

\[
\hat{x} = \frac{1}{|x|} x = \frac{1}{9} \begin{pmatrix} 3 \\ -6 \\ 6 \end{pmatrix} = \begin{pmatrix} 1/3 \\ -2/3 \\ 2/3 \end{pmatrix}.
\]

\( \square \)

### 9.4.2. Dot product

Given two vectors

\[
x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}.
\]

The scalar product or dot product is defined by

\( (9.4.5) \quad x \cdot y = x'y = \sum_{i=1}^{n} x_i y_i = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n. \)

**Example 9.4.6.** Find the dot product of

\[
x = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}, \quad y = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}.
\]

**Proof.** \( x \cdot y = 2 \times (-1) + 3 \times 2 + 1 \times 0 = -2 + 6 = 4. \)

When \( y = x \), we have

\( (9.4.6) \quad x \cdot x = \sum_{i=1}^{n} x_i^2 = |x|^2 \implies |x| = \sqrt{x \cdot x}. \)
Theorem 9.4.7. For any three vectors \( x, y, z \), we have
\[
\begin{align*}
x \cdot y &= y \cdot x, \\
x \cdot (y + z) &= x \cdot y + x \cdot z.
\end{align*}
\]

We now discuss the angle between two vectors \( x \) and \( y \). Using the law of cosines, we find that
\[
|x - y|^2 = |x|^2 + |y|^2 - 2|x||y| \cos \theta,
\]
where \( \theta \) is the angle between \( x \) and \( y \). Since
\[
|x - y|^2 = (x - y) \cdot (x - y)
= x \cdot x - x \cdot y - y \cdot x + y \cdot y
= |x|^2 - 2x \cdot y + |y|^2,
\]
it follows from (9.4.7) that
\[
|x|^2 + |y|^2 - 2|x||y| \cos \theta = |x|^2 - 2x \cdot y + |y|^2
\]
and hence
\[
x \cdot y = |x||y| \cos \theta \implies \theta = \cos^{-1}\left(\frac{x \cdot y}{|x||y|}\right).
\]

Example 9.4.8. Find the angle between
\[
x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad y = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.
\]

Proof. Since \( |x| = |y| = \sqrt{2} \) and \( x \cdot y = 0 \), we get
\[
\theta = \cos^{-1}\left(\frac{x \cdot y}{|x||y|}\right) = \cos^{-1}(0) = 90^\circ.
\]
Thus \( x \) is perpendicular to \( y \). \( \square \)
Theorem 9.4.9. \( \mathbf{x} \) and \( \mathbf{y} \) are perpendicular if and only if \( \mathbf{x} \cdot \mathbf{y} = 0 \).

The coordinate axes of \( \mathbb{R}^2 \) are

\[
\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\]

These vectors are unit, i.e., \( |\mathbf{e}_1| = |\mathbf{e}_2| = 1 \), and

\[\mathbf{e}_1 \cdot \mathbf{e}_2 = 0 \implies \mathbf{e}_1 \perp \mathbf{e}_2.\]

For any vector

\[
\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}
\]

we have

\[
\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2.
\]

Example 9.4.10. Find \( \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \) so that \( \mathbf{y} \) is perpendicular to \( \mathbf{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \).

Proof. From \( \mathbf{x} \cdot \mathbf{y} = 0 \), we have

\[y_1 + 2y_2 = 0.\]

Hence \( \mathbf{y} \) is of the form

\[
\left\{ \begin{pmatrix} -2t \\ t \end{pmatrix} : t \in \mathbb{R} \right\} = \left\{ t \begin{pmatrix} -2 \\ 1 \end{pmatrix} : t \in \mathbb{R} \right\}.
\]

\( \square \)
9.4.3. Lines in planes. Given a vector \( \mathbf{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \) and a point \( P_0 = (x_0, y_0) \), we want to a line \( l \) passing \( P_0 \) and perpendicular to \( \mathbf{n} \). Let

\[
\mathbf{r}_0 = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}
\]

and choose a point \( P = (x, y) \) on this line \( l \). Write

\[
\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}
\]

then

\[
(9.4.10) \quad \mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0
\]

which is called the vector equation of a line in the plane. Consequently,

\[
(9.4.11) \quad 0 = (a \quad b) \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} = a(x - x_0) + b(y - y_0)
\]

which is called the scalar equation of this line.

The line through \((x_0, y_0)\) and perpendicular to \(\begin{pmatrix} a \\ b \end{pmatrix}\) has the equation

\[
(9.4.12) \quad a(x - x_0) + b(y - y_0) = 0.
\]

**Example 9.4.11.** Find the equation of the line through \((4, 3)\) and perpendicular to \(\begin{pmatrix} 1 \\ 2 \end{pmatrix}\).

Proof. \(0 = 1(x - 4) + 2(y - 3) = x + 2y - 10.\) ⊡
9.4.4. Planes in space. Given a point \((x_0, y_0, z_0)\) and a vector \(\mathbf{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}\), we want to find a plane that passes through \((x_0, y_0, z_0)\) and is perpendicular to \(\mathbf{n}\). Take a point \((x, y, z)\) on this plane; write 
\[
\mathbf{r}_0 = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}, \quad \mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.
\]
Then
\[
(9.4.13) \quad \mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0
\]
which is called the vector equation of a plane. Consequently,
\[
(9.4.14) \quad 0 = (a \ b \ c) \begin{pmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{pmatrix} = a(x - x_0) + b(y - y_0) + c(z - z_0)
\]
which is called the scalar equation of a plane.

The plane through \((x_0, y_0, z_0)\) and perpendicular to \(\mathbf{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}\) has the equation
\[
(9.4.15) \quad a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.
\]

Example 9.4.12. Find the equation of the plane in \(\mathbb{R}^3\) through \((2, 0, 3)\) and perpendicular to \(\begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}\).

**Proof.** \(0 = -1(x - 2) + 4(y - 0) + 1(z - 3) = -x + 4y + z - 1. \right)

9.4.5. Parametric equations of lines. Given a point \(P_0 = (x_0, y_0)\) and a vector \(\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\), we want to find the line that goes through \(P_0\) in the direction of \(\mathbf{u}\). For a point \(P = (x, y)\), we have \(\overrightarrow{OP} = \overrightarrow{OP_0} + \overrightarrow{P_0P}\) and 
\[
\overrightarrow{PP_0} = t\mathbf{u}, \quad t \in \mathbb{R}.
\]
Hence
\[
\begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} = t \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}
\]
which is called the parametric equation of a line.

The line through \((x_0, y_0)\) in the direction of \(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\) has the equation
\[
(9.4.16) \quad x = x_0 + tu_1, \quad y = y_0 + tu_2.
\]
Example 9.4.13. Find the parametric equation of the line in the $xy$-plane that goes through the points $(-1, 2)$ and $(3, 5)$.

Proof. Since
\[ u = \begin{pmatrix} 3 - (-1) \\ 5 - 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}, \]
it follows that
\[ x = -1 + 4t, \quad y = 2 + 3t. \]
\[ \square \]

Example 9.4.14. Find a parametric form of the line in standard form $2x - 3y + 1 = 0$.

Proof. From $y = (2x + 1)/3$, we have
\[ x = t, \quad y = \frac{2}{3}t + \frac{1}{3}. \]
Another way is
\[ x = 3t + 1, \quad y = \frac{2}{3}(3t + 1) + \frac{1}{3} = 2t + 1. \]
\[ \square \]

Given a point $P_0 = (x_0, y_0, z_0)$ and a vector $\mathbf{n} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$, we want to find the line in $\mathbb{R}^3$ that goes through $P_0$ in the direction of $\mathbf{u}$. As before, we have
The line through \((x_0, y_0, z_0)\) in the direction of \(\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}\) has the equation

\[
\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + t \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}
\]

which is called the parametric equation of a line.

**Example 9.4.15.** Find the parametric equation of the line in \(xyz\)-space that goes through the points \((1, -1, 3)\) and \((2, 4, -1)\).

**Proof.** Since

\[
\mathbf{u} = \begin{pmatrix} 2 - 1 \\ 4 - (-1) \\ -1 - 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ -4 \end{pmatrix},
\]

it follows that

\[
x = 2 + t, \quad y = 4 + 5t, \quad z = -1 - 4t
\]

for \(t \in \mathbb{R}\), or

\[
x = 1 + t, \quad y = -1 + 5t, \quad z = 3 - 4t
\]

for \(t \in \mathbb{R}\). \(\square\)
CHAPTER 10

Multi-variable calculus

In this chapter, we discuss differential calculus or functions of more than one variable.

10.1. Limits and continuity

Recall an example of functions of one variable:

\[ f : [0, 4] \to \mathbb{R}, \quad x \mapsto \sqrt{x}. \]

In differential equations, we have met with functions of two variable:

\[ \frac{dy}{dx} = e^{3x} y. \]

If we define \( g(x, y) := e^{3x} y \), then \( g(x, y) \) is a function of two variable. However, in this example, \( y \) depends on \( x \), since \( y = y(x) \) is a function of \( x \). Hence, \( g(x, y) = g(x, y(x)) \) is a function of one variable.

10.1.1. Functions of more independent variables. In general, a function of two or more variables can be defined as follows.

**Definition 10.1.1.** Suppose \( D \subset \mathbb{R}^n \). Then a **real-value function** \( f \) on \( D \) assigns a real number to each element in \( D \), and we write

\[ f : D \to \mathbb{R}, \quad (x_1, \cdots, x_n) \mapsto f(x_1, \cdots, x_n). \]

The set \( D \) is the **domain** of the function \( f \), and the set

\[ \{ w \in \mathbb{R} : f(x_1, \cdots, x_n) = w \text{ for some } (x_1, \cdots, x_n) \in D \} \]

is the **range** of the function \( f \).

If a function \( f \) depends on just two independent variables, we will often denote the independent variables by \( x \) and \( y \), and write \( f(x, y) \). In the case of three variables, we will often write \( f(x, y, z) \).

**Example 10.1.2.** Evaluate the function

\[ f(x, y, z) = \frac{xy}{z^2} \]

at the points \((2, 3, -1)\) and \((-1, 2, 3)\).
PROOF. Since \( f(x, y, z) \) lists the independent variables in the order \( x, y, z \), we just substitute the points into \( f(x, y, z) \). For example
\[
f(2, 3, -1) = \frac{2 \times 3}{(-1)^2} = 6.
\]
Similarly, \( f(-1, 2, 3) = (-1) \times 2/3^2 = -2/9 \).

Example 10.1.3. Evaluate the function
\[
f(x, y, z) = \sqrt{x^2 - 6y + z}
\]
at \((3, -1, 1)\).

PROOF. \( f(3, -1, 1) = \sqrt{3^2 - 6 \times (-1) + 1} = \sqrt{16} = 4 \).

Let \( D = \{(x, y) : |x| \leq 5, |y| \leq 5 \} \) and
\[
f : D \to \mathbb{R}, \quad (x, y) \mapsto x^2 + y^2.
\]
The domain and the range of \( f \) is graphed in Figure 10.1.

Example 10.1.4. Find the largest possible domain of the function
\[
f(x, y) = \sqrt{y^2 - x}.
\]

PROOF. Since \( y^2 - x \geq 0 \), we must have that the domain is \( D = \{(x, y) : y^2 - x \geq 0 \} \).
Definition 10.1.5. If \( f \) is a function of two independent variables with domain \( D \), then the graph of \( f \) is the set of all points \((x, y, z)\) such that \( z = f(x, y) \) for \((x, y) \in D\). That is, the graph of \( f \) is the set

\[
\text{graph}(f) = \{(x, y, z) : z = f(x, y), \ (x, y) \in D\}.
\]

For example, the graph of \( f(x, y) \) in Example 10.1.4 is

\[
\text{graph}(f) = \{(x, y, \sqrt{y^2 - x}) : (x, y) \in D\}
\]

where \( D = \{(x, y) : y^2 - x \geq 0\} \).

Definition 10.1.6. Suppose that \( f : D \to \mathbb{R}, \ D \subset \mathbb{R}^2 \). Then the level curves of \( f \) comprise the set of points \((x, y)\) in the \( xy \)-plane where the function \( f \) has a constant value; that is, \( f(x, y) = c \).

The level curves of \( f \) are graphs in \( \mathbb{R}^2 \).

Example 10.1.7. Set \( D = \{(x, y) : x^2 + y^2 \leq 4\} \). Compare the level curves of

\[
f(x, y) = 4 - x^2 - y^2, \ \ (x, y) \in D
\]

and

\[
g(x, y) = \sqrt{4 - x^2 - y^2}, \ \ (x, y) \in D.
\]

Proof. The level curves of \( f \) and \( g \) are

\[
x^2 + y^2 = 4 - c, \ \ x^2 + y^2 = 4 - c^2.
\]

If \( c = 2 \), then the level curves of \( f \) and \( g \) are

\[
x^2 + y^2 = 2, \ \ x^2 + y^2 = 0.
\]

\[\square\]

10.1.2. Limits. We say that the “limit of \( f(x, y) \) as \((x, y)\) approaches \((x_0, y_0)\) is equal to \( L\)” if \( f(x, y) \) can be made arbitrarily close to \( L \) whenever the point \((x, y)\) is sufficiently close (but not equal) to the point \((x_0, y_0)\). We denote this by

\[
\lim_{(x,y) \to (x_0,y_0)} f(x, y) = L.
\]

We now give the formal definition of limits. An open disk with radius \( r \) centered at \((x_0, y_0) \in \mathbb{R}^2 \) is the set

\[
B_r(x_0, y_0) := \{(x, y) \in \mathbb{R}^2 : \sqrt{(x-x_0)^2 + (y-y_0)^2} < r\}.
\]

A closed disk with radius \( r \) centered at \((x_0, y_0) \in \mathbb{R}^2 \) is the set

\[
\overline{B}_r(x_0, y_0) := \{(x, y) \in \mathbb{R}^2 : \sqrt{(x-x_0)^2 + (y-y_0)^2} \leq r\}.
\]
Definition 10.1.8. The limit of \( f(x, y) \) as \((x, y)\) approaches \((x_0, y_0)\), denoted by
\[
\lim_{(x, y)\to(x_0, y_0)} f(x, y)
\]
is the number \( L \) such that, for every \( \epsilon > 0 \), there exists a \( \delta > 0 \) so that
\[
|f(x, y) - L| < \epsilon \quad \text{whenever } (x, y) \in B_\delta(x_0, y_0) \setminus \{(x_0, y_0)\}.
\]

Example 10.1.9. Let \( f(x, y) = x^2 + y^2 \). Prove
\[
\lim_{(x, y)\to(0,0)} f(x, y) = 0.
\]

Proof. Fix a number \( \epsilon > 0 \). We want to find a \( \delta > 0 \) so that
\[
|f(x, y) - 0| < \epsilon
\]
whenever \((x, y) \in B_\delta(0, 0) \setminus \{(0, 0)\}\). Define \( \delta = \sqrt{\epsilon} \). In this situation,
\[
0 < \sqrt{x^2 + y^2} < \delta = \sqrt{\epsilon} \implies 0 < x^2 + y^2 < \epsilon
\]
and hence \( |f(x, y)| < \epsilon \).

