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Introduction to the Theory of Analytic Spaces

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CHAPTER III. - LOCAL PROPERTIES OF ANALYTIC SETS

In this chapter, we will be concerned with the local description of analytic sets, both over $\mathbb{R}$ and over $\mathbb{C}$. In the first section we shall deal with properties that are valid in either case, and in the second with those properties that are special to complex analytic sets. A more detailed analysis of real analytic sets will be undertaken in Chapter V. The results are mostly contained in Remmert - Stein [32], Cartan [10, 12], Hervé [19].

§ 1. Germs of analytic sets.

Let k be either $\mathbb{R}$ or $\mathbb{C}$, and let Ω be an open set in k^n. Analytic functions will mean holomorphic if $k = \mathbb{C}$, real analytic if $k = \mathbb{R}$. Let S be an analytic set in Ω and let $a \in \Omega$. We denote by $\underline{S}_a$ the germ of the set S at a. We refer to $\underline{S}_a$ as an analytic germ. Let $I = I(\underline{S}_a)$ denote the set of all (germs of) analytic functions in $\mathcal{O}_{n,a}$ which vanish on the germ $\underline{S}_a$ (this statement has an obvious meaning). Clearly I is an ideal in $\mathcal{O}_{n,a}$.

We have, obviously, $\underline{S}_a \subset \underline{S}'_a$ if and only if $I(\underline{S}_a) \supset I(\underline{S}'_a)$. We say that $\underline{S}_a$ is irreducible if whenever there are two analytic germs $\underline{S}_{1a}$, $\underline{S}_{2a}$ with $\underline{S}_a = \underline{S}_{1a} \cup \underline{S}_{2a}$, one of the germs $\underline{S}_{ia}$ must be $= \underline{S}_a$.

The following lemma is obvious.

<u>Lemma 1</u>. $\underline{S}_a$ is irreducible if and only if $I(\underline{S}_a)$ is a prime ideal.

Since $\mathcal{O}_{n,a}$ is noetherian, any increasing sequence of ideals in $\mathcal{O}_{n,a}$ terminates. Hence any decreasing sequence of analytic germs $\underline{S}_{1a} \supset \underline{S}_{2a} \supset \cdots$ terminates. We deduce easily from this the following

Proposition 1. *Any analytic germ* $\underline{S}_a$ *can be written as a finite union* $\underline{S}_a = \bigcup_{\nu=1}^{k} \underline{S}_{\nu a}$ *of irreducible analytic germs* $\underline{S}_{\nu a}$ *such that, for each* ν, $\underline{S}_{\nu a}$ $\bigcup_{\mu\neq\nu} \underline{S}_{\mu a}$. *Further, this decomposition is uniquely determined upto order.*

Definition 1. *The germs* $\underline{S}_{\nu,a}$ *introduced by this decomposition* $\underline{S}_a = \cup \underline{S}_{\nu,a}$ *are called the irreducible components of* $\underline{S}_a$.

Let now I be an ideal in $\mathcal{O}_n = \mathcal{O}_{n,o}$; we suppose that $\{0\} \neq I \neq \mathcal{O}_n$. If $x_1,\ldots,x_n$ are the coordinates of k^n, we shall denote by $\mathcal{O}_p$ the subring of $\mathcal{O}_n$ consisting of functions independent of $x_{p+1},\ldots,x_n$. We have a natural injection $\mathcal{O}_p \to \mathcal{O}_n$. Let A denote the quotient ring $\mathcal{O}_n/I$. Then, we have a natural homomorphism $\eta : \mathcal{O}_p \to A$.

Proposition 2. After a linear change of coordinates in k^n, there is an integer p, $0 \leqslant p < n$, such that $\eta : \mathcal{O}_p \to A$ is injective and makes of A a finite $\mathcal{O}_p$-module.

Proof. Let $f \in I$, $f \neq 0$. We may make a linear transformation of k^n so as to ensure that $f(0,x_n) \not\equiv 0$. This condition is invariant under linear transformations of k^{n-1}. By Chapter II, Theorem 2, (2), there is a unit u and a polynomial

$$P_n = x_n^{q_n} + \sum_{0}^{q_n-1} a_\nu(x_1,\ldots,x_{n-1})x_n^\nu, \quad a_\nu(0) = 0, \quad \text{with} \quad f = uP_n;$$

then $P_n \in I$. Now, either $I_{n-1} = I \cap \mathcal{O}_{n-1} = \{0\}$, in which case we take $p = n-1$, or there is $f_{n-1} \in I_{n-1} - \{0\}$. As above, we find, after a linear change of variables in k^{n-1}, that there is a polynomial

$$P_{n-1} = x_{n-1}^{q_{n-1}} + \sum_{0}^{q_{n-1}-1} a'_\nu(x_1,\ldots,x_{n-2})x_{n-1}^\nu, \quad a'_\nu(0) = 0, \quad P_{n-1} \in I_{n-1}.$$

Continuing this process, we find an integer p such that $I_p = I \cap \mathcal{O}_p = 0$ and such that, for any $r > p$, there is a distinguished polynomial.

$$P_r = x_r^{q_r} + \sum_{0}^{q_r - 1} a_\nu^{(n-r)}(x_1,\ldots,x_{r-1})x_r^\nu, \quad a_\nu^{(n-r)}(0) = 0,$$

with $P_r \in I_r = I \cap \mathcal{O}_r$.

We claim that this integer p satisfies our requirements. In fact, trivially $I_p = \{0\}$ implies that $\eta : \mathcal{O}_p \to A$ is injective. If $f \in \mathcal{O}_n$, by Chapter II, Theorem 2, (1), we have

$$f \equiv \sum_{\nu=0}^{q_n - 1} f_{1,\nu}(x_1,\ldots,x_{n-1})x_n^\nu \pmod{P_n},$$

$$f_{1,\nu} \equiv \sum_{\mu=0}^{q_{n-1}-1} f_{2,\nu,\mu}(x_1,\ldots,x_{n-2})x_{n-1}^\mu \pmod{P_{n-1}},$$

and so on, so that

$$f \equiv \sum_{\alpha_j < q_j} f_\alpha(x_1,\ldots,x_p)x_{p+1}^{\alpha_{p+1}}\cdots x_n^{\alpha_n} \pmod{P_{p+1},\ldots,P_n},$$

so that the images of the monomials $x_{p+1}^{\alpha_{p+1}}\cdots x_n^{\alpha_n}$, $\alpha_j < q_j$ generate A over $\mathcal{O}_p$.

In what follows, we shall identify elements of $\mathcal{O}_n$ with their images in $A = \mathcal{O}_n/I$ when no confusion is likely.

Corollary. If, in addition, I is a prime ideal, K is the quotient field of $\mathcal{O}_p$, L that of $A = \mathcal{O}_n/I$, then $L = K(x_{p+1},\ldots,x_n)$.

Remark. The necessary and sufficient condition that the coordinates satisfy the assertion of Proposition 2 is that $I_p = \{0\}$ and, for $r > p$, there exists a distinguished polynomial $Q_r(x_r;x_1,\ldots,x_{r-1}) \in \mathcal{O}_{r-1}[x_r] \cap I$.

We now state two algebraic theorems that we shall use.

I. (Theorem of primitive element). If K is a field of characteristic zero and $L = K(u_1,\dots,u_r)$ a finite algebraic extension of K, then, for any infinite subset $S \subset K$, there exist elements $c_1,\dots,c_r \in S$ such that

$$L = K(\zeta) \quad \text{where} \quad \zeta = \sum_{i=1}^{r} c_i u_i .$$

II. Let K, L be as above, and in addition, suppose that K is the quotient field of a factorial ring A, that B is the integral closure of A in L, and that $\zeta \in B$ is such that $L = K(\zeta)$. Let P be the minimal polynomial of ζ over K. (Then $P \in A[X]$ since A is factorial.) If P' denotes the derivative of P, then for any $\alpha \in B$, there is $Q \in A[X]$ of degree $<$ degree P such that $\alpha P'(\zeta) = Q(\zeta)$ (note that $P'(\zeta) \neq 0$).

Now, by the theorem of primitive element, there exist complex numbers λ_j such that $y_{p+1} = \sum_{p+1}^{n} \lambda_j x_j$ is linearly independent of $x_1,\dots,x_p$ and $L = K(y_{p+1})$. Further, for any $f \in \mathcal{O}_n$, since A is a finite $\mathcal{O}_p$-module, there exists a polynomial $Q_f(X) = X^m + \sum_{0}^{m-1} b_\nu(x_1,\dots,x_p)X^\nu \in \mathcal{O}_p[X]$ with $Q_f(f) = 0$. If we choose the polynomial Q_f to have minimal degree we claim that when $f(0) = 0$, Q_f is a distinguished polynomial. In fact if not all $b_\nu(0) = 0$, then $X^m + \sum_{0}^{m-1} b_\nu(0)X^\nu$ has, at $X = 0$, a zero of order $l < m$. By Chapter II, Theorem 2, (2), $Q_f(X) = u.Q(X)$, where u is a unit, and $Q(X) = X^l + \sum_{0}^{l-1} c_\nu(x_1,\dots,x_p)X^\nu$ is a distinguished polynomial of degree 1. But then $Q(f) = 0$, and Q_f would not have minimal degree. Thus we obtain (since $\mathcal{O}_p$ is factorial)

<u>Proposition 3</u>. Given a prime ideal $I \subset \mathcal{O}_n$, $\{0\} \neq I \neq \mathcal{O}_n$, there exists, after a linear change of coordinates in k^n, an integer p, $0 \leq p \leq n$ such that

$$\eta : \mathcal{O}_p \to A = \mathcal{O}_n/I$$

is an injection which makes A a finite $\mathcal{O}_p$-module. Further, if K is the quotient field of $\mathcal{O}_p$, L that of A, we have $L = K(\overline{x}_{p+1})$, and for any $r > p$, the minimal polynomial P_r of x_r over K is in $\mathcal{O}_p[X]$, and is distinguished, so that there is a distinguished polynomial

$$P_r(x_r;x') = x_r^{q_r} + \sum_{\nu=0}^{q_r-1} a_\nu^{(r)}(x')x_r^\nu, \quad x' = (x_1,\dots,x_p),$$

$a_\nu^{(r)}(0') = 0$, with $P_r(x_r,x') \in I$.

