

# SERRE WEIGHTS FOR MOD $p$ HILBERT MODULAR FORMS: THE TOTALLY RAMIFIED CASE

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ABSTRACT. We study the possible weights of an irreducible 2-dimensional modular mod  $p$  representation of  $\text{Gal}(\overline{F}/F)$ , where  $F$  is a totally real field which is totally ramified at  $p$ , and the representation is tamely ramified at the prime above  $p$ . In most cases we determine the precise list of possible weights; in the remaining cases we determine the possible weights up to a short and explicit list of exceptions.

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## 1. INTRODUCTION

Let  $p$  be a prime number. The study of the possible weights of a mod  $p$  modular Galois representation was initiated by Serre in his famous paper [Ser87]. This proposed a concrete conjecture (“the weight part of Serre’s conjecture”) relating the weights to the restriction of the Galois representation to an inertia subgroup at  $p$ . This conjecture was resolved (at least for  $p > 2$ ) by work of Coleman-Voloch, Edixhoven and Gross (see [Edi92]).

More recently the analogous questions for Hilbert modular forms have been a focus of much investigation, beginning with the seminal paper [BDJ08]. Let  $F$  be a totally real field with absolute Galois group  $G_F$ . Then to any irreducible modular representation

$$\overline{\rho} : G_F \rightarrow \text{GL}_2(\overline{\mathbb{F}}_p)$$

there is associated a set of weights  $W(\overline{\rho})$ , the set of weights in which  $\overline{\rho}$  is modular (see section 2 for the definitions of weights and of what it means for  $\overline{\rho}$  to be modular of a certain weight). Under the assumption that  $p$  is unramified in  $F$  the paper [BDJ08] associated to  $\overline{\rho}$  a set of weights  $W^?( \overline{\rho} )$ , and conjectured that  $W^?( \overline{\rho} ) = W(\overline{\rho})$ . Many cases of this conjecture were proved in [Geeb].

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The set  $W^?(\bar{\rho})$  depends only on the restrictions of  $\bar{\rho}$  to inertia subgroups at places dividing  $p$ . In the case that these restrictions are tamely ramified, the conjecture is completely explicit, while in the general case the set depends on some rather delicate questions involving extensions of crystalline characters.

Schein [Sch08a] has proposed a generalisation of the conjecture of [BDJ08] in the tame case, removing the restriction that  $p$  be unramified in  $F$ . Not much is currently known about this conjecture; [Sch08a] proves some results towards the implication  $W(\bar{\rho}) \subset W^?(\bar{\rho})$ , but very little is known about the harder converse implication in the case that  $p$  is ramified. It is clear that the techniques of [Geeb] will not on their own extend to the general case, as they rely on combinatorial results which are false if  $p$  ramifies.

In this paper, we prove most cases of the conjecture of [Sch08a] in the case that  $p$  is totally ramified in  $F$ . Our techniques do not depend on the fact that there is only a single prime of  $F$  above  $p$ , and they would extend to the case where every prime of  $F$  above  $p$  has residue field  $\mathbb{F}_p$  (or in combination with the techniques of [Geeb], to the case where every prime of  $F$  above  $p$  is either unramified or totally ramified). We have restricted to the case that  $p$  is totally ramified in order to simplify the exposition.

We assume throughout that  $p$  is odd, and that  $\bar{\rho}|_{G_{F(\zeta_p)}}$  is irreducible. We make a mild additional assumption if  $p = 5$ . All of these restrictions are imposed by our use of the modularity lifting theorems of [Kis09b] (or rather, by their use in [Geea]). Let  $(p) = \mathfrak{p}^e$  in  $\mathcal{O}_F$ , where  $e = [F : \mathbb{Q}]$ . Under these assumptions, we are able to prove that if  $\bar{\rho}|_{G_{F_p}}$  is irreducible, then  $W(\bar{\rho}) = W^?(\bar{\rho})$ . If  $\bar{\rho}|_{G_{F_p}}$  is a sum of two characters, then we show that  $W(\bar{\rho}) \subset W^?(\bar{\rho})$ , and that equality holds if  $e \geq p$ . If  $e \leq p - 1$  then we prove that the weights in  $W^?(\bar{\rho})$  all occur except that we miss between zero and four weights; under the extra hypothesis that  $\bar{\rho}$  has an ordinary modular lift, we can usually (but not quite always) treat these exceptions as well.

We establish that  $W(\bar{\rho}) \subset W^?(\bar{\rho})$  by a computation using Breuil modules with descent data, in the same style as analogous computations in the literature; we have to use a few tricks in boundary cases, but these arguments are more or less standard.