We can extend the limit laws to the two-dimensional case.

Theorem 10.1.10. If \( a \) is a constant and if
\[
\lim_{(x, y)\to(x_0, y_0)} f(x, y) = L_1, \quad \lim_{(x, y)\to(x_0, y_0)} g(x, y) = L_2
\]
where \( L_1 \) and \( L_2 \) are real numbers, then
(1) (Addition rule)
\[
\lim_{(x, y)\to(x_0, y_0)} [f(x, y) + g(x, y)] = \lim_{(x, y)\to(x_0, y_0)} f(x, y) + \lim_{(x, y)\to(x_0, y_0)} g(x, y).
\]
(2) (Constant-factor rule)
\[
\lim_{(x, y)\to(x_0, y_0)} af(x, y) = a \lim_{(x, y)\to(x_0, y_0)} f(x, y).
\]
(3) (Multiplication rule)
\[
\lim_{(x, y)\to(x_0, y_0)} f(x, y)g(x, y) = \left[ \lim_{(x, y)\to(x_0, y_0)} f(x, y) \right] \left[ \lim_{(x, y)\to(x_0, y_0)} g(x, y) \right].
\]
(4) (Quotient rule)
\[
\lim_{(x, y)\to(x_0, y_0)} \frac{f(x, y)}{g(x, y)} = \frac{\lim_{(x, y)\to(x_0, y_0)} f(x, y)}{\lim_{(x, y)\to(x_0, y_0)} g(x, y)}
\]
provided that \( L_2 \neq 0 \).
Example 10.1.11. Compute
\[
\begin{align*}
\lim_{(x,y) \to (0,0)} (x^2 + y^2), & \quad \lim_{(x,y) \to (-1,2)} x^2y, & \quad \lim_{(x,y) \to (1,2)} (x^2 + 3x), & \quad \lim_{(x,y) \to (2,0)} \frac{4y + 2x}{x^2 + 2xy - 3}.
\end{align*}
\]

Proof. The answers are 0, 2, 5, 4.

We now consider the limits that do not exist. In the one-dimensional case, there were only two ways in which we could approach a number: from the left or from the right. If the two limits were different, we said that the limit did not exist.

In two dimensions, there are many more ways that we can approach the point \((x_0, y_0)\), namely, by any curve in the \(xy\)-plane that ends up at the point \((x_0, y_0)\). We call such curve paths.

Proposition 10.1.12. If \(f(x, y)\) approaches \(L_1\) as \((x, y) \to (x_0, y_0)\) along path \(C_1\) and \(f(x, y)\) approaches \(L_2\) as \((x, y) \to (x_0, y_0)\) along path \(C_2\), and if \(L_1 \neq L_2\), then \(\lim_{(x,y) \to (x_0, y_0)} f(x, y)\) does not exist.

A typical path is a line or a parabola.

Example 10.1.13. Show that
\[
\lim_{(x,y) \to (0,0)} \frac{x^2 - y^2}{x^2 + y^2}
\]
does not exist.

Proof. Consider the line \(L_m : y = mx\). Then
\[
\lim_{x \to 0} f(x, mx) = \lim_{x \to 0} \frac{x^2 - m^2x^2}{x^2 + m^2x^2} = \frac{1 - m^2}{1 + m^2}.
\]
If \(m = 0\), then the above limit is 1; if \(m = 1\), then the above limit is 0. Since 1 \(\neq\) 0, it follows that \(\lim_{(x,y) \to (0,0)} f(x, y)\) does not exist.

Example 10.1.14. Show that
\[
\lim_{(x,y) \to (0,0)} \frac{4xy}{xy + y^3}
\]
does not exist.

Proof. For any line \(y = mx\), we have
\[
\lim_{x \to 0} f(x, mx) = \lim_{x \to 0} \frac{4mx^2}{mx^2 + m^3x^3} = \lim_{x \to 0} \frac{4}{1 + m^2x} = 0.
\]
We now consider a parabola \(x = y^2\). Then
\[
\lim_{y \to 0} f(y^2, y) = \lim_{y \to 0} \frac{4y^3}{y^3 + y^3} = 2 \neq 4.
\]
Hence \( \lim_{(x,y) \to (0,0)} f(x,y) \) does not exist. \( \square \)

### 10.1.3. Continuity

The definition of continuity is also analogous to that in the one-dimensional case.

**Definition 10.1.15.** A function \( f(x,y) \) is **continuous** at \((x_0, y_0)\) if the following holds:

1. \( f(x,y) \) is defined at \((x_0, y_0)\),
2. the limit \( \lim_{(x,y) \to (x_0,y_0)} f(x,y) \) exists, and
3. we have \( \lim_{(x,y) \to (x_0,y_0)} f(x,y) = f(x_0, y_0) \).

For example, the function \( f(x,y) = 2 + x^2 + y^2 \) is continuous at \((0,0)\).

**Example 10.1.16.** Show that

\[
f(x, y) = \begin{cases} 
\frac{x^2 - y^2}{x^2 + y^2}, & (x, y) \neq (0,0), \\
0, & (x, y) = (0,0)
\end{cases}
\]

is discontinuous at \((0,0)\).

**Proof.** By Example 10.1.13, the limit \( \lim_{(x,y) \to (0,0)} \frac{x^2 - y^2}{x^2 + y^2} \) does not exist, hence the function is discontinuous at \((0,0)\). \( \square \)

Consider the function

\[
h(x, y) = e^{x^2 + y^2}.
\]

If we define \( z = f(x,y) = x^2 + y^2 \) and \( g(z) = e^z \), then

\[
h(x, y) = (g \circ f)(x, y) = g[f(x,y)] = e^{x^2 + y^2}.
\]

In general, if

\[
f : D \to \mathbb{R}, \quad D \subset \mathbb{R}^2
\]

and

\[
g : I \to \mathbb{R}, \quad I \subset \mathbb{R}
\]

then the **composition** \( (g \circ f)(x, y) \) is defined as

\[
h(x, y) := (g \circ f)(x, y) = g[f(x, y)].
\]

For example,

\[
h(x, y) = \sqrt{x^2 + y^2} = (g \circ f)(x, y),
\]

where \( f(x, y) = x^2 + y^2 \) with \((x, y) \in \mathbb{R}^2\) and \( g(z) = \sqrt{z} \) with \( z \geq 0 \).
10.2. Partial derivatives

Look at a function of two variables
\[ f(x, y) = x^2y. \]

Fix \( y = y_0 \) and consider the function
\[ g(x) := f(x, y_0) \]
as a function of one variable. Then
\[ \frac{d}{dx} g(x) = \frac{d}{dx} f(x, y_0) = \frac{d}{dx} (x^2y_0) = 2xy_0. \]

Such a derivative is called a \textit{partial derivative}.

10.2.1. Partial derivatives. We first consider partial derivatives of a function of two variables.

**Definition 10.2.1.** Suppose that \( f \) is a function of two independent variables \( x \) and \( y \). The \textbf{partial derivative} of \( f \) with respect to \( x \) is defined by

\[ f_x(x, y) = \frac{\partial}{\partial x} f(x, y) := \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}. \]

The partial derivative of \( f \) with respect to \( y \) is defined by

\[ f_y(x, y) = \frac{\partial}{\partial y} f(x, y) := \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}. \]

To compute \( f_x(x, y) = \partial f(x, y)/\partial x \), we differentiate \( f \) with respect to \( x \) while treating \( y \) as a constant. When we read \( \partial f(x, y)/\partial x \), we can say “the partial derivative of \( f \) of \( x \) and \( y \) with respect to \( x \).” To read \( f_x(x, y) \), we say “\( f \) sub \( x \) of \( x \) and \( y \).”

**Example 10.2.2.** Find \( \partial f/\partial x \) and \( \partial f/\partial y \) when
\[ f(x, y) = ye^{xy}. \]

**Proof.** To compute \( \partial f/\partial x \), we treat \( y \) as a constant and use the chain rule:
\[ f_x(x, y) = \frac{\partial}{\partial x} (ye^{xy}) = ye^{xy}y = y^2e^{xy}. \]

Similarly,
\[ f_y(x, y) = \frac{\partial}{\partial y} (e^{xy}) = e^{xy} + ye^{xy}x = e^{xy}(1 + xy). \]

10.2.2. Geometric interpretation. We give the following geometric interpretation of \( \partial f/\partial x \). We fix \( y = b \); then \( g(x) := f(x, b) \) as a function of \( x \) is obtained by intersecting the surface \( z = f(x, y) \) with a vertical plane that is parallel to \( xx \)-plane and goes through \( y = b \). The curve of intersection is the graph...
Figure 10.2. The surface of $f(x, y)$ intersected with the plane $y = b$

Figure 10.3. The surfaces of $f(x, y)$ intersected with the plane $y = b$ and $x = a$

of $z = g(x) = f(x, b)$, as illustrated in Figure 10.3. We project this curve onto $xz$-plane; the curve is the graph of a function that depends only on $x$. Consequently, we can find the slope of the tangent line at any point $(a, g(a)) = (a, f(a, b))$.

The partial derivative $\frac{\partial f}{\partial x}$ evaluated at $(x_0, y_0)$ is the slope of the tangent line to the curve $z = f(x, y_0)$ at the point $(x_0, y_0, z_0)$ with $z_0 = f(x_0, y_0)$.

Similarly,

The partial derivative $\frac{\partial f}{\partial y}$ evaluated at $(x_0, y_0)$ is the slope of the tangent line to the curve $z = f(x_0, y)$ at the point $(x_0, y_0, z_0)$ with $z_0 = f(x_0, y_0)$.

**Example 10.2.3.** Let $f(x, y) = 3 - x^3 - y^2$. Find $f_x(1, 1)$ and $f_y(1, 1)$, and interpret the results geometrically.

**Proof.** We have

$$f_x(x, y) = -3x^2, \quad f_y(x, y) = -2y.$$ 

Hence $f_x(1, 1) = -3$ and $f_y(x, y) = -2$. \qed
The definition of partial derivative extends in a straightforward way to functions of more than two variables.

**Example 10.2.4.** Let \( f(x,y,z) = e^{yz}(x^2 + z^3) \). Find \( f_x, f_y, f_z \).

**Proof.** By the same way, we have \( f_x = 2xe^{yz}, \ f_y = ze^{yz}(x^2 + z^3), \) and \( f_z = ye^{yz}(x^2 + z^3) + 3z^2e^{yz} = e^{yz}(x^2y + yz^3 + 2z^2) \). □

**10.2.3. Holling’s disk equation.** Holling (1959) derived an expression for the number of prey items \( P_e \) eaten by a predator during an interval \( T \) as a function of prey density \( N \) and the handling time \( T_h \) of each prey item:

\[
P_e = \frac{aNT}{1 + aT_hN},
\]

where \( a \) is a positive constant called the predator attack rate. Equation (10.2.3) is called Holling’s disk equation.

(i) We can consider \( P_e \) as a function of \( N \) and \( T_h \).
   - How handling time influences the number of prey eaten:
     
     \[
     \frac{\partial}{\partial T_h} P_e = -\frac{a^2 N^2 T}{(1 + aT_hN)^2} < 0,
     \]
     hence the number of prey items eaten decreases with increasing handling time.
   - How \( P_e \) change with \( N \):
     
     \[
     \frac{\partial}{\partial N} P_e = \frac{aT}{(1 + aT_hN)^2} > 0,
     \]
     hence the number of prey items eaten increases with increasing prey density.

(ii) We can consider \( P_e \) as a function of \( a \) and \( T \).
   - How the predator attack rate \( a \) influences the number of prey eaten per predator:
     
     \[
     \frac{\partial}{\partial a} P_e = \frac{NT}{(1 + aT_hN)^2} > 0,
     \]
     hence the number of prey eaten per predator increases with increasing predator attack rate.
   - How the length \( T \) of the interval influences the number of prey eaten per predator:
     
     \[
     \frac{\partial}{\partial T} P_e = \frac{aN}{1 + aT_hN} > 0,
     \]
     hence the number of prey eaten per predator increases with increasing length.
10.2.4. Higher-order partial derivatives. As in the case of functions of one variables, we can define higher-order partial derivatives for functions of more than one variable. For instance,

\begin{align*}
  f_{xx} &= \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right), \\
  f_{yx} &= \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right), \\
  f_{xy} &= \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right), \\
  f_{yy} &= \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial y} \right).
\end{align*}

Example 10.2.5. Let \( f(x, y) = \sin x + xe^y \). Find \( f_{xx}, f_{yx}, f_{xy} \).

\textbf{Proof.} By definitions, \( f_{xx} = -\sin x, f_{yx} = e^y = f_{xy} \). \qed

Example 10.2.5 implies that \( f_{xy} = f_{yx} \). Although, this is not always the case.