It follows that if $p = 0$, and if $I = I(\underline{S}_0)$, then $\underline{S}_0$ is the germ defined by $S = \{0\}$.

In what follows, we shall suppose the prime ideal I given, and the coordinates chosen so that Proposition 3 applies. We shall use the notation of Proposition 3.

Let δ denote the discriminant of the polynomial P_{p+1} (so that δ is the resultant of P_{p+1} and $\frac{\partial P_{p+1}}{\partial x_{p+1}}$). Then $\delta \in \mathcal{O}_p$, further since P_{p+1} is the minimal polynomial of x_{p+1} over $\mathcal{O}_p$, $\delta \neq 0$ in $\mathcal{O}_p$; since $I_p = \{0\}$, $\delta \notin I$. By the algebraic theorem II stated above, if q is the degree of P_{p+1}, we have the following

<u>Lemma 2</u>. For any $f \in \mathcal{O}_n$, there is a polynomial R_f of degree $\leq q - 1$ in $\mathcal{O}_p[X]$ such that

$$\delta f - R_f(x_{p+1}) \in I.$$

In particular, there are polynomials Q_r of degree $< q-1$ in $\mathcal{O}_p[X]$ such that, for $r > p$,

$$\delta x_r - Q_r(x_{p+1}) \in I.$$

Lemma 3. For any $f \in \mathcal{O}_n$, there exists $g \in \mathcal{O}_n - I$ and $h \in \mathcal{O}_p$ such that $gf - h \in I$.

Proof. Since A is a finite $\mathcal{O}_p$-module, we have a relationship

$$f^m + \sum_{\nu=0}^{m-1} a_\nu(x_1,\ldots,x_p) f^\nu \in I;$$

we may suppose, since I is prime, that $a_0(x') \neq 0$. We have then only to set $h = -a_0$, $g = f^{m-1} + \sum_{\nu=1}^{m-1} a_\nu(x') f^{\nu-1}$.

Definition 2. Let S be an analytic set in an open set Ω in k^n. A point $a \in S$ is called a regular point of S of dimension p if there is a neighbourhood U of a, $U \subset \Omega$, such that $S \cap U$ is an analytic submanifold of dimension p of U. A point $a \in S$ is called singular if it is not regular.

A point $a \in S$ is regular of dimension p if and only if there exist functions $f_{p+1},\ldots,f_n \in \mathcal{O}_{n,a}$ such that, in a neighbourhood of a, $S = \{x \mid f_i(x) = 0,\ i > p\}$ and $(df_{p+1})_a,\ldots,(df_n)_a$ are linearly independent.

Let S be an analytic set in an open set $\Omega \subset k^n$, $0 \in S$. We suppose that $\underline{S}_0$ is irreducible, i.e. that $I = I(\underline{S}_0)$ is a prime ideal in $\mathcal{O}_n = \mathcal{O}_{n,0}$. Choose coordinates in k^n so that Proposition 3 is satisfied. Then we have

Proposition 4. There is a fundamental system of neighbourhoods $U = U' \times U''$, $U' \subset k^p$, $U'' \subset k^{n-p}$ of 0 such that if $\pi : (S \cap U) \to U'$ denotes the restriction to $S \cap U$ of the projection of U onto U', then π is a proper map and every fibre $\pi^{-1}(x')$, $x' \in U'$, of π is a finite set.

Proof. Choose a neighbourhood V of 0 such that the polynomials $P_r(x_r, x')$ of Proposition 3 all have coefficients analytic on V and vanish on $S \cap V$. Let $V = \{x \mid |x| < \varrho\}$. Since the P_r are distinguished, there is $\sigma > 0$, such that if $|x'| \leq \sigma$ and $P_r(x_r, x') = 0$, then $|x_r| \leq \varrho/2$. Clearly if $U' = \{x' \in k^p \mid |x'| < \sigma\}$, $U'' = \{x'' \in k^{n-p} \mid |x''| \leq \varrho\}$, and π is as above, then $\pi^{-1}(E) \subset E \times \{x'' \mid |x''| \leq \varrho/2\}$, so that π is proper. Further, if $x \in \pi^{-1}(x')$, then $P_r(x_r, x') = 0$, and x_r can take at most finitely many different values.

Lemma 4. Let I be a prime ideal in $\mathcal{O}_n$, and $\underline{S}_0$ the germ of analytic set at 0 defined as the set of common zeros of a finite system of generators of I. Let P_r, $r > p$, $\delta x_r - Q_r(x_{p+1})$ be as in Proposition 3 and Lemma 2. Then there exists a fundamental system of neighbourhoods $U = U' \times U''$ of 0 such that these functions are analytic on U, $\underline{S}_0$ is induced by an analytic set S in U, and such that the following hold.

(a) $S \cap U \cap \{x \mid \delta(x') \neq 0\} = \{x \in U \mid \delta(x') \neq 0, P_{p+1}(x_{p+1}, x') = 0,$

$\delta x_r - Q_r(x_{p+1}) = 0, \quad r > p + 1\}.$

(b) If $x \in U' \times k^{n-p}$ and $P_{p+1}(x_{p+1}, x') = 0 = \delta x_r - Q_r(x_{p+1})$,

$\delta(x') \neq 0$, then $x \in U$.

Proof. Choose $V = V' \times V''$ such that all the functions considered are analytic on V; further, let $f_1, \ldots, f_m$ be analytic functions on V with $S \cap V = \{x \in V \mid f_i(x) = 0, i=1, \ldots, m\}$ (f_i generators of I). As in the proof of Proposition 2, we find that there exist $f_{\alpha,i} \in \mathcal{O}_p$, $\alpha = (\alpha_{p+1}, \ldots, \alpha_n)$, $\alpha_j < q_j$, with

$$f_i \equiv \sum_{\alpha_j < q_j} f_{\alpha,i}(x') x_{p+1}^{\alpha_{p+1}} \cdots x_n^{\alpha_n} \pmod{P_{p+1},\ldots,P_n}$$

and hence, if $N = q_{p+2} \cdots q_n$ (substitute $\delta x_r = Q_r$)

$$\delta^N f_i \equiv R_i'(x_{p+1}) \pmod{P_{p+1},\ldots,P_n, \delta x_{p+2} - Q_{p+2},\ldots,\delta x_n - Q_n},$$

where R_i' is an element in $\mathcal{O}_p[X]$. Again, since P_{p+1} is monic, we may make a polymonial division of R_i' by P_{p+1} and obtain

$$\delta^N f_i = R_i(x_{p+1}) \pmod{P_{p+1},\ldots,P_n, \delta x_{p+2} - Q_{p+2},\ldots,\delta x_n - Q_n},$$

where R_i is a polymonial of degree $< q - 1$ ($q = \deg P_{p+1}$). Now $f_i \in I$; hence $R_i(x_{p+1}) \in I$. Since P_{p+1} is the minimal polymonial of x_{p+1} over $\mathcal{O}_p$, and $\deg R_i < \deg P_{p+1}$, this implies that $R_i = 0$, so that

$$\delta^N f_i \equiv 0 \pmod{P_{p+1},\ldots,P_n, \delta x_{p+2} - Q_{p+2},\ldots}.$$

We now proceed as follows. Clearly, for each $r > p + 1$, we have, on $V' \times k^{n-p}$

$$\delta^{q_r} P_r \equiv A_r'(x_{p+1}) \pmod{\delta x_r - Q_r}$$

where $A_r' \in \mathcal{O}_p[X]$. Making a polynomial division of A_r' by P_{p+1}, we obtain

$$\delta^{q_r} P_r \equiv A_r(x_{p+1}) \pmod{P_{p+1}, \delta x_r - Q_r} \quad \text{on} \quad V' \times k^{n-p},$$

where $A_r \in \mathcal{O}_p[X]$ and has degree $< \deg P_{p+1}$. Again $A_r(x_{p+1}) \in I$ and so is zero near 0, hence 0 on $V' \times k^{n-p}$.

Hence

(2) $\delta^{q_r} P_r \equiv 0 (\text{mod } P_{p+1}, \delta x_r - Q_r)$ on $V' \times k^{n-p}$. This implies that if $\delta(x') \neq 0$, $P_{p+1}(x_{p+1}, x') = \delta x_r - Q_r = 0$, then $P_r(x_r, x') = 0$. Since P_r is distinguished, this implies that if V' is small, then any solution $x \in V' \times k^{n-p}$ of $P_{p+1}(x_{p+1}, x') = 0 = \delta x_r - Q_r$, $r > p + 1$, lies in a preassigned neighbourhood $V' \times V''$ of 0. This proves (b). Again, (1) and (2) imply that, for a fixed integer $M > 0$,

(3) $\delta^M f_i \equiv 0 (\text{mod } P_{p+1}, \delta x_{p+2} - Q_{p+2}, \ldots, \delta x_n - Q_n)$.

