MIDTERM 2

54 pts. total

Name: ____________________________
Section: _________________________

1.) Find \( f'(x) \) or \( \frac{dy}{dx} \) for each of the following. You do not need to simplify:

a.) [4 pts.] \( f(x) = x^4 + 2x + 1 \).
   \[ f'(x) = 4x^3 + 2 \]

b.) [4 pts.] \( f(x) = \frac{e^x}{x} \).
   \[ f'(x) = \frac{xe^x - e^x}{x^2} \]

c.) [4 pts.] \( y^2 + \ln(x \sin x) = 2 \). (Find \( \frac{dy}{dx} \)).
   \[
   2y \frac{dy}{dx} + \frac{1}{x \sin x} (x \cos x + (\sin x)(1)) dx = 0
   \]
   \[
   2y \frac{dy}{dx} + \frac{x \cos x + \sin x}{x \sin x} = 0
   \]
   \[
   2y \frac{dy}{dx} = -\frac{x \cos x + \sin x}{x \sin x}
   \]
   \[
   \frac{dy}{dx} = -\frac{x \cos x + \sin x}{2yx \sin x}
   \]

d.) [4 pts.] \( f(x) = \sqrt{\tan^3 x} \).
   \[ f'(x) = \frac{1}{2\sqrt{\tan^3 x}} (3 \tan^2 x)(\sec^2 x). \]
2.) [6 pts.] Use the Intermediate Value Theorem to show that the function \( f(x) = x^3 - 2x + 1 \) has three zeroes (i.e. the equation \( f(x) = 0 \) has three solutions).

\[
\begin{align*}
  f(2) &= 5 \\
  f\left(\frac{3}{4}\right) &= -5/64 \\
  f(0) &= 1 \\
  f(-2) &= -3
\end{align*}
\]

So \( f(x) = 0 \) somewhere between \( \frac{3}{4} \) and 2, between 0 and \( \frac{3}{4} \), and between \(-2\) and 0.

3.) [6 pts.] Use the formal definition of the derivative to find the derivative of \( f(x) = x^2 \).

\[
f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}
\]

\[
= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}
\]

\[
= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h}
\]

\[
= \lim_{h \to 0} \frac{2xh + h^2}{h}
\]

\[
= \lim_{h \to 0} 2x + h
\]

\[
= 2x.
\]
4.) Find the limits of the following, using whichever method you like:

a.) [4 pts.] \( \lim_{x \to 1} \frac{x^3 - 1}{x^2 - x} \)

Plugging in gives \( \frac{0}{0} \). So we use L'Hopital’s Rule:

\[
\lim_{x \to 1} \frac{x^3 - 1}{x^2 - x} = \lim_{x \to 1} \frac{3x^2}{2x - 1} = \frac{3}{2 - 1} = 3.
\]

b.) [4 pts.] \( \lim_{x \to 0} \frac{x}{\tan x} \)

Again, plugging in gives \( \frac{0}{0} \). So we use L'Hopital’s Rule:

\[
\lim_{x \to 0} \frac{x}{\tan x} = \lim_{x \to 0} \frac{1}{\sec^2 x} = \frac{1}{1} = 1.
\]

c.) [4 pts.] \( \lim_{x \to 0^+} \frac{\cos x}{x} \)

Plugging in this time gives \( \frac{1}{0} \). So the limit goes to \( \pm \infty \). To determine which one, we note that both the top and bottom are positive as \( x \) goes to zero from the positive direction. So the limit is \( \infty \).

5.) [6 pts.] Describe where \( f(x) \) is continuous, where \( \begin{cases} 
    x^2 - 9 & \text{if } x \neq 3 \\
    7 & \text{if } x = 3
\end{cases} \)

At \( x \neq 3 \), \( f(x) \) is a rational function (i.e. a polynomial divided by another polynomial). These are continuous whenever they are defined, and this one is defined for any \( x \neq 3 \). So \( f(x) \) is continuous whenever \( x \neq 3 \).

Now, we check whether \( x \) is continuous at \( x = 3 \). To see whether it is continuous, we check that 1.) The function is defined, 2.) The limit is defined, and 3.) 1 and 2 give the same value. Now,

1.) \( f(3) = 7 \)

2.) \( \lim_{x \to 3} f(x) = \lim_{x \to 3} \frac{x^2 - 9}{x - 3} = \lim_{x \to 3} \frac{(x - 3)(x + 3)}{x - 3} = \lim_{x \to 3} x + 3 = 6 \)

3.) \( f(3) \neq \lim_{x \to 3} f(x) \).

So \( f \) is not continuous at 3. Thus, \( f \) is continuous everywhere except at \( x = 3 \).
6.) [8 pts.] A cow is launched from a cannon for Fox’s new reality TV show, So You Think You Can Fly? A camera is placed 4 yards away from the cannon to catch the cow’s hilarious reaction. The cow goes straight up in the air; when the cow is 3 yards up, he is going upwards at a rate of 6 yards per minute. How fast is the cow moving away from the camera?

This will look like a triangle with base 4 and height $h$, meaning that the applicable formula is

$$a^2 + h^2 = D^2,$$

where $D$ is the hypotenuse (i.e. the distance from cow to camera). Since we are looking for how fast the cow is moving away from the camera, we must find $\frac{dD}{dt}$.

Next, we have the following information RIGHT NOW!:

$$\text{RIGHT NOW!: } h = 3, \frac{dh}{dt} = 6, \ D = 5,$$

where the first two are directly from the statement of the problem and the third can be found by plugging 3 in for $h$ above. Now, we take the derivative and then divide through by $dt$:

$$2hdh = 2DdD$$

$$2h \frac{dh}{dt} = 2D \frac{dD}{dt}$$

Plugging in the information about RIGHT NOW, we have

$$2(3)(6) = 2(5) \frac{dD}{dt}$$

$$36 = 10 \frac{dD}{dt}$$

$$\frac{36}{10} = \frac{dD}{dt}$$

$$3.6 = \frac{dD}{dt}$$

Bonus [0 pts.]: Why does Fox force these shows on us? Seriously.