\textbf{Theorem 10.2.6.} (Mixed-derivative theorem) If \( f(x, y) \) and its partial derivatives \( f_x, f_y, f_{xy}, \) and \( f_{yx} \) are continuous on an open disk centered at the point \( (x_0, y_0) \), then

\[ f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0). \]

We can similarly define partial derivatives of higher order. For instance,

\begin{align*}
  f_{yx^2} &= \frac{\partial^4 f}{\partial x^2 \partial y} = \frac{\partial}{\partial x} \left( \frac{\partial}{\partial x} \frac{\partial f}{\partial y} \right), \\
  f_{y^2x} &= \frac{\partial^4 f}{\partial y^2 \partial x} = \frac{\partial}{\partial y} \left( \frac{\partial}{\partial y} \frac{\partial f}{\partial x} \right).
\end{align*}

Example 10.2.7. Let \( f(x, y) = y^2 \sin x \). Find \( f_{yx^2}, f_{y^2x}, f_{xy^2} \).

\textbf{Proof.} We have

\begin{align*}
  f_{yx^2} &= \frac{\partial^2}{\partial x^2} (2y \sin x) = \frac{\partial}{\partial x} (2y \cos x) = -2y \sin x, \\
  f_{y^2x} &= \frac{\partial^2}{\partial y^2} (y^2 \cos x) = \frac{\partial}{\partial y} (2y \cos x) = 2 \cos x.
\end{align*}

Similarly, \( f_{xy^2} = 2 \cos x \). \qed

Theorem 10.2.6 can be extended to higher-order derivatives. The order differentiation does not matter, as long as the function and all of its derivatives through the order in question are continuous on an open disk centered at the point at which we want to compute the derivative.
Example 10.2.8. Let \( f(x,y) = \ln(x^2 + 3xy) \). Find \( f_y^2 \) and \( f_x^3 \).

**Proof.** We have
\[
 f_x = \frac{2x + 3y}{x^2 + 3xy}, \quad f_y = \frac{3x}{x^2 + 3xy},
\]
and
\[
 f_{x^2} = \frac{-2x^2 - 6xy - 9y^2}{(x^2 + 3xy)^2}, \quad f_{x^3} = \frac{4x^3 + 12x^2y + 54xy^2 + 54y^3}{(x^2 + 3xy)^3}.
\]
Similarly, \( f_y^2 = -9x^2/(x^2 + 3xy)^2 \).

Example 10.2.9. Let \( f(x,y) = x^3 \cos y \). Find \( f_{yx^2} \) and \( f_{xy^2} \).

**Proof.** As in Example 10.2.8, we can show that \( f_{yx^2} = -6x \sin y \) and \( f_{xy^2} = -3x^2 \cos y \).

10.3. Tangent planes, differentiability, and linearization

Suppose that \( z = f(x) \) is differentiable at \( x = x_0 \). Then the equation of the tangent line of \( z = f(x) \) at \( (x_0, z_0) \) with \( z_0 = f(x_0) \) is given by
\[
 z - z_0 = f'(x_0)(x - x_0). \tag{10.3.1}
\]
We now generalize the concept of tangent lines to functions of two variables. The analogue of a tangent line is called a tangent plane (see Figure 10.4). Recall the general equation of a plane
\[
 z - z_0 = A(x - x_0) + B(y - y_0) \tag{10.3.2}
\]
by (9.4.15).

10.3.1. Tangent planes. Let \( z = f(x,y) \) be a function of two variables and consider a point \( P = (x_0, y_0, z_0) \) with \( z_0 = f(x_0, z_0) \). We take the curve \( C_1 \) that is obtained as the intersection of the surface \( z = f(x,y) \) with the plane that is parallel to the \( yz \)-plane and contains the point \( P \). Likewise, we take the curve \( C_2 \)
Figure 10.5. The surfaces \( z = f(x, y) \) and its tangent plane at \( P = (x_0, y_0, z_0) \) that is obtained as the intersection of the surface \( z = f(x, y) \) with the plane that is parallel to the \( xz \)-plane and contains the point \( P \). The tangent lines associated to \( C_1 \) and \( C_2 \) determine the tangent plane at \( P \).

By (10.3.2), this tangent plane takes the form

\[
(10.3.3) \quad z - z_0 = A(x - x_0) + B(y - y_0);
\]

we will use curves \( C_1 \) and \( C_2 \) to determine the constants \( A \) and \( B \). \( C_1 \) satisfies the equation

\[
(10.3.4) \quad z = f(x_0, y).
\]

The tangent line to \( C_1 \) at \( P \) satisfies

\[
(10.3.4) \quad z - z_0 = \frac{\partial f(x_0, y_0)}{\partial y}(y - y_0).
\]

Since the tangent line of \( C_1 \) at \( P \) is contained in the tangent plane, letting \( x = x_0 \) in (10.3.3), we obtain

\[
(10.3.5) \quad z - z_0 = B(y - y_0)
\]

compared with (10.3.4) that yields

\[
(10.3.5) \quad B = \frac{\partial f(x_0, y_0)}{\partial y} = f_y(x_0, y_0).
\]

Similarly,

\[
(10.3.6) \quad A = \frac{\partial f(x_0, y_0)}{\partial x} = f_x(x_0, y_0).
\]

If the tangent plane to the surface \( z = f(x, y) \) at the point \( (x_0, y_0, z_0) \) exists, then that tangent plane had the equation

\[
(10.3.7) \quad z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).
\]
Example 10.3.1. Find the tangent plane to the surface 
\[ z = f(x, y) = 4x^2 + y^2 \]
at the point \((1, 2, 8)\).

**Proof.** Since \(f_x(x, y) = 8x\) and \(f_y(x, y) = 2y\), it follows that 
\[ f_x(1, 2) = 8, \quad f_y(1, 2) = 4. \]
Then the tangent plane given by (10.3.7) is 
\[ z - 8 = 8(x - 1) + 4(y - 2) = 8x + 4y - 16 \]
or \[8x + 4y - z = 8. \]

10.3.2. Differentiability. Recall that the linear approximation of a differentiable function \(f(x)\) at \(x = x_0\) is given by (see (4.2.19))
\[(10.3.8)\]
\[ L(x) = f(x_0) + f'(x_0)(x - x_0). \]
The distance between \(f(x)\) and \(L(x)\) is
\[ |f(x) - L(x)| = \left| f(x) - f(x_0) - f'(x_0)(x - x_0) \right| \]
and then
\[ \frac{|f(x) - L(x)|}{|x - x_0|} = \left| \frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \right| ; \]
taking the limit \(x \to x_0\), we obtain
\[(10.3.9)\]
\[ \lim_{x \to x_0} \left| \frac{f(x) - L(x)}{x - x_0} \right| = 0. \]
We say that \(f(x)\) is differentiable at \(x = x_0\) if (10.3.9) holds.

**Definition 10.3.2.** Suppose that \(f(x, y)\) is a function of two independent variables and that both \(\partial f/\partial x\) and \(\partial f/\partial y\) are defined throughout an open disk containing \((x_0, y_0)\). Set
\[ L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0). \]
Then \(f(x, y)\) is **differentiable** at \((x_0, y_0)\) if
\[ \lim_{(x, y) \to (x_0, y_0)} \left| \frac{f(x, y) - L(x, y)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} \right| = 0. \]
Furthermore, if \(f(x, y)\) is differentiable at \((x_0, y_0)\), then \(z = L(x, y)\) defines the tangent plane to the graph of \(f\) at \((x_0, y_0, f(x_0, y_0))\). We say that \(f(x, y)\) is differentiable if it is differentiable at every point of its domain.

As in Theorem 4.1.2 the following theorem holds.

**Theorem 10.3.3.** If \(f(x, y)\) is differentiable at \((x_0, y_0)\), then \(f\) is continuous at \((x_0, y_0)\).
That \( f(x, y) \) is differentiable at \((x_0, y_0)\) means that the function \( f(x, y) \) is close to the tangent plane at \((x_0, y_0)\) for all \((x, y)\) close to \((x_0, y_0)\). The mere existence of the partial derivatives \( \partial f / \partial x \) and \( \partial f / \partial y \) at \((x_0, y_0)\), however, is not enough to guarantee differentiability (and, consequently, the existence of a tangent plane at a certain point).

**Example 10.3.4.** Consider the function

\[
f(x, y) = \begin{cases} 
0, & xy \neq 0, \\
1, & xy = 0.
\end{cases}
\]

Show that \( f_x(0, 0) \) and \( f_y(0, 0) \) exist, but \( f(x, y) \) is not continuous and not differentiable at \((0, 0)\).

**Proof.** Since \( f(x, 0) = 1 \), we have \( f_x(0, 0) = 0 \). Likewise, from \( f(0, y) = 1 \) we get \( f_y(0, 0) = 0 \). To prove that \( f(x, y) \) is discontinuous, we consider two special paths \( C_1 : y = 0 \) and \( C_2 : y = x \). Then

\[
\lim_{(x, y) \to (0, 0) \text{ along } C_1} f(x, y) = 1
\]

and

\[
\lim_{(x, y) \to (0, 0) \text{ along } C_2} f(x, y) = \lim_{x \to 0} f(x, x) = 0.
\]

Therefore, \( f(x, y) \) is not continuous at \((0, 0)\). \(\square\)

We have the following sufficient condition for differentiability.

**Theorem 10.3.5.** Suppose that \( f(x, y) \) is defined on an open disk centered at \((x_0, y_0)\) and the partial derivatives \( \partial f / \partial x \) and \( \partial f / \partial y \) are continuous on an open disk centered at \((x_0, y_0)\). Then \( f(x, y) \) is differentiable at \((x_0, y_0)\).

**Example 10.3.6.** Show that \( f(x, y) = 2x^2y - y^2 \) is differentiable for all \((x, y) \in \mathbb{R}^2\).

**Proof.** Since \( f_x(x, y) = 4xy \) and \( f_y(x, y) = 2x^2 - 2y \), it follows from Theorem 10.3.5 that \( f(x, y) \) is differentiable on \( \mathbb{R}^2 \). \(\square\)

### 10.3.3. Linearization

From Definition 10.3.2 we give the following

**Definition 10.3.7.** Suppose that \( f(x, y) \) is differentiable at \((x_0, y_0)\). The linearization of \( f(x, y) \) at \((x_0, y_0)\) is the function

\[
L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).
\]

The approximation

\[
f(x, y) \approx L(x, y)
\]

is the **standard linear approximation** or the **tangent plane approximation** of \( f(x, y) \) at \((x_0, y_0)\).
Example 10.3.8. Find the linear approximation of
\[ f(x, y) = x^2y + 2xe^y \]
at the point (2, 0).

**Proof.** From \( f_x(x, y) = 2xy + 2e^y \) and \( f_y = x^2 + 2xe^y \), we obtain
\[ L(x, y) = 4 + 2(x - 2) + 8(y - 0) = 2x + 8y. \]

Example 10.3.9. Find the linear approximation of
\[ f(x, y) = \ln(x - 2y^2) \]
at the point (3, 1) and use it to find an approximation for \( f(3.05, 0.95) \).

**Proof.** Since \( f_x(x, y) = \frac{1}{x - 2y^2} \) and \( f_y(x, y) = \frac{-4}{x - 2y^2} \), it follows that
\[ L(x, y) = 0 + 1(x - 3) + (-4)(y - 1) = x - 4y + 1. \]
In particular, \( L(3.05, 0.95) = 3.05 - 4 \times 0.95 + 1 = 0.25 \). Using the calculator, we have \( f(3.05, 0.95) \approx 0.2191 \) and the error of approximation is \( |f(3.05, 0.95) - L(3.05, 0.95)| \approx 0.031 \).

Example 10.3.10. Find the linear approximation of
\[ f(x, y) = \ln(x^2 - 3y) \]
at (1, 0), and use it to approximate \( f(1.1, 0.1) \).

**Proof.** Since \( f_x(x, y) = \frac{2x}{x^2 - 3y} \) and \( f_y(x, y) = \frac{-3}{x^2 - 3y} \), it follows that
\[ L(x, y) = 0 + 2(x - 1) + (-3)(y - 0) = 2x - 3y - 2. \]
In particular, \( L(1.1, 0.1) = 2 \times 1.1 - 3 \times 0.1 - 2 = -0.1 \). Using the calculator, we get \( f(1.1, 0.1) = \ln(1.21 - 0.3) = \ln 0.91 \approx -0.094 \).

10.3.4. Vector-valued functions. We now consider the vector-valued functions
\[ f : \mathbb{R}^n \to \mathbb{R}^m \]
given by
\[ \mathbf{x} = (x_1, \cdots, x_n) \mapsto \begin{bmatrix} f_1(\mathbf{x}) = f_1(x_1, \cdots, x_n) \\ \vdots \\ f_m(\mathbf{x}) = f_m(x_1, \cdots, x_n) \end{bmatrix} \]
Here, each function \( f_i(x_1, \cdots, x_n) \) is a real-valued function:
\[ f_i : \mathbb{R}^n \to \mathbb{R}, \ (x_1, \cdots, x_n) \mapsto f_i(x_1, \cdots, x_n). \]
We will consider vector-valued functions where $n = m = 2$:

$$(u, v) \mapsto \begin{bmatrix} f(u, v) \\ g(u, v) \end{bmatrix}$$

**Example 10.3.11.** Find the linearization of

$$f(x) = 2\ln x$$

at $x_0 = 1$.

**Proof.** The linearization of $f(x)$ at $x_0 = 1$ is $L(x) = f(1) + f'(1)(x - 1) = 0 + 2(x - 1) = 2x - 2$. □

The linearization of a real-valued function $f : \mathbb{R}^2 \to \mathbb{R}$ is

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

$$= f(x_0, y_0) + \begin{bmatrix} f_x(x_0, y_0) & f_y(x_0, y_0) \end{bmatrix} \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix}$$

**Example 10.3.12.** Find the linearization of

$$f(x, y) = \ln x + \ln y$$

at $(1, 1)$.