If we now choose $U \subset V$, $U = U' \times U''$ such that (b) holds, and all the above congruences are represented by linear relations with coefficients analytic on U, then

$$S \cap U \cap \{x \mid \delta(x') \neq 0\} = \{x \in U \mid \delta(x') \neq 0,\ f_1(x) = \ldots = f_m(x) = 0\},$$

and, by (3), this is

$$= \{x \in U \mid \delta(x') \neq 0,\ P_{p+1}(x_{p+1}, x') = 0 = \delta x_r - Q_r(x_{p+1}), r > p + 1\}.$$

This proves Lemma 4.

We remark that in Proposition 4 and Lemma 4, given any $V'' \subset k^{n-p}$ (which is a neighbourhood of 0), then we can find a neighbourhood V' of 0 in k^p such that for any open set $U' \subset V'$, $0 \in U'$, the assertions in Proposition 4 and Lemma 4 are true.

<u>Proposition 5</u>. Let U be a neighbourhood of 0 such that Lemma 4 is true relative to the ideal $I = I(\underline{S}_0)$ where $\underline{S}_0$ is irreducible. Then any point $x \in S \cap U$ with $\delta(x') \neq 0$ is a regular point of S of dimension p, and the projection π has a jacobian of rank p at x.

Proof. Since $\delta(x') \neq 0$ and $P_{p+1}(x) = 0$, we conclude that, at the point x, $\frac{\partial P_{p+1}}{\partial x_{p+1}} \neq 0$. Hence, S is defined near x by the system of equations

$P_{p+1}(x_{p+1},x') = 0$, $x_r = \frac{Q_r(x_{p+1})}{\delta(x')}$, $r > p+1$, which has the property that $dP_{p+1},\ldots,d\left(x_r - \frac{Q_r}{\delta(x')}\right)$ are k-independent at x. This proves Proposition 5.

Remark further that if X,Y are analytic manifolds countable at ∞ and $f : X \to Y$ is an analytic map such that $f^{-1}(y)$ is discrete for every $y \in f(X)$, then $\dim X \leqslant \dim Y$ (apply the rank theorem to a point where the differential of f has maximal rank). Combining this with Proposition 5 we obtain

Proposition 6. The integer p of Proposition 2 and 3 relative to $I = I(\underline{S}_a)$, $\underline{S}_a$ being irreducible, is the largest integer m such that ($\underline{S}_a$ is induced by an analytic set S), every neighbourhood of 0 contains points at which S is regular of dimension m.

This characterisation of the integer p is clearly invariant under analytic automorphism of a neighbourhood of 0 in k^n.

Definition 3. The dimension of an irreducible analytic germ $\underline{S}_a$ at $a \in k^n$ is the integer p of Proposition 2. The dimension of an arbitrary analytic germ $\underline{S}_a$ is the maximum dimension of of the irreducible components $\underline{S}_{\nu,a}$ of $\underline{S}_a$. The dimension of an analytic set S in an open set Ω in k^n is $\max_{a \in S} \dim \underline{S}_a$, where $\underline{S}_a$ is the germ at a defined by S.

Theorem 1. *Let* S *be an analytic set in an open set* Ω *in* k^n. *Let* $a \in S$ *and* $\dim \underline{S}_a = p$. *Then any neighbourhood of* a *contains points at which* S *is regular of dimension* p. *In particular, the set of regular points of* S *is dense in* S.

Proof. Let $\underline{T}_a$ be an irreducible component of $\underline{S}_a$ of dimension p. Let $\underline{T}'_a$ be the union of the other irreducible components of S_a. Let $\underline{T}''_a = \underline{T}_a \cap \underline{T}'_a$ and T, T'' be analytic sets in an open set U' containing a inducing the germs $\underline{T}_a$, $\underline{T}''_a$ at a; then $S \cap U' = T \cup T''$. Since a regular point of T of dimension p which does not lie on T'' is clearly a regular point of S of dimension p, it suffices to prove that U contains a regular point of T of dimension p not on T''. Let U be so chosen that Propositions 3, 4, 5 apply to $\underline{T}_a$. Then, there is $f \in \mathcal{O}_n$, $f = 0$ on $\underline{T}''_a$, $f \notin I = I(\underline{T}_a)$ (since $\underline{T}''_a \subset \underline{T}'_a$). Let δ have the significance of Proposition 4 (relative to $\underline{T}_a$). Then $\delta \notin I$. Since I is prime, $f\delta \notin I$. Hence, arbitrarily near a, there are points $x \in T$ with $f(x) \neq 0$. Theorem 1 follows from Proposition 5.

Proposition 7. If $\underline{S}_a$ is an irreducible germ of dimension p and $\underline{S}_a \supset \underline{S}'_a$, where $\underline{S}'_a$ is any analytic germ at a, then $\dim \underline{S}'_a < \dim \underline{S}_a$.

Proof. Clearly, we may suppose that $\underline{S}'_a$ is irreducible. We choose the coordinates $x_1, \ldots, x_n$ in k^n so that if $I = I(\underline{S}_a)$, $I' = I(\underline{S}'_a)$ (so that $I \subset I'$), then we have $I_p = \{0\}$, there exists a distinguished pseudopolynomial $P_r(x_r; x') \in I$, $r > p$. If we show that, after a linear change of variables in $k^p(x_1, \ldots, x_p)$, there is a distinguished polynomial $P_p(x_p; x_1, \ldots, x_{p-1}) \in I'$, the result follows from the remark after Proposition 2. Now, by the preparation theorem, it suffices to prove that there exists $h \in \mathcal{O}_p \cap I' = I'_p$, $h \neq 0$. Let $g \in \mathcal{O}_n$, $g \notin I$, $g \in I'$. By Lemma 3, there is $g_1 \in \mathcal{O}_n - I$

such that $gg_1 \equiv h(\text{mod } I)$, where $h \in \mathcal{O}_p$. But then clearly, since $g \notin I'$, $h \in I'$, and, since I is prime, $gg_1 \notin I$, so that $h \notin I$, and in particular $h \neq 0$.

§ 2. Complex analytic sets.

In this section we shall deal only with complex analytic sets, so that $k = \mathbb{C}$.

Let I be a prime ideal in $\mathcal{O}_{n,o} = \mathcal{O}_n$, and suppose that the coordinates $(x_1, \ldots, x_n)$ are so chosen that Proposition 3, 4 and Lemma 4 are valid. Let $\pi : S \cap U \to U'$ be the projection defined in Proposition 4.

For any ideal $I \subset \mathcal{O}_{n,a}$ we denote by $S(I)$ the germ at a of analytic set defined as the set of common zeros of a finite system of generators of I. Clearly, $S(I)$ is independent of the system of generators chosen.

<u>Proposition</u> 8. We have $\pi(S \cap U) = U'$.

<u>Proof</u>. Since π is proper, its image is closed in U'. Hence it suffices to show that $\pi(S \cap U)$ is dense in U'. For any $x' \in U'$, $\delta(x') \neq 0$, the polynomial $P_{p+1}(x_{p+1}, x')$ has a complex zero x_{p+1}. Let $x = \left(x', x_{p+1}, \frac{Q_{p+2}(x_{p+1})}{\delta(x')}, \ldots, \frac{Q_n(x_{p+1})}{\delta(x')}\right)$. By Lemma 4, $x \in S \cap U$, and clearly $\pi(x) = x'$. Hence $\pi(S \cap U)$ contains the dense set $\{x' \in U' \mid \delta(x') \neq 0\}$.

<u>Remarks</u>. 1. Note that, by the remark following Proposition 3, this implies that, if the coordinates are so chosen that Proposition 2 is valid, then there is a fundamental system of neighbourhoods $\{U_\nu\}$ of 0 such that $\pi_p(S \cap U_\nu)$ is a neighbourhood of 0 in $\mathbb{C}^p$, π_p being the projection of $\mathbb{C}^n$ onto $\mathbb{C}^p$.

2. Actually, the map $\pi : S \cap \{x \in U \mid \delta(x') \neq 0\} \to U' - \{x' \in U' \mid \delta(x') \neq 0\}$ is a covering map.

Lemma 5. Let $f \in \mathcal{O}_n$. If, for sufficiently small U as above, for any $x' \in U'$, $\delta(x') \neq 0$, there is $x \in S \cap U$ such that $\pi(x) = x'$ and $f(x) = 0$, then $f \in I$.

Proof. If $f \notin I$, there is $g \notin I$ such that $gf \equiv h (\text{mod } I)$ where $h \in \mathcal{O}_p$. Then $h \notin I$, and for any sufficiently small x, $\delta(x') \neq 0$, $h(x') = f(x)\, g(x) = 0$ $\big($if $x \in S \cap U$, $\pi(x) = x'\big)$, so that $h = 0$ and so $h \in I$, a contradiction.

Theorem 2. (Hilbert's Nullstellensatz). Let $\mathfrak{a}$ be any ideal of $\mathcal{O}_n$ and $\underline{S}_o = S(\mathfrak{a})$ the germ of analytic set defined as the set of common zeros of a finite system of generators of $\mathfrak{a}$. Then $I(\underline{S}_o) = \text{rad}\,\mathfrak{a} = \{f \in \mathcal{O}_n \mid f^m \in \mathfrak{a}$ for some integer $m > 0\}$.