For the harder converse, our techniques are roughly a combination of those of [Geeb] and an argument due to Kevin Buzzard, which uses a technique known as “weight cycling”. This argument was first written up in section 5 of [Tay06] in the case that  $p$  splits completely in  $F$ . The argument essentially depends only on the residue field of primes dividing  $p$ , and thus applies equally well in our totally ramified setting. It is the use of this argument that entails our restriction to the totally ramified case, rather than permitting arbitrary ramification. We note that while it might be possible to prove some additional results for small residue fields in cases with little ramification, it is clear that if one wishes to work with arbitrarily ramified fields, then our methods will only give complete results in the case of residue field  $\mathbb{F}_p$  (for example, there will be many situations where even if one is told that  $\bar{\rho}$  is modular of all but one weight in  $W^?(\bar{\rho})$ , our methods will be unable to prove modularity in this remaining weight).

As in [Geeb], the plan is to construct modular lifts of  $\bar{\rho}$  which are potentially Barsotti-Tate of specific type, using the techniques of Khare-Wintenberger, as explained in [Geea]. These techniques reduce the construction of such lifts to the purely local problem of exhibiting a single potentially Barsotti-Tate lift of  $\bar{\rho}|_{G_{F_p}}$  of

the appropriate type. In the case that  $\bar{\rho}|_{G_{F_p}}$  is irreducible, writing down such a lift is rather non-trivial; in fact, as far as we are aware, no-one has written down such a lift in any case in which  $e > 1$ . We accomplish this by means of an explicit construction of a corresponding strongly divisible module.

The immediate consequence of the existence of these lifts is that  $\bar{\rho}$  is modular of one of two weights, the constituents of a certain principal series representation. In [Geeb] we were able to conclude that only one of these two weights was actually possible, but in the totally ramified case  $\bar{\rho}$  is frequently modular of both of these weights, so no such argument is possible. It is at this point that we employ weight cycling. Crucially, we can frequently ensure that our lift is non-ordinary, and when this holds weight cycling ensures that  $\bar{\rho}$  is modular of both weights. The cases where we cannot guarantee a non-ordinary lift are certain of those for which  $\bar{\rho}|_{G_{F_p}}$  is reducible and  $e < p$ , which is why our results are slightly weaker in this case.

We note that our methods should also be applicable in the non-tame case, and should give similar results, subject to the appropriate local calculations. For explicit conjectures in this case (“explicit” in the sense of [BDJ08], i.e., in terms of certain crystalline extensions) see the forthcoming [GHS].

We now detail the outline of the paper. In section 2 we give our initial definitions and notation. In particular, we introduce spaces of algebraic modular forms on definite quaternion algebras, and we explain what it means for  $\bar{\rho}$  to be modular of a specific weight. Note that we work throughout with these spaces of forms, rather than their analogues for indefinite quaternion algebras as used in [Sch08a] or [BDJ08]. While our results do not immediately go over to their setting, our proofs do; both the results on the existence of Barsotti-Tate lifts of specified type and the weight cycling argument are available in that case (for the latter, see [Sch08b]). We restrict ourselves to the case of definite quaternion algebras for ease of exposition, for example in order to use the results of [Gee] as a black box.

In section 3 we explain which tame lifts we will need to consider, and the relationship between the existence of modular lifts of specified types and the property of being modular of a certain weight. This amounts to recalling certain concrete instances of the local Langlands correspondence for  $GL_2$  and local-global compatibility. All of this material is completely standard.

We give an exposition of the weight cycling result in section 4, adapted to the situation at hand. In particular, we combine weight cycling with the results of earlier sections to give a result establishing that  $\bar{\rho}$  is modular of a particular weight provided that it has a modular lift which is potentially Barsotti-Tate of a particular type and is non-ordinary.

Having done this, we now need some concrete results on the existence of (local) potential Barsotti-Tate representations of particular type that lift  $\bar{\rho}|_{G_{F_p}}$ . We warm up for these calculations by establishing (in the tame case) the inclusion  $W(\bar{\rho}) \subset W^?(\bar{\rho})$  in section 5. This uses a calculation with Breuil modules. In section 6.1 we produce the required lifts in the case that  $\bar{\rho}|_{G_{F_p}}$  is reducible. This case is relatively straightforward, as we are able to use reducible lifts. The irreducible case is considerably more challenging, and is completed in sections 6.2 and 6.3, where we explicitly construct the lifts by writing down the corresponding strongly divisible modules.

Finally, in section 7 we combine these results with the lifting techniques of [Gee] and some combinatorial arguments to prove the main theorems.