**Proof.** Since $f(1, 1) = 0, f_x(x, y) = 1/x, \text{ and } f_y(x, y) = 1/y$, we obtain

$$L(x, y) = 0 + \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x - 1 \\ y - 1 \end{bmatrix} = x - 1 + y - 1 = x + y - 2.$$ □

Suppose that

$$h : \mathbb{R}^2 \to \mathbb{R}^2, \quad (x, y) \mapsto \begin{bmatrix} f(x, y) \\ g(x, y) \end{bmatrix}$$

and assume that all first partial derivatives are continuous on a disk centered at $(x_0, y_0)$. The linearization of $f$ is

(10.3.10) \quad \alpha(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \quad \text{and the linearization of } g \text{ is}

(10.3.11) \quad \beta(x, y) = g(x_0, y_0) + g_x(x_0, y_0)(x - x_0) + g_y(x_0, y_0)(y - y_0);$

we define the vector-valued function $L(x, y) = \begin{bmatrix} \alpha(x, y) \\ \beta(x, y) \end{bmatrix}$ and then

$$L(x, y) = \begin{bmatrix} f(x_0, y_0) \\ g(x_0, y_0) \end{bmatrix} + \begin{bmatrix} f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \\ g_x(x_0, y_0)(x - x_0) + g_y(x_0, y_0)(y - y_0) \end{bmatrix}$$

$$= \begin{bmatrix} f(x_0, y_0) \\ g(x_0, y_0) \end{bmatrix} + \begin{bmatrix} f_x(x_0, y_0) & f_y(x_0, y_0) \\ g_x(x_0, y_0) & g_y(x_0, y_0) \end{bmatrix} \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix}.$$ We call the $2 \times 2$ matrix

(10.3.12) \quad (Dh)(x_0, y_0) = \begin{bmatrix} f_x(x_0, y_0) & f_y(x_0, y_0) \\ g_x(x_0, y_0) & g_y(x_0, y_0) \end{bmatrix}
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The Jacobi matrix or the derivative matrix. Consequently,

\[ L(x, y) = h(x_0, y_0) + (Dh)(x_0, y_0) \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix}. \]

Example 10.3.13. Assume that

\[ f : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad (x, y) \mapsto \begin{bmatrix} u \\ v \end{bmatrix} \]

with

\[ u(x, y) = x^2y - y^3, \quad v(x, y) = 2x^3y^2 + y. \]

Compute the Jacobi matrix and evaluate it at \((1, 2)\).

**Proof.** Since \( u_x(x, y) = 2xy, u_y(x, y) = x^2 - 3y^2, v_x(x, y) = 6x^2y^2, \) and \( v_y(x, y) = 4x^3y + 1, \) we obtain

\[ (Df)(1, 2) = \begin{bmatrix} 2xy & x^2 - 3y^2 \\ 6x^2y^2 & 4x^3y + 1 \end{bmatrix}_{(1,2)} = \begin{bmatrix} 4 & -11 \\ 24 & 9 \end{bmatrix}. \]

Example 10.3.14. Assume that

\[ f : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad (x, y) \mapsto \begin{bmatrix} u \\ v \end{bmatrix} \]

with

\[ u(x, y) = 2x^2y, \quad v(x, y) = \frac{1}{xy}. \]

Compute the linear approximation to \( f(x, y) \) at \((1, 1)\).

**Proof.** The Jacobi matrix is

\[ (Df)(x, y) = \begin{bmatrix} 4xy \\ x^2 - 3y^2 \end{bmatrix} \implies (Df)(1,1) = \begin{bmatrix} 4 & 2 \\ -1 & -1 \end{bmatrix}. \]

Then \( L(x, y) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 & 2 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x - 1 \\ y - 1 \end{bmatrix} = \begin{bmatrix} 4x + 2y - 4 \\ -x - y \end{bmatrix}. \]

Example 10.3.15. Let \( f : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) with

\[ u(x, y) = ye^{-x}, \quad v(x, y) = \sin x + \cos y. \]

Find the linear approximation to \( f(x, y) \) at \((0, 0)\) and compute \( f(0.1, -0.1) \) with its linear approximation.

**Proof.** The Jacobi matrix of \( f(x, y) \) is

\[ (Df)(x, y) = \begin{bmatrix} -ye^{-x} & e^{-x} \\ \cos x & -\sin y \end{bmatrix} \]
and hence

\[ L(x, y) = \begin{bmatrix} u(0, 0) \\ v(0, 0) \end{bmatrix} + (Df)(0, 0) \begin{bmatrix} x - 0 \\ y - 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x + 1 \end{bmatrix}. \]

In particular,

\[ L(0.1, -0.1) = \begin{bmatrix} -0.1 \\ 1.1 \end{bmatrix}. \]

Using calculator, we obtain \( f(0.1, -0.1) \approx \begin{bmatrix} -0.09 \\ 1.09 \end{bmatrix}. \quad \square \]

**Example 10.3.16.** Find a linear approximation to

\[ f(x, y) = \begin{bmatrix} x^2 - xy \\ 3y^2 - 1 \end{bmatrix} \]

at (1, 2) and an approximation for \( f(1.1, 1.9). \)

**Proof.** The Jacobi matrix is

\[ (Df)(x, y) = \begin{bmatrix} 2x - y & -x \\ 0 & 6y \end{bmatrix} \implies (Df)(1, 2) = \begin{bmatrix} 0 & -1 \\ 0 & 12 \end{bmatrix} \]

and therefore

\[ L(x, y) = f(1, 2) + (Df)(1, 2) \begin{bmatrix} x - 1 \\ y - 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 11 \end{bmatrix} + \begin{bmatrix} -(y - 2) \\ 12(y - 2) \end{bmatrix} = \begin{bmatrix} -y + 1 \\ 12y - 13 \end{bmatrix}, \]

in particular, \( L(1.1, 1.9) = \begin{bmatrix} -0.9 \\ 9.8 \end{bmatrix} \) and \( f(1.1, 1.9) = \begin{bmatrix} -0.88 \\ 9.83 \end{bmatrix}. \quad \square \]

We can generalize the Jacobi matrix to functions \( f : \mathbb{R}^n \rightarrow \mathbb{R}^m. \) If

\[ f(x_1, \cdots, x_n) = \begin{bmatrix} f_1(x_1, \cdots, x_n) \\ \vdots \\ f_m(x_1, \cdots, x_n) \end{bmatrix} \]

where \( f_i : \mathbb{R}^n \rightarrow \mathbb{R}, \) \( i = 1, \cdots, n, \) are real-valued functions of \( n \) independent variables, then the Jacobi matrix is an \( m \times n \) matrix of the form

\[ (10.3.14) \quad Df(x) = \begin{bmatrix} \frac{\partial}{\partial x_1} f_1(x) & \cdots & \frac{\partial}{\partial x_n} f_1(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial x_1} f_m(x) & \cdots & \frac{\partial}{\partial x_n} f_m(x) \end{bmatrix} \]

The linearization of \( f(x) \) about the point \( (x^*) \) is

\[ (10.3.15) \quad L(x^*) = f(x^*) + (Df)(x^*) \cdot (x - x^*). \]

### 10.4. Chain rule

Recall the chain rule (see Theorem 4.2.4) that

\[ (f \circ g)'(x) = f'[g(x)]g'(x) \]

where \( g \) is differentiable at \( x \) and \( f \) is differentiable at \( y = g(x) \).
10.4.1. **Chain rule for functions of two variables.** We now consider the chain rule for functions of two variables.

**Theorem 10.4.1.** If \( w = f(x, y) \) is differentiable and \( x \) and \( y \) are differentiable functions of \( t \), then \( w \) is a differentiable function of \( t \) and

\[
\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt}.
\]

**Proof.** We approximate \( w = f(x, y) \) at \((x_0, y_0)\) by its linear approximation

\[
L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).
\]

If we set \( \Delta x = x - x_0 \), \( \Delta y = y - y_0 \), and \( \Delta w = f(x, y) - f(x_0, y_0) \), we can approximate \( \Delta w \) by its linear approximation and find that

\[
\Delta w \approx f_x(x_0, y_0) \Delta x + f_y(x_0, y_0) \Delta y.
\]

Dividing both sides by \( \Delta t \), we obtain

\[
\frac{\Delta w}{\Delta t} \approx f_x(x_0, y_0) \frac{\Delta x}{\Delta t} + f_y(x_0, y_0) \frac{\Delta y}{\Delta t}.
\]

Letting \( \Delta t \to 0 \) yields \(10.4.1\). \( \square \)

**Example 10.4.2.** Let \( w = f(x, y) = x^2y^3 \) with \( x(t) = \sin t \) and \( y(t) = e^{-t} \). Find the derivative of \( w = f(x, y) \) with respect to \( t \) when \( t = \pi/2 \).

**Proof.** By \(10.4.1\),

\[
\frac{dw}{dt} = 2xy^3 \frac{dx}{dt} + 3x^2y^2 \frac{dy}{dt} = 2xy^3 \cos t - 3x^2y^2e^{-t} = \sin t \cdot e^{-3t}(2 \cos t - 3 \sin t).
\]

Hence \( dw/dt \big|_{t=\pi/2} = e^{-3\pi/2}(0 - 3) = -3e^{-3\pi/2} \). \( \square \)

10.4.2. **Implicit differentiation.** Consider the equation

\[
x^2y - e^{-y} = 0.
\]

We differentiate both sides of \(10.4.2\) with respect to \( x \) and find that

\[
0 = 2xy + \frac{dy}{dx} \left( x^2 + e^{-y} \right).
\]

Solving for \( dy/dx \), we obtain

\[
\frac{dy}{dx} = -\frac{2xy}{x^2 + e^{-y}}.
\]

We now consider the general case. We think of \( y \) as a function of \( x \) and define a function

\[
F(x, y) = 0.
\]

This defines \( y \) implicitly as a function of \( x \). To find the derivative of \( y \) with respect to \( x \), we set

\[
w = F(u, v) \text{ with } u(x) = x \text{ and } v(x) = y.
\]
By (10.4.1), we obtain
\[ \frac{dw}{dx} = \frac{\partial F}{\partial u} \frac{du}{dx} + \frac{\partial F}{\partial v} \frac{dv}{dx}. \]

Since \( \frac{du}{dx} = 1 \) and \( \frac{dv}{dx} = \frac{dy}{dx} \), it follows that
\[ \frac{dw}{dx} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} \frac{dy}{dx}. \]

From \( w(x) = 0 \), we conclude that
\[ 0 = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} \frac{dy}{dx}. \]

Thus

Suppose that \( w = F(x, y) \) is differentiable and \( F(x, y) = 0 \) defines \( y \) implicitly as a differentiable function of \( x \). Then, at any point where \( F_y \neq 0 \),

\[ (10.4.4) \quad \frac{dy}{dx} = \frac{F_x}{F_y}. \]

Example 10.4.3. Find \( \frac{dy}{dx} \) if \( x^2y - e^{-y} = 0 \).

Proof. We set \( F(x, y) = x^y - e^{-y} \). Then
\[ F_x(x, y) = 2xy, \quad F_y(x, y) = x^2 + e^{-y} \]
and \( \frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{2xy}{x^2 + e^{-y}} \).

Example 10.4.4. Find \( \frac{dy}{dx} \) for \( y = \arcsin x \).

Proof. \( y = \arcsin x \) for \( -\pi/2 \leq y \leq \pi/2 \) is equivalent to \( x = \sin y \) for \( -1 \leq x \leq 1 \). Let \( F(x, y) = x - \sin y \). Then
\[ F_x(x, y) = 1, \quad F_y(x, y) = -\cos y. \]
Hence \( \frac{dy}{dx} = 1/\cos y = \frac{1}{\sqrt{1-x^2}} \) for \( x \in (-1, 1) \).

Example 10.4.5. Find \( \frac{dy}{dx} \) if \( \ln(x^2 + y^2) = 3xy \).

Proof. Let \( F(x, y) = \ln(x^2 + y^2) - 3xy \). Then
\[ F_x(x, y) = \frac{2x}{x^2 + y^2} - 3y, \quad F_y(x, y) = \frac{2y}{x^2 + y^2} - 3x. \]
So
\[ \frac{dy}{dx} = -\frac{2x - 3x^2y - 3y^3}{2y - 3xy^2 - 3x^3}. \]
10.4.3. Directional derivatives and gradient vectors. Consider two vectors \( \mathbf{r}_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \) and \( \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \), where \( \mathbf{u} \) is a unit vector. The line through \((x_0, y_0)\) in the direction of \( \mathbf{u} \) is

\[
\mathbf{r} = \mathbf{r}_0 + t \mathbf{u}, \quad t \in \mathbb{R}.
\]

Then

\[
\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + \begin{bmatrix} tu_1 \\ tu_2 \end{bmatrix}, \quad t \in \mathbb{R}.
\]

Since \( x = x_0 + tu_1 \) and \( y = y_0 + tu_2 \), it follows that

\[
\frac{dx}{dt} = u_1, \quad \frac{dy}{dt} = u_2.
\]

Let \( f(x, y) \) be a function of two variables and let \( z(t) := f(x(t), y(t)) \).

By the chain rule and (10.4.7), we conclude that

\[
\frac{dz}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} = \frac{\partial f}{\partial x} u_1 + \frac{\partial f}{\partial y} u_2 = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.
\]

**Definition 10.4.6.** Assume that \( z = f(x, y) \) is a function of two independent variables and that \( \partial f/\partial x \) and \( \partial f/\partial y \) exist. Then the vector

\[
\text{grad}(f)(x, y) = \nabla f(x, y) = \begin{bmatrix} f_x(x, y) \\ f_y(x, y) \end{bmatrix}
\]

is called the **gradient** of \( f \) at \((x, y)\).

The **directional derivative** of \( f(x, y) \) at \((x_0, y_0)\) in the direction of the unit vector \( \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \) is

\[
D_{\mathbf{u}} f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \mathbf{u}.
\]

Note that in the definition of the directional derivative, we assume that \( \mathbf{u} \) is a unit vector. Choosing a unit vector ensures that the directional derivative of \( f(x, y) \) agrees with the partial derivatives when we go along the positive \( x \)- or \( y \)-axis.