Proof. We first remark that, if $\mathfrak{a}$ is prime, $I(\underline{S}_o) = \mathfrak{a}$. This is a trivial consequence of Lemma 5. Hence, if $\mathfrak{a}$ is primary (i.e. $\text{rad}\,\mathfrak{a}$ is prime), we deduce, since $S(\mathfrak{a}) = S(\text{rad}\,\mathfrak{a})$, that $I(\underline{S}_o) = \text{rad}\,\mathfrak{a}$. If $\mathfrak{a}$ is arbitrary, $\neq \{0\}$, since $\mathcal{O}_n$ is noetherian, we obtain by the Noether decomposition theorem,

$$\mathfrak{a} = \bigcap_{\nu=1}^{k} \mathfrak{q}_\nu, \quad \mathfrak{q}_\nu \text{ being primary.}$$

Clearly then

$$S(\mathfrak{a}) = \bigcup_{\nu=1}^{k} S(\mathfrak{q}_\nu),$$

so that

$$I\big(S(\mathfrak{a})\big) = \bigcap_{\nu=1}^{k} I\big(S(\mathfrak{q}_\nu)\big) = \bigcap_{\nu=1}^{k} \text{rad}\,\mathfrak{q}_\nu = \text{rad}\,\mathfrak{a}.$$

We now give a very important application of the results obtained above. We begin with a definition.

Definition 4. Let S be an analytic set in an open set Ω in $\mathbb{C}^n$. A function f on S is said to be holomorphic at $a \in S$ if there is a neighbourhood U of a in Ω and a holomorphic function F in U with $F|U \cap S = f|U \cap S$.

We may define germs of holomorphic functions in the obvious way. If $a \in S$, let $\mathcal{O}_{S,a}$ denote the ring of germs of holomorphic functions at a on S. Clearly, we have

$$\mathcal{O}_{n,a}/I(\underline{S}_a) \simeq \mathcal{O}_{S,a}.$$

Hence $\mathcal{O}_{S,a}$ is an analytic ring over $\mathbb{C}$.

Definition 5. A map $f : S_1 \to S_2$ (S_i analytic set in an open set in $\mathbb{C}^{n_i}$) is called holomorphic if the map $j \circ f$, where $j : S_2 \to \mathbb{C}^{n_2}$ is the natural injection, has the form $j \circ f = (f_1, \ldots, f_{n_2})$ where the f_ν are holomorphic on S_1.

Clearly, a holomorphic map $f : S_1 \to S_2$ induces, for $a \in S_1$, an algebra homomorphism

$$f^* : \mathcal{O}_{S_2, f(a)} \to \mathcal{O}_{S_1, a}$$

viz, $f^*(\varphi) = \varphi \circ f$.

Theorem 3. Let $f : S_1 \to S_2$ be holomorphic. Then the homomorphism

$$f^* : \mathcal{O}_{S_2, f(a)} \to \mathcal{O}_{S_1, a}$$

is finite (see Chapter II) if and only if a is an isolated point of the fibre $f^{-1}f(a)$.

Proof. Let $S_1 \subset \Omega_1 \subset \mathbb{C}^n(x_1, \ldots, x_n)$, $S_2 \subset \Omega_2 \subset \mathbb{C}^m(y_1, \ldots, y_m)$. We may suppose that $a = 0$, $f(a) = 0$. We set

$$R_1 = \mathcal{O}_{S_1, 0}, \quad R_2 = \mathcal{O}_{S_2, f(0)}.$$

Suppose that

$$f^* : R_2 \to R_1$$

is finite. Then every element of R_1 is integral over R_2.

Hence, if $\varphi \in \mathcal{O}_{S_1,0}$, there exist holomorphic germs $a_1,\ldots,a_r \in \mathcal{O}_{S_2,0}$ such that

$$\varphi^r(x) + \sum_{\nu=1}^{r} a_\nu\big(f(x)\big)\varphi^{r-\nu}(x) = 0.$$

In particular, we have, in some neighbourhood of 0 on S_1,

$$x_k^r + \sum_{\nu=1}^{r} a_\nu^{(k)}\big(f(x)\big)x_k^{r-\nu} = 0, \quad k = 1,\ldots,n, a_\nu^{(k)} \in \mathcal{O}_{S_2,0}.$$

Hence, if $f(x) = 0$, and x is near 0 on S_1, x_k satisfies a polynomial relation and so can be at most one of finitely many complex numbers. Hence 0 is isolated in $f^{-1}f(0)$.

Suppose conversely that 0 is an isolated point of $f^{-1}f(0)$. This means precisely that if $\mathfrak{a}$ is the ideal of $\mathcal{O}_{n,0}$ generated by $\big(f_1,\ldots,f_m, I(\underline{S}_0)\big)$, then $S(\mathfrak{a}) = \{0\}$. (For notation $S(\mathfrak{a})$, see Theorem 2). Hence, by the Nullstellensatz, for any k, $1 \leq k \leq n$, there is an integer r such that

$$x_k^r \equiv \sum_{\nu=1}^{m} \alpha_\nu(x) f_\nu(x) \big(\text{mod } I(\underline{S}_0)\big), \quad \alpha_\nu \in \mathcal{O}_{n,0}.$$

This implies clearly that there is an integer $q > 0$ such that $[\mathfrak{m}(R_1)]^q \subset f^*\big(\mathfrak{m}(R_2)\big).R_1$ (for notation see Chapter II). This implies that $f^* : R_2 \to R_1$ is quasi-finite. By Theorem 1, Chapter II, f^* is finite, q.e.d.

Corollary 1. The necessary and sufficient condition that a system of coordinates $(x_1,\ldots,x_n)$ of $\mathbb{C}^n$ satisfy the assertion of Proposition 2 relative to an ideal $I \subset \mathcal{O}_n$ is that 0 is an isolated point of the set $\{x_1 = \ldots = x_p = 0\} \cap S(I)$ and $I \cap \mathcal{O}_p = \{0\}$.

<u>Corollary</u> 2. If X, Y are analytic sets in open sets in $\mathbb{C}^n$, $\mathbb{C}^m$ respectively, $f : X \to Y$ a holomorphic map for which $a \in X$ is an isolated point of $f^{-1}f(a)$, then there is a neighbourhood U of a such that any $b \in U$ is an isolated point of $f^{-1}f(b)$.

<u>Proof</u>. Let X be an analytic set in an open set Ω in $\mathbb{C}^n$ and suppose that $a = 0$. By Theorem 3, there exist $a_{\mu,i} \in \mathcal{O}_{Y,b}$, $b = f(0)$, such that

$$x_i^{p_i} + \sum_{\mu=1}^{p_i} a_{\mu,i}\, f(x)\, x_i^{p_i-\mu} = 0 \quad \text{in } \mathcal{O}_{X,0} ;$$

here $x_1, \ldots, x_n$ are the coordinates in $\mathbb{C}^n$. There is a neighbourhood U of 0 and an open set $V \subset Y$, $b \in V$, so that $a_{\mu,i}$ are holomorphic in $V, f(U) \subset V$, and the above equations hold on U. Then, given $f(x) \in V$, each x_i can have only finitely many values, which proves our assertion.

<u>Corollary</u> 3. If S is an analytic set in an open set $\Omega \subset \mathbb{C}^n$ of dimension p at every point and the restriction to S of the projection π of $\mathbb{C}^n$ onto $\mathbb{C}^p$ is a proper map with finite fibres into $\Omega' = \pi(\Omega)$, then $\pi|S$ is an open map.

<u>Proof</u>. Let $0 \in S$ and U a neighbourhood of 0; we have to show that $\pi(U \cap S)$ is a neighbourhood of $0 = \pi(0) \in \mathbb{C}^p$. This is an immediate consequence of Corollary 1 above and the Remark 1 after Proposition 8.

<u>Corollary</u> 4. If S is an analytic set in an open set Ω in $\mathbb{C}^n$ and $0 \in S$, then $\dim \underline{S}_0$ is the smallest integer k such that there exists a subspace H of $\mathbb{C}^n$ of dimension $n - k$ such that 0 is an isolated point of $H \cap S$.

This follows easily from Corollary 1 above and the definition of $\dim \underline{S}_0$.

<u>Corollary</u> 5. If X is an analytic set in an open set in $\mathbb{C}^n$ and $f : X \to \mathbb{C}^k$ is a holomorphic map, then any point $a \in X$

has a neighbourhood U in X such that, for $x \in U$

$$\dim_x f^{-1}f(x) \leqslant \dim_a f^{-1}f(a).$$

Proof. If $p = \dim_a f^{-1}f(a)$, then, if the coordinates at a are suitably chosen, a is an isolated point of the set

$$f^{-1}f(a) \cap \{z \in \mathbb{C}^n \mid z_1 = \ldots = z_p = 0\}.$$

Let $g : X \to \mathbb{C}^{k+p}$ be the map $g(z) = (f(z), z_1, \ldots, z_p)$. Then a is an isolated point of $g^{-1}g(a)$; by Corollary 2 above, there is an open set U' in $\mathbb{C}^n$ containing a such that, for $z \in U \cap X$, x is an isolated point of $g^{-1}g(x)$. This means precisely that x is an isolated point of

$$f^{-1}f(x) \cap \{z \in \mathbb{C}^n \mid z_1 = x_1, \ldots, z_p = x_p\}.$$

By Corollary 4 above, $\dim_x f^{-1}f(x) \leqslant p$.

We now continue our study of complex analytic sets.

Theorem 4. Let Ω be an open set $\mathbb{C}^n$ and π, the restriction to Ω of the projection of $\mathbb{C}^n$ onto $\mathbb{C}^p$ (first p variables). Let $\Omega' = \pi(\Omega)$ and A' be a thin subset of Ω'. Let X be an analytic set in $\Omega - \pi^{-1}(A')$ and suppose that $\pi | X$ is a finite covering of $\Omega' - A'$ (i.e. $\pi | X$ is proper and locally biholomorphic) and that $\pi | \overline{X}$ is a proper map into Ω'. Then $\overline{X}$ is an analytic set in Ω of dimension p at each of its points.

Proof. Let $A = \pi^{-1}(A')$. We may suppose that Ω (and hence Ω') is connected. Hence $\Omega' - A'$ is connected (Chapter I, Proposition 11) and hence there is an integer k such that for any $x' \in \Omega' - A'$, there are exactly k points $x^{(1)}, \ldots, x^{(k)} \in X$ with $\pi(x^{(j)}) = x'$.

Let f be any holomorphic function on Ω. We define holomorphic functions $a_{1,f}, \ldots, a_{k,f}$ on Ω' as follows.

For $x'\epsilon\Omega' - A'$, let $a_{l,f}$ be the l-th elementary symmetric function

$$a_{l,f}(x') = (-1)^l \sum_{1\leq j_1<\dots<j_l\leq k} f\left(x^{(j_1)}\right)\dots f\left(x^{(j_l)}\right),$$

where $x^{(1)},\dots,x^{(k)}$ are the points of X with $\pi\left(x^{(j)}\right) = x'$. Clearly $a_{l,f}$ is holomorphic in $\Omega' - A'$, and further, since $\pi : \overline{X} \to \Omega'$ is proper, for any compact set $K'\subset\Omega'$, $a_{l,f}$ is bounded on $K' - A'$, and hence (Chapter I, Proposition 10) can be extended to a holomorphic function on Ω'. Let P_f be the holomorphic function on Ω defined by

$$P_f(x) = f^k(x) + \sum_{l=1}^{k} a_{l,f}(x') f^{k-l}(x), \quad \pi(x) = x'.$$

By construction, $P_f(x) = 0$ if $x\epsilon X$, and hence $P_f(x) = 0$ if $x\epsilon\overline{X}$. We claim that

(4) $\overline{X} = \{x\epsilon\Omega | P_f(x) = 0$ for any holomorphic f on $\Omega\}$.

Let X' be the set of common zeros of the P_f, let $x'\epsilon\Omega'$, and set $E' = \{x\epsilon X' | \pi(x) = x'\}$, $\overline{E} = \{x\epsilon\overline{X} | \pi(x) = x'\}$. Then $\overline{E}\subset E'$, and to prove (4), it suffices to prove that $\overline{E} = E'$. For this, it suffices to prove that $f(\overline{E}) = f(E')$ for any holomorphic f on Ω. Let $\alpha\epsilon f(E')$; because of the continuity of the roots of a polynomial, there is a sequence x'_ν of points, $x'_\nu \to x'$, $x'_\nu\epsilon A'$, such that there is a zero α_ν of the polynomial $\zeta^k + \sum_1^k a_{l,f}(x'_\nu)\zeta^{k-l}$ such that the sequence $\alpha_\nu \to \alpha$; but since $x'_\nu\epsilon A'$, there is $x_\nu\epsilon X$, $\pi(x_\nu) = x'_\nu$, with $\alpha_\nu = f(x_\nu)$. Further, since $\pi : \overline{X} \to \Omega'$ is proper, we can find a subsequence $\{\nu_k\}$ such that $x_{\nu_k} \to x\epsilon\overline{X}$; clearly then $x\epsilon\overline{E}$ and $f(x) = \lim f(x_{\nu_k}) = \alpha$. Hence $\alpha\epsilon f(\overline{E})$, and (4) is proved. Because of Corollary 2 to Theorem 5, Chapter II, $\overline{X}$ is analytic in Ω. It is clear, since $\pi:X \to \Omega' - A'$ is locally biholomorphic, that

$\overline{X}$ is regular of dimension p at every point of X. Since X is dense in $\overline{X}$, Proposition 5 implies that $\overline{X}$ has dimension p at every point.

Proposition 9. Let S be an analytic set in Ω such that $\underline{S}_o$ is irreducible. Let U be a neighbourhood of 0 as in Proposition 4 and Lemma 4. Then the set $X = \{x \in S \mid \delta(x') \neq 0\}$ is dense in some neighbourhood of 0 on S.

Proof. We have already remarked that the projection $\pi : X \to U' - \{x' \in U' \mid \delta(x') \neq 0\}$ is a covering. Further $\pi : S \to U'$ is proper, hence so is $\pi : \overline{X} \to U'$. Hence, by Theorem 4, $\overline{X}$ is an analytic set of dimension p at 0. But since $\overline{X} \subset S$, Proposition 7 implies that there is a neighbourhood V of 0 with $\overline{X} \cap V = S \cap V$.

It is also possible to avoid Proposition 7 and use directly the argument used in Theorem 4. This method leads, in fact, to a somewhat stronger form of Proposition 9. Of course, Proposition 9 is stronger than Propositions 5 and 6.

If $f \in \mathcal{O}_{n,o}$, $f \in I(\underline{S}_o)$, there is $g \in I(\underline{S}_o)$, $gf \equiv h \left(\text{mod } I(\underline{S}_o)\right)$ where $h \in \mathcal{O}_p$. Replacing X in the above proof by the set $\{x \in S \mid h(x')\delta(x') \neq 0\}$, we deduce

Proposition 9'. If S is as in Proposition 9 and $f \notin I(\underline{S}_o)$, then the set of points $\{x \in S \mid f(x) = 0\}$ is dense in some neighbourhood of 0 in S.

It is trivial matter to extend Proposition 9' to sets S for which $\underline{S}_o$ is not necessarily irreducible (with the obvious conditions on f).

Definition 6. Let S be an analytic set in an open set Ω in $\mathbb{C}^n$. A holomorphic function f on Ω is called a universal denominator for S at a point $a \in S$ if a has a neighbourhood U in Ω such that the following holds: if h is a holomorphic function on the set S' of points of $S \cap U$ at which S is regular, and if h is bounded on S', then there is a neighbourhood V of a such that $f\,h$ is the restriction to $S' \cap V$ of a holomorphic function on V.

Theorem 5. Let S be an analytic set in Ω, and suppose that S has dimension p at each point. Let π denote the projection of $\mathbb{C}^n$ onto $\mathbb{C}^p$ (first p variables), and let $\Omega' = \pi(\Omega)$. Suppose that $\pi|S$ is a proper mapping into Ω' with finite fibres $\{$i.e. $\pi^{-1}(x') \cap S$ is finite for any $x' \in \Omega'\}$. Then, given a point $a \in S$ which is regular on S at which $x_1 - a_1, \ldots, x_p - a_p$ form a system of local coordinates, there exists a linear function l on $\mathbb{C}^n$ and holomorphic functions $\alpha_1, \ldots, \alpha_k$ on Ω' such that, if we set

$$P(\zeta, x') = \zeta^k + \sum_{\nu=1}^{k} \alpha_\nu(x')\zeta^{k-\nu},$$

the following conditions are satisfied.

(a) $P\big(l(x), \pi(x)\big) \equiv 0$ on S.

(b) $\frac{\partial P}{\partial \zeta}\big(l(a), \pi(a)\big) \neq 0$, $P'(x) = \frac{\partial P}{\partial \zeta}\big(l(x), \pi(x)\big)$ is a universal denominator for S at every one of its points.

(c) If h is holomorphic and bounded on the set S' of regular points of S there exist holomorphic functions $\beta_0, \ldots, \beta_{k-1}$ on Ω' such that

(d) $P'(x)h(x) = \sum_{\nu=0}^{k-1} \beta_\nu\big(\pi(x)\big)\big(l(x)\big)^\nu$ on S', and there is a constant M (independent of h) such that

(e) $\|\beta_\nu\|_{\Omega'} \leqslant M\|h\|_{S'}$.

Proof. We begin by proving the following. There exists a thin subset A' of Ω' such that if $A = S \cap \pi^{-1}(A')$ then $\pi : S - A \to \Omega' - A'$ is a (finite) covering. Note that, by Corollary 3 to Theorem 3, A is nowhere dense in S.

Let B be the union of the set of singular points of S with the set of points of S' where the jacobian matrix of the map $\pi : S' \to \Omega'$ is not invertible. (Since $\pi : S' \to \Omega'$ has finite fibres, this is a nowhere dense analytic set in S'.)

Clearly, B is closed, hence so is $\pi(B) = A'$. We claim that A' is thin. For this, it is clearly sufficient to prove the following: if $b' \in \Omega'$ and $b \in S$, $\pi(b) = b'$, then, there is a neighbourhood U of b and a thin set E' in a neighbourhood U' of b' in Ω' such that S is regular at any point of $U \cap \left(S - \pi^{-1}(E')\right)$, and $\pi | U \cap \left(S - \pi^{-1}(E')\right)$ is of maximal rank. Let $\underline{S}_{\nu,b}$ be the irreducible components of S at b. By Corollary 1 to Theorem 3 and Lemma 4, there is a holomorphic function g'_ν near b' (which is a multiple of the discriminant δ_ν corresponding to $\underline{S}_{\nu,b}$) such that $g'_\nu \notin I(\underline{S}_{\nu,b})$, while $\{x \in \underline{S}_{\nu,b} | g'_\nu(x') = 0\}$ contains the two sets $\bigcup_{\mu \neq \nu} (\underline{S}_{\mu,b} \cap \underline{S}_{\nu,b})$ and $\{x \in \underline{S}_{\nu,b} | \delta_\nu(x') = 0\}$. Because of Proposition 5, we may take for E' the zeros of $g' = \prod_\nu g'_\nu$.