(i) If \( \mathbf{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \), then

\[
D \begin{bmatrix} 1 \\ 0 \end{bmatrix} f(x_0, y_0) = \begin{bmatrix} f_x(x_0, y_0) \\ f_y(x_0, y_0) \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = f_x(x_0, y_0).
\]

(ii) If \( \mathbf{u} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \), then

\[
D \begin{bmatrix} 0 \\ 1 \end{bmatrix} f(x_0, y_0) = \begin{bmatrix} f_x(x_0, y_0) \\ f_y(x_0, y_0) \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = f_y(x_0, y_0).
Example 10.4.7. Compute the directional derivative of 
\( f(x, y) = \sqrt{x^2 + 2y^2} \)
at the point \((-1, 2)\) in the direction \([-1/3\]

PROOF. The gradient of \( f \) is
\[
\nabla f(x, y) = \begin{bmatrix}
\frac{x}{\sqrt{x^2 + 2y^2}} \\
\frac{2y}{\sqrt{x^2 + 2y^2}}
\end{bmatrix}
\]
and the gradient of \( f \) at \((-1, 2)\) is
\[
\nabla f(-1, 2) = \begin{bmatrix}
\frac{-1}{\sqrt{1+8}} \\
\frac{4}{\sqrt{1+8}}
\end{bmatrix} = \begin{bmatrix}
-1/3 \\
4/3
\end{bmatrix}.
\]
Since \([-1/3\] is not a unit vector, we normalize it by
\[
u = \frac{1}{\sqrt{10}} \begin{bmatrix}
-1 \\
3
\end{bmatrix}.
\]
Therefore
\[
D_u f(-1, 2) = \nabla f(-1, 2) \cdot u = \begin{bmatrix}
-1/3 \\
4/3
\end{bmatrix} \cdot \begin{bmatrix}
\frac{-1}{\sqrt{10}} \\
\frac{4}{\sqrt{10}}
\end{bmatrix} = \frac{13}{3\sqrt{10}}.
\]
□

Example 10.4.8. Compute the directional derivative of 
\( f(x, y) = x^2 y - 2y^2 \)
at the point \((-3, 2)\) in the direction of \((-1, 1)\).

PROOF. The gradient of \( f \) is
\[
\nabla f(x, y) = \begin{bmatrix}
2xy \\
x^2 - 4y
\end{bmatrix}
\]
and the gradient of \( f \) at \((-3, 2)\) is
\[
\nabla f(-3, 2) = \begin{bmatrix}
-12 \\
1
\end{bmatrix}.
\]
The vector that goes from \((-3, 2)\) to \((-1, 1)\) has the form
\[
\begin{bmatrix}
-1 - (-3) \\
1 - 2
\end{bmatrix} = \begin{bmatrix}
2 \\
-1
\end{bmatrix}.
\]
Since \([2\] is not a unit vector, we normalize it by
\[
u = \frac{1}{\sqrt{5}} \begin{bmatrix}
2 \\
-1
\end{bmatrix}.
\]
Therefore
\[ D_u f(-3, 2) = \nabla f(-3, 2) \cdot u = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \end{bmatrix} = -5\sqrt{5}. \]

\[ \square \]

**Example 10.4.9.** Compute the directional derivative of \( f(x, y) = 2x^2y - 3x \) at the point \( P = (2, 1) \) in the direction of the point \( Q = (3, 2) \).

**Proof.** The gradient of \( f \) at \((2, 1)\) is
\[ \nabla f(2, 1) = \begin{bmatrix} 4xy - 3 \\ 2x^2 \end{bmatrix} \bigg|_{(2, 1)} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}. \]

The vector that goes from \((2, 1)\) to \((3, 2)\) has the form
\[ \begin{bmatrix} 3 - 2 \\ 2 - 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \]

Since \( \begin{bmatrix} 1 \\ 1 \end{bmatrix} \) is not a unit vector, we normalize it by
\[ u = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \]

Therefore
\[ D_u f(2, 1) = \nabla f(2, 1) \cdot u = \frac{1}{\sqrt{2}} \begin{bmatrix} 5 \\ 8 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{13}{\sqrt{2}}. \]

\[ \square \]

We will now show that, geometrically, the gradient of \( f \) at \((x_0, y_0)\) is perpendicular to the level curve \( f(x, y) = c \) that passes through this point.

**Theorem 10.4.10.** Suppose that \( f(x, y) \) is a differentiable function. The gradient vector \( \nabla f(x, y) \) has the following properties:

(i) at each point \((x_0, y_0)\), \( f(x, y) \) increases most rapidly in the direction of the gradient vector \( \nabla f(x_0, y_0) \);

(ii) the gradient vector of \( f \) at a point \((x_0, y_0)\) is perpendicular to the level curve through \((x_0, y_0)\).

**Proof.** (i) Recall that
\[ D_u f(x, y) = \nabla f(x, y) \cdot u = |\nabla f(x, y)||u|\cos \theta, \]
where \( \theta \) is the angle between \( \nabla f(x, y) \) and \( u \). Since \(|u| = 1 \) (\( u \) is a unit vector), we have
\[ D_u f(x, y) = |\nabla f(x, y)|\cos \theta. \]

The angle \( \theta \) is in the interval \([0, 2\pi)\) and \( \cos \theta \) is maximal when \( \theta = 0 \). We therefore find that \( D_u f(x, y) \) is maximal when \( u \) is in the direction of \( \nabla f(x, y) \).
(ii) Consider a parameterized curve

\[ \mathbf{r}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \]

of the level curve \( f(x, y) = c \). Then

\[ 0 = \frac{d}{dt} f(x(t), y(t)) = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} = \nabla f \cdot \frac{d\mathbf{r}}{dt}. \]

This equation shows that the gradient of \( f \) at \((x_0, y_0)\) is perpendicular to the level curve at \((x_0, y_0)\). \( \square \)

**Example 10.4.11.** Let \( f(x, y) = x^2y + y^2 \). In what direction does \( f(x, y) \) increase most rapidly at \((1, 1)\)?

**Proof.** By Theorem 10.4.10, the function \( f(x, y) \) increases most rapidly at \((1, 1)\) in the direction of \( \nabla f(1, 1) \). Since

\[ \nabla f(x, y) = \begin{bmatrix} 2xy \\ x^2 + 2y \end{bmatrix}, \]

it follows that

\[ \nabla f(1, 1) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}. \]

That is, the function \( f(x, y) \) increases most rapidly in the direction \( \begin{bmatrix} 2 \\ 3 \end{bmatrix} \) at the point \((1, 1)\). \( \square \)

**Example 10.4.12.** Find a unit vector that is perpendicular to the level curve of the function \( f(x, y) = x^2 - y^2 \) at \((1, 2)\).

**Proof.** The gradient of \( f \) at \((1, 2)\) is perpendicular to the level curve at \((1, 2)\). The gradient of \( f \) is given as

\[ \nabla f(x, y) = \begin{bmatrix} 2x \\ -2y \end{bmatrix}, \]

hence

\[ \nabla f(1, 2) = \begin{bmatrix} 2 \\ -4 \end{bmatrix} \]

and its normalized vector is

\[ \mathbf{u} = \frac{1}{\sqrt{20}} \begin{bmatrix} 2 \\ -4 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} \sqrt{5} \\ -\frac{2}{5} \sqrt{5} \end{bmatrix}. \]
Example 10.4.13. Find a unit vector that is normal to the level curve of the function \( f(x, y) = x^2 - y^3 \) at the point (1, 3).

**Proof.** The gradient vector of \( f \) at (1, 3) is
\[
\nabla f(1, 3) = \begin{bmatrix} 2x \\ -3y^2 \end{bmatrix}_{(1,3)} = \begin{bmatrix} 2 \\ -27 \end{bmatrix}.
\]
and the desired unit vector is given by
\[
u = \frac{1}{\sqrt{2^2 + 27^2}} \begin{bmatrix} 2 \\ -27 \end{bmatrix} = \frac{1}{\sqrt{733}} \begin{bmatrix} 2 \\ -27 \end{bmatrix}.
\]
□

10.5. Applications

In Chapter 5, we introduced local extrema for functions of one variable. Local extrema can also be defined for functions of more than one independent variables.

10.5.1. Maxima and minima. Recall that we denote by \( B_\delta(x_0, y_0) \) the open disk with radius \( \delta \) centered at \((x_0, y_0)\).

**Definition 10.5.1.** A function \( f(x, y) \) defined on a set \( D \subset \mathbb{R}^2 \) has a local (or relative) maximum at a point \((x_0, y_0)\) if there exists a \( \delta > 0 \) such that
\[
f(x, y) \leq f(x_0, y_0) \quad \text{for all } (x, y) \in B_\delta(x_0, y_0) \cap D.
\]

A function \( f(x, y) \) defined on a set \( D \subset \mathbb{R}^2 \) has a local (or relative) minimum at a point \((x_0, y_0)\) if there exists a \( \delta > 0 \) such that
\[
f(x, y) \geq f(x_0, y_0) \quad \text{for all } (x, y) \in B_\delta(x_0, y_0) \cap D.
\]

We can define global (or absolute) extrema as well: if the inequalities in the definition hold for all \((x, y) \in D\), then \( f \) has an absolute maximum (minimum) at \((x_0, y_0)\).

As Theorem 5.1.3, we have

**Theorem 10.5.2.** If \( f(x, y) \) has a local extremum at \((x_0, y_0)\) and if \( f \) is differentiable at \((x_0, y_0)\), then
\[
(10.5.1) \quad \nabla f(x_0, y_0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
\]

A point \((x_0, y_0)\) that satisfies \((10.5.1)\) is called a (smooth) critical point; points where \( f(x, y) \) is not differentiable are also called (singular) critical points. Note that \((10.5.1)\) is a necessary condition.
**Example 10.5.3.** Find the all critical points of \( f(x, y) = x^2 + y^2 + 1 \) and determine equations of the tangent planes at those points.

**Proof.** Since \( f(x, y) \) is differentiable, all critical points must satisfy (10.5.1). From

\[
\begin{bmatrix}
0 \\
0
\end{bmatrix} = \begin{bmatrix}
2x \\
2y
\end{bmatrix}
\]

we obtain the only critical point is \((0, 0)\). Then the tangent plane at \((0, 0)\) is

\[
z = f(0, 0) + f_x(0, 0)(x - 0) + f_y(0, 0)(y - 0) = 1.
\]

\[\Box\]

**Example 10.5.4.** Find all critical points \( f(x, y) = x^2 + y^2 + xy, \ (x, y) \in \mathbb{R}^2 \).

**Proof.** From

\[
\begin{bmatrix}
0 \\
0
\end{bmatrix} = \nabla f(x, y) = \begin{bmatrix}
2x + y \\
2y + x
\end{bmatrix} \implies 2x + y = 0, \quad x + 2y = 0
\]

we find that the only critical points is \((0, 0)\).

\[\Box\]

We now give a sufficient condition that will allow us to determine whether a candidate for a local extremum is indeed a local extremum.

Suppose that the second partial derivatives of \( f \) are continuous in a disk centered at \((x_0, y_0)\). Consider the **Hessian matrix** of \( f \) at \((x_0, y_0)\):

\[
(10.5.2) \quad \text{Hess}(f)(x_0, y_0) := \begin{bmatrix}
f_{xx}(x_0, y_0) & f_{xy}(x_0, y_0) \\
f_{yx}(x_0, y_0) & f_{yy}(x_0, y_0)
\end{bmatrix}.
\]

**Example 10.5.5.** Compute the Hessian matrices of

\( f_1(x, y) = x^2 + y^2, \ f_2(x, y) = x^2 - y^2, \ f_3(x, y) = -x^2 - y^2. \)

**Proof.** By (10.5.2), we have

\[
\text{Hess}(f_1)(x, y) = \begin{bmatrix}
2 & 0 \\
0 & 2
\end{bmatrix}, \quad \text{Hess}(f_2)(x, y) = \begin{bmatrix}
2 & 0 \\
0 & -2
\end{bmatrix}, \quad \text{Hess}(f_3)(x, y) = \begin{bmatrix}
-2 & 0 \\
0 & -2
\end{bmatrix}.
\]

\[\Box\]

It can be showed that \((0, 0)\) is a candidate for a local extremum for all three functions in Example 10.5.5. However, \((0, 0)\) is a local minimum for \( f_1(x, y) \) and is a local maximum for \( f_3 \).

**Theorem 10.5.6.** Suppose that the second partial derivatives of \( f \) are continuous in a disk centered at \((x_0, y_0)\). Suppose that \( \nabla f(x_0, y_0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \).
10.5. APPLICATIONS

(1) If \( \det \text{Hess}(f)(x_0, y_0) > 0 \) and \( f_{xx}(x_0, y_0) > 0 \), then \( f \) has a local minimum at \((x_0, y_0)\).

(2) If \( \det \text{Hess}(f)(x_0, y_0) > 0 \) and \( f_{xx}(x_0, y_0) < 0 \), then \( f \) has a local maximum at \((x_0, y_0)\).

(3) If \( \det \text{Hess}(f)(x_0, y_0) < 0 \), then \( f \) does not have a local extremum at \((x_0, y_0)\). The point \((x_0, y_0)\) is then called a saddle point.

Example 10.5.7. (Example 10.5.4 is revisited) The Hessian matrix of \( f(x, y) = x^2 + y^2 + xy \) is
\[
\text{Hess}(f)(x, y) = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.
\]
Since \( \det \text{Hess}(f)(0, 0) = 2 \times 2 - 1 \times 1 = 3 > 0 \), it follows from Theorem 10.5.6 that \((0, 0)\) is a local minimum.

Example 10.5.8. Find all local extrema of \( f(x, y) = 3xy - x^3 - y^3 \), \((x, y) \in \mathbb{R}^2\) and classify them according to whether each is a local maximum, a local minimum, or neither.

Proof. The Hessian matrix of \( f \) is
\[
\text{Hess}(f)(x, y) = \begin{bmatrix} -6x & 3 \\ 3 & -6y \end{bmatrix}
\]
and its determinant is \( \det \text{Hess}(f)(x, y) = 36xy - 9 \). The critical points satisfy
\[
\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \nabla f(x, y) = \begin{bmatrix} 3y - 3x^2 \\ 3x - 3y^2 \end{bmatrix}
\]
and this set of equations has the solutions \((0, 0)\) or \((1, 1)\). Since \( \det \text{Hess}(f)(1, 1) = 36 - 9 = 27 > 0 \) and \( f_{xx}(1, 1) = -6 < 0 \), \( f(x, y) \) has a local maximum at \((1, 1)\). At \((0, 0)\), \( \det \text{Hess}(f)(0, 0) = -9 < 0 \), hence \((0, 0)\) is a saddle point.

Suppose that the second partial derivatives of \( f \) are continuous in a disk centered at \((x_0, y_0)\). According to Theorem 10.2.6, we have
\[
f_{yx}(x_0, y_0) = f_{xy}(x_0, y_0).
\]
Hence the Hessian matrix of \( f \) at \((x_0, y_0)\) is of the form
\[
H := \begin{bmatrix} a & c \\ c & b \end{bmatrix},
\]
where \( a = f_{xx}(x_0, y_0), b = f_{yy}(x_0, y_0), \) and \( c = f_{xy}(x_0, y_0) \).

(i) The matrix \( H \) is called symmetric.
(ii) The eigenvalues of a symmetric matrix are always real. Let $\lambda$ be an eigenvalue of $H$; then

$$
0 = \lambda^2 - \text{tr}(H)\lambda + \det(H) = \lambda^2 - (a + b)\lambda + (ab - c^2)
$$

$$
= \left(\lambda - \frac{a + b}{2}\right)^2 + ab - c^2 - \frac{(a + b)^2}{4}
$$

$$
= \left(\lambda - \frac{a + b}{2}\right)^2 - \left(c^2 + \frac{a^2 + b^2 - 2ab}{4}\right)
$$

$$
= \left(\lambda - \frac{a + b}{2}\right)^2 - \left(c^2 + \left(\frac{a - b}{2}\right)^2\right).
$$

Therefore the eigenvalues of $H$ are

$$
\lambda = \frac{a + b}{2} \pm \sqrt{c^2 + \left(\frac{a - b}{2}\right)^2} \in \mathbb{R}.
$$

**Theorem 10.5.9.** Suppose that the second partial derivatives of $f$ are continuous in a disk centered at $(x_0, y_0)$. Suppose also that $\nabla f(x_0, y_0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Then

1. If the two eigenvalues of $\text{Hess}(f)(x_0, y_0)$ are positive, then $f$ has a local minimum at $(x_0, y_0)$.
2. If the two eigenvalues of $\text{Hess}(f)(x_0, y_0)$ are negative, then $f$ has a local maximum at $(x_0, y_0)$.
3. If the two eigenvalues of $\text{Hess}(f)(x_0, y_0)$ are of opposite signs, then $f$ does not have a local extremum at $(x_0, y_0)$. The point $(x_0, y_0)$ is then called a saddle point.

If one or both eigenvalues are equal to zero, we cannot say anything about the nature of the critical point on the basis of the Hessian matrix.

**Example 10.5.10.** Find the local extrema of

$$
f(x, y) = 2x^2 - xy + y^4, \quad (x, y) \in \mathbb{R}^2.
$$

**Proof.** We compute

$$
\nabla f(x, y) = \begin{bmatrix} 4x - y \\ -x + 4y^3 \end{bmatrix}, \quad \text{Hess}(f)(x, y) = \begin{bmatrix} 4 & -1 \\ -1 & 12y^2 \end{bmatrix}.
$$

Since $f(x, y)$ is differentiable on $\mathbb{R}^2$, all critical points are

$$
(0, 0), \quad \left(\frac{1}{16}, \frac{1}{4}\right), \quad \left(-\frac{1}{16}, -\frac{1}{4}\right).
$$

We evaluate the Hessian matrix at each candidate and compute its eigenvalues.

1. $\text{Hess}(f)(0, 0) = \begin{bmatrix} 4 & -1 \\ -1 & 0 \end{bmatrix}$. The eigenvalues satisfy

$$
\lambda^2 - 4\lambda - 1 = 0;
$$
thus
\[ \lambda = \frac{4 \pm \sqrt{16 + 4}}{2} = 2 \pm \sqrt{5} \approx \{4.2361, -0.2361\} \]
implies that \( f \) has a saddle point at \((0, 0)\).

(ii) \( \text{Hess}(f)(1/16, 1/4) = \begin{bmatrix} 4 & -1 \\ -1 & 3/4 \end{bmatrix} \). The eigenvalues satisfy
\[ \lambda^2 - \frac{19}{4} \lambda + 2 = 0; \]
thus
\[ \lambda = \frac{19}{4} \pm \frac{\sqrt{361 - 8}}{2} = \frac{19}{8} \pm \frac{1}{8} \sqrt{233} \approx \{4.2830, 0.4670\} \]
implies that \( f \) has a local minimum at \((1/16, 1/4)\).

(iii) \( \text{Hess}(f)(-1/16, -1/4) = \begin{bmatrix} 4 & -1 \\ -1 & 3/4 \end{bmatrix} \). This is the same matrix as that for case (ii). We thus conclude that \( f \) has a local minimum at \((-1/16, 1/4)\) as well.

Let \( A \) be a \( 2 \times 2 \) symmetric matrix and recall that
\[ \det(A) = \lambda_1 \lambda_2, \quad \text{tr}(A) = \lambda_1 + \lambda_2, \]
where \( \lambda_1, \lambda_2 \) are eigenvalues of \( A \). We have showed that the eigenvalues of \( A \) are both real. If \( \det(A) > 0 \), then either both \( \lambda_1 \) and \( \lambda_2 \) are positive or both are negative. If, in addition, \( \text{tr}(A) > 0 \), then both \( \lambda_1 \) and \( \lambda_2 \) are positive. We thus conclude from Theorem 10.5.9 that

**Theorem 10.5.11.** Suppose that the second partial derivatives of \( f \) are continuous in a disk centered at \((x_0, y_0)\). Suppose also that \( \nabla f(x_0, y_0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \). Then

1. If \( \det \text{Hess}(f)(x_0, y_0) > 0 \) and \( \text{tr} \text{Hess}(f)(x_0, y_0) > 0 \), then \( f \) has a local minimum at \((x_0, y_0)\).
2. If \( \det \text{Hess}(f)(x_0, y_0) > 0 \) and \( \text{tr} \text{Hess}(f)(x_0, y_0) < 0 \), then \( f \) has a local maximum at \((x_0, y_0)\).
3. If \( \det \text{Hess}(f)(x_0, y_0) = 0 \), then \((x_0, y_0)\) is not a local extremum; instead, \((x_0, y_0)\) is a saddle point.

Recall that if one of the eigenvalues of \( \text{Hess}(f)(x_0, y_0) \) is equal to 0 (or, equivalently, if \( \det \text{Hess}(f)(x_0, y_0) = 0 \)), then we cannot say anything about the nature of the critical point on the basis of the Hessian matrix.

**Example 10.5.12.** In this problem, we will illustrate that if one of the eigenvalues of the Hessian matrix at a point where the gradient vanishes is equal to 0, then we
cannot make any statements about whether the point is a local extremum just on
the basis of the Hessian matrix.

Consider the following functions:
\[ f_1(x, y) = x^2, \quad f_2(x, y) = x^2 + y^3, \quad f_3(x, y) = x^2 + y^4. \]

For each \( i = 1, 2, 3 \), we have
\[ \nabla f_i(0, 0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \text{Hess}(f_i)(0, 0) = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}. \]

The eigenvalues of \( \text{Hess}(f_i)(0, 0) \) are 0 and 2, hence we cannot use the criterion
stated in Theorem 10.5.11. However, \( f_1 \) and \( f_2 \) have a local (actually global)
minimum at \((0, 0)\).

Example 10.5.13. Find and classify the critical points of
\[ f(x, y) = x^3 - 4xy + y, \quad (x, y) \in \mathbb{R}^2. \]

**Proof.** Compute
\[ \nabla f(x, y) = \begin{bmatrix} 3x^2 - 4y \\ -4x + 1 \end{bmatrix}, \quad \text{Hess}(f)(x, y) = \begin{bmatrix} 6x & -4 \\ -4 & 0 \end{bmatrix}. \]
The only critical point is \((1/4, 3/64)\), since \( f \) is differentiable on \( \mathbb{R}^2 \). From
\[ \text{Hess}(f) \left( \frac{1}{4}, \frac{3}{64} \right) = \begin{bmatrix} 3 & -4 \\ -4 & 0 \end{bmatrix}, \]
we see that \( \det \text{Hess}(f)(1/4, 3/64) = -16 < 0 \) and hence \( f \) has a saddle point
at \((1/4, 3/64)\). \( \square \)

Example 10.5.14. Find and classify the critical points of
\[ f(x, y) = \sqrt{x^2 + y^2}, \quad (x, y) \in \mathbb{R}^2. \]

**Proof.** Compute
\[ \nabla f(x, y) = \begin{bmatrix} \frac{x}{\sqrt{x^2 + y^2}} \\ \frac{y}{\sqrt{x^2 + y^2}} \end{bmatrix} = \begin{bmatrix} \frac{x}{\sqrt{x^2 + y^2}} \\ \frac{y}{\sqrt{x^2 + y^2}} \end{bmatrix}, \quad (x, y) \neq (0, 0). \]

Since the gradient of \( f \) is undefined at \((0, 0)\), the point \((0, 0)\) is a critical point.
There are no other critical points, because \( \nabla f(x, y) \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \) for \((x, y) \neq (0, 0)\).
Since \( f(x, y) \geq 0 \), it follows that \( f(x, y) \) has a local (actually global) minimum at
\((0, 0)\). \( \square \)
10.5.2. **Global extrema.** Recall that, for functions of one variable, the extreme-value theorem, Theorem 5.1.1, guarantees the existence of global extrema for functions defined on a closed interval. The analogue of closed intervals in the two-dimensional plane is a closed set; similarly, the analogue of an open interval is an open set.

Let \( D \subset \mathbb{R}^2 \) be a set.

(i) A point \((x, y)\) is called an **interior point** of \( D \) if there exists a \( \delta > 0 \) such that the disk centered at \((x, y)\) with radius \( \delta \) is contained in \( D \)—that is, if \( B_\delta(x, y) \subset D \).

(ii) A point \((x, y)\) is called a **boundary point** of \( D \) if every disk centered at \((x, y)\) contains both points in \( D \) and points not in \( D \); the boundary point \((x, y)\) need not be contained in \( D \).

(iii) The **interior** of \( D \) consists of all interior points of \( D \), written as \( \text{Int}(D) \) or \( D^\circ \); the **boundary** of \( D \) consists of all boundary points of \( D \), written as \( \text{bdy}(D) \) or \( \partial D \).

(iv) A set \( D \subset \mathbb{R}^2 \) is **open** if \( D = D^\circ \); a set \( D \subset \mathbb{R}^2 \) is **closed** if \( D = \overline{D} := D^\circ \cup \partial D \).

Consider the open unit disk
\[
B := \{ (x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1 \}.
\]
The boundary of \( B \) is the unit circle \( \{ (x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1 \} \) and \( \overline{B} = \{ (x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1 \} \).

(v) A set \( D \subset \mathbb{R}^2 \) is **bounded** if it is contained within some disk.

**Theorem 10.5.15. (Extreme-value theorem in \( \mathbb{R}^2 \))** If \( f \) is continuous on a closed and bounded set \( D \subset \mathbb{R}^2 \), then \( f \) has both a global maximum and a global minimum on \( D \).

Global extrema can occur in the interior of \( D \) or on the boundary of \( D \). We have discussed how to find candidates for local extrema in the interior. To find local extrema on the boundary, we consider the restricted function \( f|D \) that can be viewed as a function of one variable; we can then use the tools of single-variable calculus to find all candidates for local extrema on the boundary of \( D \). To find global extrema for continuous functions defined on a closed and bounded set, we thus proceed as follows:

1. Determine all candidates for local extrema in the interior of \( D \).
2. Determine all candidates for local extrema on the boundary of \( D \).
3. Select the global maximum and the global minimum from the set of points determined in steps (1) and (2).

**Example 10.5.16.** Find the global extrema of
\[
f(x, y) = x^2 - 3y + y^2, \quad -1 \leq x \leq 1, \quad 0 \leq y \leq 2.
\]

**Proof.** By Theorem 10.5.15, the function \( f(x, y) \) has global extrema. Compute
\[
\nabla f(x, y) = \begin{bmatrix} 2x \\ -3 + 2y \end{bmatrix}, \quad \text{Hess}(f)(x, y) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.
\]
Letting $\nabla f(x, y) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ we find that $(0, 3/2)$ is in the interior of the domain of $f$ and by Theorem 10.5.11 $f(x, y)$ has a local minimum $-2.25$ at $(0, 3/2)$.

We now check the boundary values of $f(x, y)$.

(i) Consider the line segment $C_1 = \{(x, y) \in \mathbb{R}^2 : -1 \leq x \leq 1 \text{ and } y = 0\}$.

On $C_1$, the function $f$ is of the form

$$f(x, 0) = x^2, \quad -1 \leq x \leq 1.$$ 

Hence, the critical point of $f$ on $C_1$ is $(0, 0)$, and then $f$ has the global minimum 0 at $(0, 0)$ and has the global maximum 1 at $(-1, 0)$ and $(1, 0)$, on the line segment $C_1$.