Thus A' is thin. Since $\pi : S - A \to \Omega' - A'$ is proper and a local homeomorphism, it is a finite covering.

We now choose a linear function l on $\mathbb{C}^n$ such that:

(i) for a (countable) dense set of points $x' \in \Omega' - A'$, l separates the points of $\pi^{-1}(x')$;
(ii) for the given point a, $l(a)$ is different from $l(c)$ for any $c \neq a$, $c \in \pi^{-1}\pi(a)$.

Suppose that $\pi : S - A \to \Omega' - A'$ is a covering of k sheets (note that, by the Corollary to Theorem 3 and Proposition 8

$\pi : S \to \Omega'$ is an open map). Let $\alpha_1, \ldots, \alpha_k$ be the holomorphic functions on $\Omega' - A'$ which are the elementary symmetric functions of the values of l on the fibres of π. They have holomorphic extensions to Ω'. We set then

$$P(\zeta, x') = \zeta^k + \sum_{\nu=1}^{k} \alpha_\nu(x')\zeta^{k-\nu}.$$

Clearly, $P\big(l(x), \pi(x)\big) = 0$ on $S - A$, hence, this latter set being dense in S, $= 0$ on S. This is (a).

Since by assumption $x_1 - a_1, \ldots, x_p - a_p$ form local coordinates at a on S, π is a homeomorphism in a neighbourhood of a; hence, by our assumption (ii), $l(a)$ is a simple root of $P(\zeta, a')$, and hence $\frac{\partial P}{\partial \zeta}\big(l(a), \pi(a)\big) \neq 0$.

To complete the proof of Theorem 5, it suffices now to prove (c); to obtain the second part of (b), we have only to apply (c) to a neighbourhood of a given point of S.

We have only to find holomorphic functions β_ν on Ω' such that (e) holds and (d) holds on $S - A$. Let $x' \in \Omega' - A'$, and let $x^{(1)}, \ldots, x^{(k)}$ be the points of $S - A$ with $\pi\big(x^{(j)}\big) = x'$. Consider the sum

$$\sum_{j=1}^{k} \frac{P(\zeta; x')}{\zeta - l\big(x^{(j)}\big)}\, h\big(x^{(j)}\big) = \sum_{j=1}^{k} \frac{P(\zeta; x') - P\big(l\big(x^{(j)}\big); x'\big)}{\zeta - l\big(x^{(j)}\big)}\, h\big(x^{(j)}\big)$$

this is clearly of the form

$$\sum_{\nu=0}^{k-1} \beta_\nu(x')\zeta^\nu$$

where the β_ν are holomorphic on $\Omega' - A'$, and for any x', $\beta_\nu(x')$ is a linear combination of the $h\big(x^{(j)}\big)$ with coefficients depending only on the $l\big(x^{(j)}\big)$. Hence

$$|\beta_\nu(x')| \leqslant M \max_j \big|h\big(x^{(j)}\big)\big|.$$

In particular, the β_ν are bounded on $\Omega' - A'$ (since h is bounded on S') and so admits a holomorphic extension to Ω' such that (e) holds. If in the identity

$$\sum_{j=1}^{k} \frac{P(\zeta;x')}{\zeta - l\left(x^{(j)}\right)} h\left(x^{(j)}\right) = \sum_{\nu=o}^{k-1} \beta_\nu(x')\zeta^\nu$$

we substitute $\zeta = l(x)$ where x is a point such that $\pi(x) = x' \epsilon \Omega' - A'$ and l separates the points of $\pi^{-1}(x')$ [such points are dense in $S - A$], we obtain (d) on a dense subset of $S - A$; hence (d) holds on S'.

Remark. The set of points where S is not regular is clearly contained in the set $P'(x) = \frac{\partial P}{\partial \zeta}\left(l(x), \pi(x)\right) = 0$.

Corollary. If S is an analytic set such that $\underline{S}_o$ is irreducible there exists $f \epsilon \mathcal{O}_{n,o} - I(\underline{S}_o)$ which is a universal denominator at any point sufficiently near 0.

Of course, this corollary can be proved directly in a somewhat simpler fashion. In fact, the last part of our arguments shows that with the notation of Proposition 3, $\frac{\partial P_{p+1}}{\partial x_{p+1}}$ is such a universal denominator.

We shall see later that the finiteness of the fibres of $\pi | S$ is a consequence of the hypothesis that $\pi | S$ is proper.

Remark. Theorem 4 remains valid if we replace the assumption that $\pi : X \to \Omega' - A'$ is an unramified covering by the assumption that $\pi : X \to \Omega' - A'$ is a proper map with finite fibres and that X has dimension p at each point. In fact, we have seen in the proof of Theorem 5 that there is a thin subset $B' \subset \Omega' - A'$ such that $X - \pi^{-1}(B')$ is dense in X and $\pi : X - \pi^{-1}(B') \to \Omega' - A' - B'$ is an unramified covering. Then, the constuction of the functions $P_f(x)$ can be done, first for $x' \epsilon \Omega' - A' - B'$, and, by successive applications of the continuation theorem, extended first to

$x'\in\Omega' - A'$, then to $x'\in\Omega'$. The rest of the proof remains the same.

Proposition 10. (Maximum Principle) Let S be an analytic set in Ω and f a holomorphic function on Ω. Let $\underline{S}_a$ be irreducible, and suppose that f is not constant on S in any neighbourhood of a. Then $f(S)$ is a neighbourhood of $f(a)$ in $\mathbb{C}$.

Proof. We may suppose that $a = 0$, $f(a) = 0$. Let U be a neighbourhood of 0, and let the coordinates in $\mathbb{C}^n$ be so chosen that $U = U' \times U''$, $U'\subset \mathbb{C}^p$, $p = \dim \underline{S}_0$ and $\pi : S\cap U \to U'$ is proper, has finite fibres (and satisfies Proposition 3). For $x'\in U'$, $\delta(x') \neq 0$, let $a_1,\dots,a_k$ be the elementary symmetric functions of the values of f on $\pi^{-1}(x')\cap S$; then the a_ν admit holomorphic extensions to U'. Let

$$P(\zeta;x') = \zeta^k + \sum_{\nu=1}^{k} a_\nu(x')\zeta^{k-\nu}.$$

Then, for $x'\in U'$, $\delta(x') \neq 0$, we have $f\left(\pi^{-1}(x')\right) = \{\zeta\in\mathbb{C} \mid P(\zeta,x') = 0\}$. Hence, by the continuity of the roots of a polynomial, we have $f\left(\pi^{-1}(x')\right) = \{\zeta\in\mathbb{C} \mid P(\zeta,x') = 0\}$ for any $x'\in U'$.

By assumption, we have $f\notin I(\underline{S}_0)$. Hence, by Lemma 5, $a_k(x') \not\equiv 0$ near $0\in U'$. Further, $a_\nu(0) = 0$ for each ν $\left(\text{since } f(0) = 0\right)$. We may suppose, after a linear change of variable in $\mathbb{C}^p$, that $a_k(0,\dots,0,x_p) \not\equiv 0$ near $x = 0$. By the preparation theorem, there exists a distinguished polynomial

$$Q(x_p,x_1,\dots,x_{p-1},\zeta) = x_p^m + \sum_{\nu=1}^{m} b_\nu(x_1,\dots,x_{p-1},\zeta)x_p^{m-\nu}$$

such that P and Q have the same zeros near 0. But clearly, since $b_\nu(0) = 0$, for any ζ near 0, there is x_p near 0 with $Q(x_p,0,\ldots,0,\zeta) = 0$; hence the set $\zeta\in\mathbb{C}$ near 0 for which there exists x' near 0 with $P(\zeta,x') = 0$ is a neighbourhood of 0. Since $f\left(\pi^{-1}(x')\right) = \{\zeta\in\mathbb{C} \mid P(\zeta,x') = 0\}$, $f(S)$ is a neighbourhood of 0.

Corollary 1. A compact analytic set S in $\mathbb{C}^n$ consists of a finite number of points.

Proof. It suffices to prove that for any holomorphic function f on $\mathbb{C}^n$, $f(S)$ is finite. If it were infinite, there would exist $\alpha\in f(S) - \left(f(S)\right)^o$ and $\alpha_\nu\in f(S)$, $\alpha_\nu \neq \alpha_\mu$ if $\nu \neq \mu$, such that $\alpha_\nu \to \alpha$. Let $s_\nu\in S$, $f(s_\nu) = \alpha_\nu$; by passing to a subsequence if necessary, we may suppose that $s_\nu \to s_0\in S$, since S is compact. If $\underline{S}_{s_0} = \cup\underline{S}_{k,s_0}$, then there are infinitely many s_ν on at least one $\underline{S}_{k,s_0}$. Hence f is not constant on this component, so that $f(S)$ is a neighbourhood of $f(s_0) = \alpha$, and then $\alpha\notin f(S) - \left(f(S)\right)^o$, a contradiction.

Corollary 2. Let φ be a proper holomorphic map of an analytic set S in an open set in $\mathbb{C}^n$ into an open set in $\mathbb{C}^m$. Then, for $y\in\mathbb{C}^m$, $\varphi^{-1}(y)$ is a finite set (being a compact analytic subset of $\mathbb{C}^n$).

Corollary 3. Let Ω be an open set in $\mathbb{C}^n$, $\pi : \Omega \to \mathbb{C}^p$ the projection. Let $\Omega' = \pi(\Omega)$ and A' be a thin subset of Ω'. If X is an analytic set in $\Omega - \pi^{-1}(A')$ of dimension p at any point and $\pi|\overline{X}$ is proper, then $\overline{X}$ is an analytic set in Ω of dimension p at each point. This follows from Corollary 2 and the remark preceding Proposition 10.