(ii) Consider the line segment $C_2 = \{(x, y) \in \mathbb{R}^2 : x = 1 \text{ and } 0 \leq y \leq 2\}$. On $C_2$, the function $f$ is of the form

$$f(1, y) = 1 - 3y + y^2, \quad 0 \leq y \leq 2.$$ 

Hence, the critical point of $f$ on $C_2$ is $(1, 3/2)$, and then $f$ has the global minimum $-1.25$ and has the global maximum 1 at $(1, 0)$, on the line segment $C_2$.

(iii) Consider the line segment $C_3 = \{(x, y) \in \mathbb{R}^2 : -1 \leq x \leq 1 \text{ and } y = 2\}$. On $C_3$, the function $f$ is of the form

$$f(x, 2) = x^2 - 2, \quad -1 \leq x \leq 1.$$ 

Hence, the critical point of $f$ on $C_3$ is $(0, 2)$, and then $f$ has the global minimum $-2$ at $(0, 0)$ and has the global maximum $-1$ at $(-1, 0)$ and $(1, 0)$, on the line segment $C_3$. 
(iv) Consider the line segment $C_4 = \{(x, y) \in \mathbb{R}^2 : x = -1 \text{ and } 0 \leq y \leq 2\}$. On $C_4$, the function $f$ is of the form
\[ f(-1, y) = 1 - 3y + y^2, \quad 0 \leq y \leq 2. \]
Hence, the critical point of $f$ on $C_4$ is $(-1, 3/2)$, and then $f$ has the global minimum $-1.25$ at $(-1, 3/2)$ and has the global maximum 1 at $(-1, 0)$. Therefore, the function has the global maximum 1 at $(1, 0)$ and $(1, 0)$, and the global minimum $-2.25$ at $(0, 3/2)$.

\[ \square \]

**Example 10.5.17.** Find the absolute maxima and minima of $f(x, y) = x^2 + y^2 - 2x + 4$ on the disk $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 4\}$.

**Proof.** By Theorem [10.5.15], the function $f(x, y)$ has global extrema. Compute
\[ \nabla f(x, y) = \begin{bmatrix} 2x - 2 \\ 2y \end{bmatrix}, \quad \text{Hess}(f)(x, y) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}. \]
The critical point of $f$ in the domain $D$ is $(1, 0)$ and $f(1, 0) = 3$.
We now seek extrema on the boundary of the domain: the circle $x^2 + y^2 = 4$.
Consider the parameterization of the circle:
\[ x = 2 \cos \theta, \quad y = 2 \sin \theta, \quad 0 \leq \theta < 2\pi. \]
On the boundary $\partial D$, we have
\[ f(x, y) = 4 - 4 \cos \theta + 4 = 8 - 4 \cos \theta = g(\theta), \quad 0 \leq \theta < 2\pi. \]
Since $g'(\theta) = 4 \sin \theta$, it follows that the critical points of $g(\theta)$ are $\theta = 0$ and $\theta = \pi$. Consequently, $f(x, y)$ has the minimum $f(2, 0) = 4$ at $(2, 0)$ and the maximum $f(-2, 0) = 12$ at $(-2, 0)$.
Therefore the functions has the global maximum at $(-2, 0)$ and the global minimum at $(1, 0)$.

**Example 10.5.18.** Find the absolute maxima and minima of $f(x, y) = x^2 + y^2 + x - y$ on the disk $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$.

**Proof.** Compute
\[ \nabla f(x, y) = \begin{bmatrix} 2x + 1 \\ 2y - 1 \end{bmatrix}, \quad \text{Hess}(f)(x, y) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}. \]
Hence the critical point of $f$ in the interior of $D$ is $(-1/2, 1/2)$ and $f(x, y)$ has a local minimum $f(-1/2, 1/2) = -1/2$ at $(-1/2, 1/2)$. We now consider the boundary of $D$: the unit circle $x^2 + y^2 = 1$. On the unit circle, we have
\[ f(x, y) = 1 + \cos \theta - \sin \theta = g(\theta), \quad 0 \leq \theta < 2\pi. \]
Since $g'(\theta) = -\sin \theta - \cos \theta$, we find that the critical points of $g(\theta)$ are
\[ \theta = \frac{\pi}{2}, \quad \frac{3\pi}{4}, \quad \frac{3\pi}{2}, \quad \frac{7\pi}{4}. \]
the corresponding values are

\[ g(\theta) = 0, \quad 1, \quad 1 + \sqrt{2}. \]

Therefore, the function \( f(x, y) \) has the global minimum \(-1/2\) at \((-1/2, 1/2)\) and the global maximum \(1 + \sqrt{2}\) at \((\sqrt{2}/2, -\sqrt{2}/2)\).

\[ \square \]

**Example 10.5.19.** Determine the values of three nonnegative numbers whose sum is 90 and whose product is maximal.

**Proof.** We denote the three numbers by \(x, y, z\). Then

\[ x + y + z = 90, \quad x, y, z \geq 0. \]

Consider the function

\[ f(x, y) = xyz = xy(90 - x - y), \quad x + y \leq 90, \quad x \geq 0, \quad y \geq 0, \]

and the domain \(D = \{(x, y) \in \mathbb{R}^2 : x + y \leq 90, \quad x \geq 0, \quad y \geq 0\}\). Compute

\[
\nabla f(x, y) = \begin{bmatrix}
90y - 2xy - y^2 \\
90x - x^2 - 2xy
\end{bmatrix}, \quad \text{Hess}(f)(x, y) = \begin{bmatrix}
-2y & 90 - 2x - 2y \\
90 - 2x - 2y & -2x
\end{bmatrix}.
\]

The critical points of \(f\) in the interior of \(D\) satisfy

\[ 0 = y(90 - 2x - y), \quad 0 = x(90 - 2y - x). \]

The solutions to above system is

\[ (x, y) = (0, 0), \quad (0, 90), \quad (90, 0), \quad (30, 30). \]

Since we consider the critical points of \(f\) in the interior of \(D\), it follows that such a critical point is \((30, 30)\). From

\[
\text{Hess}(f)(30, 30) = \begin{bmatrix}
-60 & -30 \\
-30 & -60
\end{bmatrix}
\]

we see that \(f(x, y)\) has a local maximum \(f(30, 30) = 27000\) at \((30, 30)\).

We now consider the boundary of \(D\). The boundary of \(D\) consists of three line segments:

(i) \(\{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 90 \text{ and } y = 0\}\).

(ii) \(\{(x, y) \in \mathbb{R}^2 : x + y = 90 \text{ and } 0 \leq x \leq 90\}\).

(iii) \(\{(x, y) \in \mathbb{R}^2 : x = 0 \text{ and } 0 \leq y \leq 90\}\).

In each case, we have \(f(x, y) = 0\). Therefore, the maximum of the product of three nonnegative numbers is 27000. \[ \square \]

**Example 10.5.20.** Suppose crop yield \(Y\) depends on nitrogen \((N)\) and phosphorus \((P)\) concentrations as

\[ Y(N, P) = NP e^{-(N + P)}. \]

Find the value of \((N, P)\) that maximizes crop yield.
**Proof.** Compute
\[
\nabla Y(N, P) = e^{-(N+P)} \begin{bmatrix} (1 - N)P \\ (1 - P)N \end{bmatrix},
\]
and
\[
\text{Hess}(Y)(N, P) = e^{-(N+P)} \begin{bmatrix} (N - 2)P & (1 - N)(1 - P) \\ (1 - N)(1 - P) & (P - 2)N \end{bmatrix}.
\]
The solutions of \(\nabla Y(N, P) = 0\) are \((N, P) = (0, 0), (1, 1)\).

Since \(\text{Hess}(Y)(0, 0) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \text{Hess}(Y)(1, 1) = e^{-2} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\)

it follows that \(Y(N, P)\) has the global maximum \(e^{-2}\) at \((1, 1)\). □

### 10.5.3. Extrema with constraints.

Consider the function
\[
\text{f}(x, y) = e^{-xy}, \quad x^2 + 4y^2 \leq 1.
\]

Compute
\[
\nabla f(x, y) = \begin{bmatrix} -ye^{-xy} \\ -xe^{-xy} \end{bmatrix}, \quad \text{Hess}(f)(x, y) = \begin{bmatrix} y^2e^{-xy} & (xy - 1)e^{-xy} \\ (xy - 1)e^{-xy} & x^2e^{-xy} \end{bmatrix}.
\]

Letting \(\nabla f(x, y) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\) yields \((x, y) = (0, 0)\). then

\[
\text{Hess}(f)(0, 0) = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.
\]

Since \(\det \text{Hess}(f)(0, 0) < 0\), \((0, 0)\) is a saddle point. Hence, the global extrema must occur on the boundary of \(\{(x, y) \in \mathbb{R}^2 : x^2 + 4y^2 \leq 1\}\):

\[
\text{f}(x, y) = e^{-xy}, \quad g(x, y) = x^2 + 4y^2 - 1 = 0.
\]

To find the extrema of \(f\) on the boundary, we need the following

**Theorem 10.5.21.** (Lagrange’s theorem) Assume that \(f\) and \(g\) have continuous first partial derivatives and that \(f(x, y)\) has an extremum at \((x_0, y_0)\) subject to the constraint \(g(x, y) = 0\). If \(\nabla g(x_0, y_0) \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}\), then there exists a number \(\lambda\) such that
\[
\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0).
\]

The number \(\lambda\) is called a **Lagrange multiplier**. Using Lagrange multipliers to find candidates for extrema subject to a constraint is called the **method of Lagrange multiplier**. The condition \(\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)\) is a necessary condition.

**Example 10.5.22.** Find all extrema of \(f(x, y) = e^{-xy}\) subject to the constraint \(x^2 + 4y^2 = 1\).
Proof. As in (10.5.6), let \( g(x, y) = x^2 + 4y^2 - 1 \). By Theorem 10.5.21, we are looking for \((x, y)\) and \(\lambda\) satisfying
\[
\nabla f(x, y) = \lambda \nabla g(x, y), \quad g(x, y) = 0.
\]
Since
\[
\nabla f(x, y) = e^{-xy} \begin{bmatrix} -y \\ -x \end{bmatrix}, \quad \nabla g(x, y) = \begin{bmatrix} 2x \\ 8y \end{bmatrix},
\]
we arrive at
\[-ye^{-xy} = 2\lambda x, \quad -xe^{-xy} = 8\lambda y, \quad x^2 + 4y^2 = 1.
\]
Eliminating \(\lambda\) yields
\[x^2 - 4y^2 = 0, \quad x^2 + 4y^2 = 1.
\]
There are four solutions \((x, y)\) where
\[\left(\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}\right), \left(-\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}\right), \left(-\sqrt{\frac{1}{2}}, -\sqrt{\frac{1}{2}}\right), \left(\sqrt{\frac{1}{2}}, -\sqrt{\frac{1}{2}}\right).
\]
with
\[f\left(\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}\right) = f\left(-\sqrt{\frac{1}{2}}, -\sqrt{\frac{1}{2}}\right) = e^{-1/4}
\]
and
\[f\left(\sqrt{\frac{1}{2}}, -\sqrt{\frac{1}{2}}\right) = f\left(-\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}\right) = e^{1/4}.
\]
The maxima are \((-\sqrt{1/2}, \sqrt{1/2}/2)\) and \((\sqrt{1/2}, -\sqrt{1/2}/2)\), and the minima are \((\sqrt{1/2}, \sqrt{1/2}/2)\) and \((-\sqrt{1/2}, -\sqrt{1/2}/2)\).

We have mentioned that the condition \(\nabla f = \lambda \nabla g\) is a necessary condition. This means that finding \((x_0, y_0)\) so that \(\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)\) only identifies candidates for local extrema.

Example 10.5.23. Use Lagrange multipliers to identify candidates for local extrema of \(f(x, y) = y\) subject to the constraint \(y - x^3 = 0\), and show that there is one such candidate that turns out not to be a local extremum. Furthermore, show that the function \(f(x, y)\) subject to the constraint \(y - x^3 = 0\) has no global extrema.

Proof. We define \(g(x, y) = y - x^3\). Then
\[
\nabla f(x, y) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \nabla g(x, y) = \begin{bmatrix} -3x^2 \\ 1 \end{bmatrix}.
\]
From Theorem 10.5.21, we have
\[0 = -3\lambda x^2, \quad 1 = \lambda, \quad y = x^3.
\]
Hence \((x, y) = (0, 0)\).

We claim that \((0, 0)\) is not a local extremum. Along the curve \(y - x^3 = 0\), we have
\[f(x, y) = f(x, x^3) = x^3;
\]
we know from single-variable calculus that \(h(x) = x^3\) has no local extrema on \(\mathbb{R}\), and hence \((0, 0)\) is not a local minimum. Because \(\lim_{x \to \infty} h(x) = \infty\) and
Example 10.5.24. Suppose you wish to enclose a rectangular plot. You have 1600 ft of fencing. Using that material, what are the dimensions of the plot that will have the largest area?

Proof. We wish to maximize

\[ A = xy \]

subject to the constraint \(2x + 2y = 1600\). Consider

\[ f(x, y) = xy, \quad g(x, y) = 2x + 2y - 1600 = 0. \]

From

\[ \nabla f(x, y) = \begin{bmatrix} y \\ x \end{bmatrix}, \quad \nabla g(x, y) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \]

and the equation \( \nabla f(x, y) = \lambda \nabla g(x, y) \), we have

\[ y = 2\lambda, \quad x = 2\lambda, \quad x + y = 800, \]

from which we get \( \lambda = 200 \) and \( x = y = 400 \), with \( f(400, 400) = 160000 \).

We now look at the boundary. By the physical reason, \( x, y > 0 \), so that we need only to consider the line segment \( x + y = 800 \) with \( 0 < x < 800 \). On this line segment, we have

\[ f(x, y) = f(x, 800 - x) = x(800 - x) = -x^2 + 800x, \quad 0 < x < 800. \]

The maximum value takes at \((400, 400)\). Consequently, the largest area is \( f(400, 400) = 160000 \). □

Example 10.5.25. Let

\[ f(x, y) = x + y \]

with constraint function

\[ \frac{1}{x} + \frac{1}{y} = 1, \quad x \neq 0, \quad y \neq 0. \]

Use Lagrange multipliers to find all local extrema. Are these global extrema?

Proof. Let \( g(x, y) = \frac{1}{x} + \frac{1}{y} - 1 \). From

\[ \nabla f(x, y) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \nabla g(x, y) = \begin{bmatrix} -1/x^2 \\ -1/y^2 \end{bmatrix}, \]

we have

\[ 1 = -\frac{\lambda}{x^2}, \quad 1 = -\frac{\lambda}{y^2}, \quad \frac{1}{x} + \frac{1}{y} = 1, \quad x \neq 0, \quad y \neq 0. \]

Thus \( y = x \) or \( y = -x \). In the second case, we obtain \( 1 = \frac{1}{x} + \frac{1}{-x} = 0 \), a contradiction. Hence \( y = x \) and hence \( 1 = \frac{1}{x} + \frac{1}{x} = \frac{2}{x} \). Consequently, \( x = y = 2 \). So, there is only one local extrema \((2, 2)\) with \( f(2, 2) = 4 \).
It is clear to see that the local extrema $(0, 0)$ is not global, since

$$\lim_{x \to 1} f(x, y) = \lim_{x \to 1} \left( x + \frac{x}{x - 1} \right) = \lim_{x \to 1} \frac{x^2}{x - 1} = \pm \infty.$$
CHAPTER 11

Systems of differential equations

In this chapter, we develop the theory of systems of differential equations.

11.1. Linear systems

A system of differential equations is

\[
\begin{align*}
\frac{dx_1}{dt} &= g_1(t, x_1, x_2, \cdots, x_n), \\
\frac{dx_2}{dt} &= g_2(t, x_1, x_2, \cdots, x_n), \\
&\vdots \\
\frac{dx_n}{dt} &= g_n(t, x_1, x_2, \cdots, x_n).
\end{align*}
\]

(11.1.1)

On the left-hand side of (11.1.1) are the derivatives of \(x_i(t)\) with respect to \(t\); on the right-hand side of each equation is a function \(g_i\) that depends on the variables \(x_1, x_2, \cdots, x_n\) and on \(t\).

We first look at the case when the functions \(g_i\) are linear in the variables \(x_1, x_2, \cdots, x_n\)—that is, when, for \(i = 1, \cdots, n\),

\[
(11.1.2) \quad g_i(t, x_1, x_2, \cdots, x_n) = a_{i1}(t)x_1 + a_{i2}(t)x_2 + \cdots + a_{in}(t)x_n + f_i(t).
\]

We can write the linear system in matrix form as

\[
(11.1.3) \quad \frac{dx}{dt} = A(t)x(t) + f(t),
\]

where

\[
(11.1.4) \quad x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}, A(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{bmatrix}, f(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{bmatrix}
\]

Equation (11.1.4) is called a system of linear first-order equations. We will investigate only the case when \(f(t) = 0\) and \(A(t)\) does not depend on \(t\). In this case, (11.1.3) reduces to

\[
(11.1.5) \quad \frac{dx}{dt} = Ax(t),
\]

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where

\[
A = \begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{bmatrix}
\]

Equation (11.1.5) is called a **homogeneous linear first-order system with constant coefficients**, since \( f(t) = 0 \). Since the matrix \( A \) does not depend on \( t \), all the coefficients are constant; such a system is **autonomous**.

In this chapter, we restrict to the case \( n = 2 \):

\[
\frac{dx}{dt} = Ax(t),
\]

where

\[
x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.
\]

**Example 11.1.1.** Write

\[
\begin{align*}
\frac{dx_1}{dt} &= 4x_1 - 2x_2 \\
\frac{dx_2}{dt} &= -3x_1 + x_2
\end{align*}
\]
in matrix notation.

**Proof.** We have

\[
\frac{dx(t)}{dt} = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} x(t),
\]

where \( x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \).

A solution is an ordered \( n \)-tuple of functions \((x_1(t), x_2(t), \ldots, x_n(t))\) that satisfies (11.1.5). An **equilibrium** is a point \( \hat{x} = (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n) \) such that \( A\hat{x} = 0 \).

**11.1.1. Direction fields.** Consider

\[
(11.1.6) \quad \frac{dx_1}{dt} = x_1 - 2x_2, \quad \frac{dx_2}{dt} = x_2.
\]

Consider the point \((2, -1)\) in the \( x_1x_2 \)-plane. From (11.1.6), we have

\[
\frac{dx_2}{dx_1} = \frac{dx_2/dt}{dx_1/dt} = \frac{x_2}{x_1 - 2x_2};
\]

then along a curve whose tangent line at \((2, -1)\) has slope

\[
\frac{dx_2}{dx_1} \bigg|_{(2,-1)} = \frac{-1}{2 - 2(-1)} = \frac{1}{4},
\]

and this tangent line is \( y = -\frac{1}{4}x - \frac{1}{2} \).

We can draw tangent lines at each point \((x_1, x_2)\) in the \( x_1x_2 \)-plane. Knowing all the tangent lines then allows us to sketch the corresponding solution curve. This is done by assigning each point \((x_1, x_2)\) in the \( x_1x_2 \)-plane a vector \( \begin{bmatrix} dx_1/dt \\ dx_2/dt \end{bmatrix} \) which
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has the property that it is tangential to the solution curve that passes through the point \((x_1, x_2)\) and it points in the direction of the solution.

In the example (11.1.6), the vector is of the form \([x_1 - 2x_2 \atop x_2]\) and the slope of the solution curve that goes through \((x_1, x_2)\) is

\[
\frac{dx_2}{dx_1} = \frac{x_2}{x_1 - 2x_2}.
\]

The collection of these vectors is called a direction field or slope field, and each vector of the direction field is called a direction vector.

The point \((0, 0)\) is special: when we compute the direction vector at \((0, 0)\), we find that

\[
\begin{bmatrix}
\frac{dx_1}{dt} \\
\frac{dx_2}{dt}
\end{bmatrix} = \begin{bmatrix}
0 \\
0
\end{bmatrix}.
\]

That is, if we start at this point, neither \(x_1(t)\) nor \(x_2(t)\) will change. We call such points equilibria. We will discuss their significance in Section 11.3.

11.2. Solving linear systems

We now consider the solutions of a linear system.

11.2.1. Specific solutions. Consider the following differential equation

\[
\frac{dx}{dt} = ax.
\]

From (11.2.1), we get

\[
x(t) = c e^{at}.
\]

We now consider a system of two linear equations

\[
\frac{dx_1}{dt} = a_{11}x_1(t) + a_{12}x_2(t), \quad \frac{dx_2}{dt} = a_{21}x_1(t) + a_{22}x_2(t),
\]

which, in matrix form, is written as

\[
\frac{dx}{dt} = A x(t).
\]

A solution of (1.2.3) is a vector-valued function.

**Example 11.2.1.** Consider

\[
\frac{dx_1}{dt} = x_1(t), \quad \frac{dx_2}{dt} = 2x_2(t).
\]

Thus,

\[
A = \begin{bmatrix}
1 & 0 \\
0 & 2
\end{bmatrix}.
\]

Using (11.2.2) implies

\[
x_1(t) = c_1 e^t, \quad x_2(t) = c_2 e^{2t}
\]

or

\[
x(t) = \begin{bmatrix} c_1 e^t \\ c_2 e^{2t} \end{bmatrix} = e^t \begin{bmatrix} c_1 \\ 0 \end{bmatrix} + e^{2t} \begin{bmatrix} 0 \\ c_2 \end{bmatrix}.
\]
On the other hand, if $\lambda$ is an eigenvalue of the matrix $A$, we have $\lambda_1 = 1$ and $\lambda_2 = 2$. Hence

$$x(t) = e^{\lambda_1 t} \begin{bmatrix} c_1 \\ 0 \end{bmatrix} + e^{\lambda_2 t} \begin{bmatrix} 0 \\ c_2 \end{bmatrix}.$$  

This example suggests us that the general solution is of the form

$$x(t) = u e^{\lambda_1 t} + v e^{\lambda_2 t},$$

where $\lambda_1, \lambda_2$ are eigenvalues of $A$.

We now claim that (11.2.3) has the form (11.2.4).

**Lemma 11.2.2.** If $\lambda$ is an eigenvalue of $A$ with an eigenvector $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, then

$$x(t) = e^{\lambda t} u$$

is a solution of (11.2.3).

**Proof.** Plugging (11.2.5) into (11.2.3) implies

$$\frac{dx}{dt} = \begin{bmatrix} u_1 \lambda e^{\lambda t} \\ u_2 \lambda e^{\lambda t} \end{bmatrix} = \lambda e^{\lambda t} u = e^{\lambda t} Au = Ax(t).$$

□

In this subsection we will look only at differential equations of the form (11.2.3) for which the eigenvalues of $A$ are both real and distinct. We will discuss complex eigenvalues in the next subsection. We will not discuss the case when both eigenvalues are identical.

**Example 11.2.3.** Find specific solutions of

$$\begin{align*}
\frac{dx_1}{dt} &= 2x_1 - 2x_2, \\
\frac{dx_2}{dt} &= 2x_1 - 3x_2.
\end{align*}$$

**Proof.** The coefficient matrix of (11.2.6) is

$$A = \begin{bmatrix} 2 & -2 \\ 2 & -3 \end{bmatrix}.$$  

Since $\det(A) = 2 \times (-3) - 2 \times (-2) = -2$ and $\text{tr}(A) = 2 - 3 = -1$, it follows that $0 = \lambda^2 + \lambda - 2 \implies \lambda_1 = 1$ and $\lambda_2 = -2$.

For $\lambda_1 = 1$, an eigenvector $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ satisfies

$$0 = (A - \lambda_1 I_2)u = \begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \implies u_1 - 2u_2 = 0 \implies u = u_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$  

Hence

$$x(t) = u_2 e^{t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad u_2 \in \mathbb{R},$$
is a solution of (11.2.6).

For \( \lambda_2 = -2 \), an eigenvalue \( v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \) satisfies

\[
0 = (A - \lambda_2 I_2)v = \begin{bmatrix} 4 & -2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \implies 2v_1 - v_2 = 0 \implies v = u_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix}.
\]

Hence

\[
x(t) = v_1 e^{-2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad v_1 \in \mathbb{R},
\]

is a solution of (11.2.6). \( \Box \)

### 11.2.2. General solutions.

Suppose that \( y(t) \) and \( z(t) \) are two solutions of

\[
\frac{dx}{dt} = Ax(t).
\]

That is

\[
\frac{dy}{dt} = Ay(t), \quad \frac{dz}{dt} = Az(t).
\]

Since

\[
\frac{d}{dt} [c_1y(t) + c_2z(t)] = c_1 \frac{dy}{dt} + c_2 \frac{dz}{dt} = c_1 Ay(t) + c_2 Az(t) = A[c_1y(t) + c_2z(t)],
\]

it follows that \( c_1y(t) + c_2z(t) \) is also a solution.

**Lemma 11.2.4.** Suppose that

\[
\begin{bmatrix} \frac{dx}{dt} \\ \frac{dx}{dt} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}.
\]

If

\[
y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} \quad \text{and} \quad z(t) = \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix}
\]

are solutions, then

\[
x(t) = c_1y(t) + c_2z(t)
\]

is also a solution.

We only consider the case that the coefficient matrix \( A \) has two real and distinct eigenvalues. When \( A \) has repeated eigenvalues or complex eigenvalues, we do not give the general solution here.

**Theorem 11.2.5.** Let

\[
\frac{dx}{dt} = Ax(t)
\]

where \( A \) is a \( 2 \times 2 \) matrix with two real and distinct eigenvalues \( \lambda_1 \) and \( \lambda_2 \) with corresponding eigenvectors \( u \) and \( v \). Then

\[
x(t) = c_1 e^{\lambda_1 t} u + c_2 e^{\lambda_2 t} v
\]

is the general solution, where the constants \( c_1 \) and \( c_2 \) depend on the initial condition.
The general solution of (11.2.6) is
\[ x(t) = c_1 e^{t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}. \]

**Example 11.2.6.** If (11.2.6) has the initial condition \( x(0) = \begin{bmatrix} -1 \\ 4 \end{bmatrix} \), then, by (11.2.8),
\[ c_1 e^0 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{-2\times0} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}; \]
thus
\[ 2c_1 + c_2 = -1, \quad c_1 + 2c_2 = 4 \implies c_1 = -2, \quad c_2 = 3. \]
Hence the general solution of (11.2.6) is
\[ x_1(t) = -4e^t + 3e^{-2t}, \quad x_2(t) = -2e^t + 6e^{-2t}. \]

**Example 11.2.7.** Solve
\[ \frac{dx}{dt} = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix} x(t) = Ax(t) \]
with the initial condition \( x(0) = \begin{bmatrix} 3 \\ -1 \end{bmatrix} \).

**Proof.** Since \( \det(A) = -1 \) and \( \text{tr}(A) = 0 \), it follows that
\[ 0 = \lambda^2 - 1 \implies \lambda_1 = 1 \text{ and } \lambda_2 = -1. \]
For \( \lambda_1 = 1 \) we have
\[ 0 = (A - \lambda_1 I_2)u = \begin{bmatrix} 1 & -3 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \implies u_1 - 3u_2 = 0 \implies u = u_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix}; \]
for \( \lambda_2 = -1 \) we have
\[ 0 = (A - \lambda_2 I_2)v = \begin{bmatrix} 3 & -3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \implies v_1 - v_2 = 0 \implies v = v_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \]
Hence the general solution is
\[ x(t) = c_1 e^{t} \begin{bmatrix} 3 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \]
By the initial condition, we have
\[ 3c_1 + c_2 = 3, \quad 2c_1 = 4 \implies c_1 = 2, \quad c_2 = -3. \]
Thus
\[ x_1(t) = 6e^t - 3e^{-t}, \quad x_2(t) = 2e^t - 3e^{-t}. \]

11.3. Equilibria and stability