Proposition 11. Let S be an analytic set in Ω such that $\underline{S}_a$ is irreducible. Then a has a fundamental system of neighbourhoods U such that the set of regular points of S in U is connected.

Proof. Choose U such that Proposition 4 is valid and further, with the notation as before, the set $S\cap\{x\in U' | \delta(x') \neq 0\}$ is dense in $S\cap U$. It suffices to prove that $X = S\cap\{x\in U | \delta(x') \neq 0\}$ is connected. Now $\pi : X \to U' - \{x'\in U' | \delta(x') = 0\}$ is a finite covering. Hence if X is not connected, and Y is a connected component of X, $\pi|Y$ is also a covering. Let h be the function $= 0$ on Y, $= 1$ on $X - Y$. If S' is the set of regular points of S, $S' - X$ is thin, and hence h has a holomorphic extension to S'. By the corollary to Theorem 5, there is $f\notin I(\underline{S}_0)$ such that $F = fh$ is holomorphic on $\underline{S}_0$. Now, for any $x'\in U'$, $\delta(x') \neq 0$, clearly there is $x\in S$ (viz $x\in Y$) with $F(x) = 0$. By Lemma 5, $F\in I(\underline{S}_0)$, so that F vanishes at all points near 0. But clearly there are points arbitrarily close to 0 on $X - Y$ where $f \neq 0$, which implies that F $I(\underline{S}_0)$. This proves Proposition 11.

Corollary. If S, a are as above, and $\dim \underline{S}_a = p$, then there is a neighbourhood V of such that S has dimension p at any point of $V\cap S$.

This follows at once from Proposition 11 and Proposition 5.

Theorem 6. Let S be an analytic set in an open set Ω in $\mathbb{C}^n$ and let $0\in S$. Suppose that $\underline{S}_0$ is irreducible. Then there exists a neighbourhood V of 0 and finitely many holomorphic functions $f_1,\ldots,f_m$ in V such that:

(a) the set of singular points of $S\cap V$ is precisely the set $\{x\in V | f_1(x) = \ldots = f_m(x) = 0\}$;

(b) each f_i is a universal denominator at every point of V.

Proof. Choose the coordinates $(x_1,\ldots,x_n)$ in $\mathbb{C}^n$ and $U' = \{x'\in\mathbb{C}^p | |x'| < \varrho'\}$, $U'' = \{x''\in\mathbb{C}^{n-p} | |x''| < \varrho''\}$ such that if $U = U' \times U''$, then the projection $\pi : U\cap S \to U'$ is proper with finite fibres.

Let $\varepsilon > 0$ be sufficiently small, and set, for $|\alpha_{ij}| < \varepsilon$, $i = 1,\ldots,p$, $j = 1,\ldots,n$,

$$\xi_i^{(\alpha)} = x_i + \sum_{j=1}^{n} \alpha_{ij} x_j .$$

Then, if ε is small, $\left(\xi_1^{(\alpha)},\ldots,\xi_p^{(\alpha)}, x_{p+1},\ldots,x_n\right)$ are linearly independent. Let $\varrho_1 < \varrho'$ be fixed, and ε small enough. Let $U_1' = \{x'\in U' \mid |x'| < \varrho_1\}$ and $U_1 = U_1' \times U''$. Let $U_\alpha = \{x\in U \mid |\xi_i^{(\alpha)}(x)| < \varrho_1\}$, and let $\pi_\alpha : S\cap U_\alpha \to U_1'$ be the map $\pi_\alpha(x) = \left(\xi_1^{(\alpha)}(x),\ldots,\xi_p^{(\alpha)}(x)\right)$. If ε is small enough, then π_α is proper. To prove this, we remark that if K' is a compact subset of U_1', $\pi_\alpha^{-1}(K')$ is closed in U, since clearly it cannot be adherent to any point of ∂U_α in U and is closed in U_α. Further, if $\pi_\alpha(x)\in K'$, then $\pi(x)$ lies in a compact neighbourhood of U_1' in U if ε is small enough; hence $\pi_\alpha^{-1}(K')$ is contained in a compact subset of U, and being closed in U, is itself compact. Because of Corollary 1 to Proposition 11, $\pi_\alpha^{-1}(x')$ is finite for any $x'\in U_1'$.

Now, for any regular point a of S in U, there exist α_{ij}, $|\alpha_{ij}| < \varepsilon$, such that $\xi_1^{(\alpha)},\ldots,\xi_p^{(\alpha)}$ form a system of coordinates at a. Let W be a neighbourhood of 0 such that $W\subset \bigcap_{|\alpha_{ij}|<\varepsilon} U_\alpha$ (such a W exists if ε is small). If $a\in W\cap S$ and S is regular at a, let α be so chosen that the $\xi^{(\alpha)}$ form coordinates at a. There exists, by Theorem 5, a function P_α' which is a universal denominator at any point of U_α, $P_\alpha'(a) \neq 0$, which vanishes on the singular set of S in U_α. Hence, there exists a family of holomorphic functions $\{f_t\}$ in W which are universal denominators at any point of W such that the singular set of S in W is given by $\{x\in V \mid f_t(x) = 0 \,\forall t\}$. Theorem 6 follows easily from this and Chapter II, Theorem 5 (if we replace W by a smaller open set V).

Corollary 1. Let S be an analytic set in an open set Ω in $\mathbb{C}^n$. Then the set of singular points of S is an analytic set in Ω.

Proof. Let $a \in S$ and $\underline{S}_a = \cup \underline{S}_{\nu,a}$, where the $\underline{S}_{\nu,a}$ are the irreducible components of $\underline{S}_a$. Let S_ν be an analytic set in a neighbourhood W of a inducing $\underline{S}_{\nu,a}$. Since the $\underline{S}_{\nu,a}$ are irreducible and none of them is contained in the union of the others, we have $\dim (\underline{S}_{\nu,a} \cap \underline{S}_{\mu,a}) < \min (\dim \underline{S}_{\nu,a}, \dim \underline{S}_{\mu,a})$ if $\mu \neq \nu$. Hence, by the Corollary to Proposition 11, we may choose W so small that for $b \in S_\nu \cap S_\mu \cap W$ we have $\dim (\underline{S}_{\nu,b} \cap \underline{S}_{\mu,b}) < \min (\dim \underline{S}_{\nu,b}, \dim \underline{S}_{\mu,b})$ for $\mu \neq \nu$. In particular, no germ $\underline{S}_{\nu,b}$ is contained in the union of the others. We claim that the set of singular points of S which lie on S_ν is the union T_ν of the set of singular points of S with $\bigcup_{\mu \neq \nu} (S_\nu \cap S_\mu)$. In fact, it is clear that S is regular at any point of S_ν not on T_ν. If $b \in S_\nu \cap S_\mu$, $\nu \neq \mu$, then since an analytic set is clearly irreducible at regular point, b is singular. If $b \notin \bigcup_{\mu \neq \nu} (S_\mu \cap S_\nu)$, then $\underline{S}_b = \underline{S}_{\nu,b}$ so that b is singular on S if and only if it is singular on S_ν. The corollary clearly follows from this and Theorem 6.

If $\underline{S}_o$ is not irreducible and if $\underline{S}_o = \cup \underline{S}_{\nu,o}$ is the decomposition of $\underline{S}_o$ into irreducible components, let S, S_ν be representatives of these germs in a neighbourhood of o. If f is a universal denominator for $S_{\nu,b}$ for any $b \in S_\nu \cap U$, and if g is holomorphic on U, $g = 0$ on S_μ, $\mu \neq \nu$, $g \neq 0$ on a dense subset of S_ν, then $h = gf$ is a universal denominator at any point of $S \cap U$ for S. It follows from this that we have

Corollary 2. Theorem 6 remains valid if we drop the hypothesis that $\underline{S}_o$ is irreducible.

Corollary 3. If S is an analytic set in an open set Ω in $\mathbb{C}^n$, and A is the set of singular points of S, and if $a \in S$, then for any $f \in \mathcal{O}_{\Omega,a}$ which vanishes on $\underline{A}_a$, there is an integer $k \geq 1$ such that f^k is a universal for S at all points near a.

This follows at once from Corollary 2 above and the Hilbert Nullstellensatz.

Proposition 12. Let Ω be an open set in $\mathbb{C}^n$ and $A \subset \Omega$ an analytic subset of dimension $\leq n - 2$. Then, for any holomorphic function on $\Omega - A$, there is a unique holomorphic function F on Ω with $F|\Omega - A = f$.

Proof. The uniqueness of F, if it exists, is obvious. Hence we have only to prove that for every $a \in A$, there is a neighbourhood V and an F holomorphic in V with $F|V - A = f|V - A$. We may suppose that $a = 0$, and, after a linear change of coordinates in $\mathbb{C}^n$, that 0 is an isolated point of the set

$$A \cap \{x | x_1 = \dots = x_{n-2} = 0\}.$$

If $\varrho > 0$ is sufficiently small, and

$$a_\nu = (0,\dots,0,\tfrac{1}{\nu},0),$$

it is clear that, if ν is large, the set

$$\overline{D}_\nu = \{x \in \mathbb{C}^n | x_1 = \dots = x_{n-2} = 0,\ x_{n-1} = \tfrac{1}{\nu},\ |x_n| \leq \varrho\} \subset \Omega - A,$$

and

$$K_0 = \{x \in \mathbb{C}^n | x_1 = \dots = x_{n-2} = 0,\ x_{n-1} = 0,\ |x_n| = \varrho\} \subset \Omega - A.$$

Hence, by Chapter I, Proposition 12, There is a connected neighbourhood V of 0 (containing K_0) and F holomorphic in V such that $F = f$ near K_0. Since $V - A$ is connected, it follows from the principle of analytic continuation that $F = f$ in $V - A$.

Proposition 13. Let S be an analytic in an open set Ω in $\mathbb{C}^n$, and let f be holomorphic in Ω. Suppose that, for $a \in S$, $f \not\equiv 0$ on any irreducible component of the germ $\underline{S}_a$ induced by S at a. Then, if $S' = \{x \in S \mid f(x) = 0\}$, we have

$$\dim \underline{S}'_a = \dim \underline{S}_a - 1.$$

Proof. 1. We first suppose that $S = \Omega$ and $a = 0$. Then, by the preparation theorem, we may suppose that

$$f(x) = x_n^p + \sum_{\nu=1}^{p} a_\nu(x') x_n^{p-\nu}, \quad x' = (x_1, \ldots, x_{n-1}), \quad a_\nu(0) = 0.$$

Clearly, if $U = U' \times U''$, $U' \subset \mathbb{C}^{n-1}$, $U'' \subset \mathbb{C}$, is a neighbourhood of 0 such that $x' \in U'$, $f(x', x_n) = 0$ imply that $x_n \in U''$, then the projection $\pi : S' = \{x \in U \mid f(x) = 0\} \to U'$ is surjective; hence if $g = g(x_1, \ldots, x_{n-1}) \in I(\underline{S}'_0)$, then $g \equiv 0$, {since g clearly vanishes on $\pi(S')$}, so that, by definition of dimension, $\dim \underline{S}'_0 = n - 1$.

2. For the general case, we may suppose that $a = 0$ and that $\underline{S}_0$ is irreducible. If $p = \dim \underline{S}_0$, we can find a neighbourhood $U = U' \times U''$ of 0, $U' \subset \mathbb{C}^p$, $U'' \subset \mathbb{C}^{n-p}$ such that $\pi : S \cap U \to U'$ is surjective, proper, and has finite fibres. Further, if δ is as in Lemma 2, $W = \{x \in S \cap U \mid \delta(x') \neq 0\}$ is dense in $S \cap U$. Now $\pi : W \to U' - \{x' \in U' \mid \delta(x') = 0\}$ is a finite covering, of say, q sheets. Let, for $x' \in U'$, $\delta(x') \neq 0$, $h(x')$ be the product of the values of f at the points of S lying over x'. Then h is bounded (if U is small enough) and so admits a holomorphic extension, also denoted by h, to U'. Clearly $\{x' \in U', \delta(x') \neq 0, h(x') = 0\}$ is the projection of $W \cap S'$ onto U'. Further, if $x \in S$, $\delta(x') = 0$, we can find a sequence of points $x_\nu \in S$, $\delta(x'_\nu) \neq 0$, $x_\nu \to x$. If $f(x) = 0$, then $f(x_\nu) \to 0$, hence $h(x'_\nu) \to 0$ so that $h(x') = 0$. Hence

$\pi(S') \subset \{x' \in U' | h(x') = 0\}$, and one proves, in the same way, the converse inclusion , so that $\pi(S') = \{x' \in U' | h(x') = 0\}$. Since $f \not\equiv 0$ on $\underline{S}_o$, $h \not\equiv 0$. By the argument given in 1. above, we can suppose, after a linear change of coordinates in $\mathbb{C}^p$ (and shrinking of U') that the following holds: let π_{p-1} be the restriction to U' of the projection of $\mathbb{C}^p$ onto $\mathbb{C}^{p-1}$. Then $T \equiv \pi_{p-1}\big(\pi(S')\big) = \pi_{p-1}(U')$. It follows since any function $g(x_1,\ldots,x_{p-1}) \in I(\underline{S}'_o)$ must vanish on T, that $I(\underline{S}'_o) \cap \mathcal{O}_{p-1} = \{0\}$, so that $\dim \underline{S}'_o = p - 1 = \dim \underline{S}_o - 1$.

Theorem 5 has another important application.

Theorem 7. Let S be an analytic set in an open set in $\mathbb{C}^n$. The space of holomorphic functions on S is complete; i.e. if $\{f_p\}$ is a sequence of holomorphic functions on S and $f_p \to f$, uniformly on compact subsets of S, then f is holomorphic on S.

Proof. Let $a \in S$ and S_ν be analytic sets in an open neighbourhood Ω of a such that $\underline{S}_a = \bigcup_{\nu=1}^{k} \underline{S}_{\nu,a}$ is the decomposition of $\underline{S}_a$ into irreducible components. By Theorem 5, applied to $\underline{S}_{\nu,a}$ after a linear change of coordinates in $\mathbb{C}^n$, there is a neighbourhood V_ν of a in Ω and a holomorphic function u_ν in V_ν such that if φ is holomorphic on $V_\nu \cap S$, there is a holomorphic function ψ'_ν in V_ν such that, with $M' > 0$ independent of φ, we have $\{x \in V_\nu \cap S_\nu \; u_\nu(x) \neq 0\}$ is dense in $V_\nu \cap S_\nu$,

$$u_\nu \varphi = \psi'_\nu \quad \text{in} \quad V_\nu \cap S_\nu,$$

and

$$\|\psi'_\nu\|_{V_\nu} \leq M' \|\varphi\|_{V_\nu \cap S}.$$

If g_ν is a holomorphic function in V_ν vanishing on $\bigcup_{\mu \neq \nu} S_\mu$, while $\{x \in S_\nu | g_\nu(x) \neq 0\}$ is dense in $S_\nu \cap V_\nu$ (such a g_ν exists, by Proposition 9' if V_ν is small enough), then, for any φ holomorphic on $V_\nu \cap S$, there is a holomorphic ψ_ν $(= g_\nu \psi'_\nu)$ on V with

(a) $$v_\nu \varphi = \psi_\nu \quad \text{on} \quad V_\nu \cap S,$$

(b) $$\|\psi_\nu\|_{V_\nu} \leq M \|\varphi\|_{V_\nu \cap S};$$

here $v_\nu = g_\nu u_\nu$ and $M > 0$ is independent of φ. Further, $\{x \in S_\nu \cap V_\nu | v_\nu(x) \neq 0\}$ is dense in $S_\nu \cap V_\nu$.

Let $\mathcal{O}_a = \mathcal{O}_{\Omega,a}$ be the ring of germs of holomorphic functions on Ω at a, and $\mathcal{J}_a = \mathcal{J}_a(S)$, the ideal in $\mathcal{O}_a$ of functions vanishing on $\underline{S}_a$.

Consider the homomorphism

$$\alpha : \mathcal{O}_a \to \mathcal{O}_a^k$$

given by $\alpha(f) = (v_1 f, \ldots, v_k f)$. Let $E_a = \alpha(\mathcal{O}_a) + \mathcal{J}_a^k \subset \mathcal{O}_a^k$. By Cartan's theorem (Chapter II, Corollary 1 to Theorem 5), if a sequence (h_p) of k-tuples of holomorphic functions in a neighbourhood U of a converges uniformly on U (to h say) and $(h_p)_a \in E_a$ for each p, then $(h)_a \in E_a$.

Let $\{f_p\}$ be a sequence of holomorphic functions on Ω, which converges uniformly on $\Omega \cap S$. Let $\varphi_0 = f_0$, $\varphi_p = f_p - f_{p-1}$, $p \geq 1$. We may suppose that $\sum_{p=0}^{\infty} \|\varphi_p\|_{\Omega \cap S} < \infty$. By our remarks above, there are holomorphic functions $\psi_{p,\nu} (\nu = 1, \ldots, k)$ on V_ν such that $v_\nu \varphi_p = \psi_{p,\nu}$ on $V_\nu \cap S$, $\|\psi_{p,\nu}\|_{V_\nu} \leq M \|\varphi\|_{V_\nu \cap S}$. If $\psi_p = (\psi_{p,1}, \ldots, \psi_{p,k})$, then $(\alpha(\varphi_p))_a - (\psi_p)_a \in \mathcal{J}_a^k$, hence $(\psi_p)_a \in E_a$. Further, $\sum \psi_p$ converges on $V = \cap V_\nu$ since $\sum \|\psi_p\|_V \leq M \sum \|\varphi_p\|_{\Omega \cap S} < \infty$.

Let $\psi = \left(\psi^{(1)},\ldots,\psi^{(k)}\right) = \sum \psi_p$. Then, by our remark above, $(\psi)_a \in E_a$, so that there is a neighbourhood W of a and a holomorphic function φ on W such that $v_\nu\varphi = \psi^{(\nu)}$ on $W \cap S$.

Let $f = \lim f_p = \sum \varphi_p$. Then, since $v_\nu\varphi_p = \psi_{p,\nu}$, we have $v_\nu f = \psi^{(\nu)}$ on $W \cap S$, so that $v_\nu(f - \varphi) = 0$ on $W \cap S$ for $\nu = 1,\ldots,k$. But since $\{x \in V_\nu \cap S_\nu \mid v_\nu(x) \neq 0\}$ is dense in $V_\nu \cap S$, this implies that $f = \varphi$ on $W \cap S_\nu$ for each ν, so that $\varphi = f$ on $W \cap S$. Since φ is holomorphic on W, this proves the theorem.

Remark. The idea of this proof is essentially that of Bungart-Rossi [7].