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## 2. NOTATION AND ASSUMPTIONS

Let  $p$  be an odd prime. Fix an algebraic closure  $\overline{\mathbb{Q}}$  of  $\mathbb{Q}$ , an algebraic closure  $\overline{\mathbb{Q}}_p$  of  $\mathbb{Q}_p$ , and an embedding  $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ . We will consider all finite extensions of  $\mathbb{Q}$  (respectively  $\mathbb{Q}_p$ ) to be contained in  $\overline{\mathbb{Q}}$  (respectively  $\overline{\mathbb{Q}}_p$ ). If  $K$  is such an extension, we let  $G_K$  denote its absolute Galois group  $\text{Gal}(\overline{K}/K)$ . Let  $F$  be a totally real field in which  $p$  is totally ramified, say  $(p) = \mathfrak{p}^e$ . Choose a uniformiser  $\pi_{\mathfrak{p}} \in \mathfrak{p}$ . Let  $\overline{\rho} : G_F \rightarrow \text{GL}_2(\overline{\mathbb{F}}_p)$  be a continuous Galois representation. Assume from now on that  $\overline{\rho}|_{G_F(\zeta_p)}$  is absolutely irreducible. If  $p = 5$  and the projective image of  $\overline{\rho}$  is isomorphic to  $\text{PGL}_2(\mathbb{F}_5)$ , assume further that  $[F(\zeta_5) : F] = 4$ . We normalise the isomorphisms of local class field theory so that a uniformiser corresponds to a geometric Frobenius element.

We wish to discuss the Serre weights of  $\overline{\rho}$ . We choose to work with totally definite quaternion algebras. We recall the basic definitions and results that we need, adapted to the particular case where  $F$  is totally ramified at  $p$ .

Let  $D$  be a quaternion algebra with center  $F$  which is ramified at all infinite places of  $F$  and at a set  $\Sigma$  of finite places, which does not contain  $\mathfrak{p}$ . Fix a maximal order  $\mathcal{O}_D$  of  $D$  and for each finite place  $v \notin \Sigma$  fix an isomorphism  $(\mathcal{O}_D)_v \xrightarrow{\sim} M_2(\mathcal{O}_{F_v})$ . For any finite place  $v$  let  $\pi_v$  denote a uniformiser of  $F_v$ .

Let  $U = \prod_v U_v \subset (D \otimes_F \mathbb{A}_F^f)^\times$  be a compact open subgroup, with each  $U_v \subset (\mathcal{O}_D)_v^\times$ . Furthermore, assume that  $U_v = (\mathcal{O}_D)_v^\times$  for all  $v \in \Sigma$ .

Take  $A$  a topological  $\mathbb{Z}_p$ -algebra. Fix a continuous representation  $\sigma : U_{\mathfrak{p}} \rightarrow \text{Aut}(W_\sigma)$  with  $W_\sigma$  a finite free  $A$ -module. We regard  $\sigma$  as a representation of  $U$  in the obvious way (that is, we let  $U_v$  act trivially if  $v \nmid p$ ). Fix also a character  $\psi : F^\times \backslash (\mathbb{A}_F^f)^\times \rightarrow A^\times$  such that for any finite place  $v$  of  $F$ ,  $\sigma|_{U_v \cap \mathcal{O}_{F_v}^\times}$  is multiplication by  $\psi^{-1}$ . Then we can think of  $W_\sigma$  as a  $U(\mathbb{A}_F^f)^\times$ -module by letting  $(\mathbb{A}_F^f)^\times$  act via  $\psi^{-1}$ .

Let  $S_{\sigma, \psi}(U, A)$  denote the set of continuous functions

$$f : D^\times \backslash (D \otimes_F \mathbb{A}_F^f)^\times \rightarrow W_\sigma$$

such that for all  $g \in (D \otimes_F \mathbb{A}_F^f)^\times$  we have

$$\begin{aligned} f(gu) &= \sigma(u)^{-1} f(g) \text{ for all } u \in U, \\ f(gz) &= \psi(z) f(g) \text{ for all } z \in (\mathbb{A}_F^f)^\times. \end{aligned}$$

We can write  $(D \otimes_F \mathbb{A}_F^f)^\times = \prod_{i \in I} D^\times t_i U(\mathbb{A}_F^f)^\times$  for some finite index set  $I$  and some  $t_i \in (D \otimes_F \mathbb{A}_F^f)^\times$ . Then we have

$$S_{\sigma, \psi}(U, A) \xrightarrow{\sim} \bigoplus_{i \in I} W_\sigma^{(U(\mathbb{A}_F^f)^\times \cap t_i^{-1} D^\times t_i) / F^\times},$$

the isomorphism being given by the direct sum of the maps  $f \mapsto f(t_i)$ . From now on we make the following assumption:

For all  $t \in (D \otimes_F \mathbb{A}_F^f)^\times$  the group  $(U(\mathbb{A}_F^f)^\times \cap t^{-1} D^\times t) / F^\times$  is trivial.

Without changing  $U_{\mathfrak{p}}$ , one can always replace  $U$  by a subgroup satisfying the above assumptions (*cf.* section 3.1.1 of [Kis09a]). Under these assumptions  $S_{\sigma, \psi}(U, A)$  is a finite free  $A$ -module, and the functor  $W_\sigma \mapsto S_{\sigma, \psi}(U, A)$  is exact in  $W_\sigma$ .

We now define some Hecke algebras. Let  $S$  be a set of finite places containing  $\Sigma$ ,  $\mathfrak{p}$ , and the primes  $v$  of  $F$  such that  $U_v \neq (\mathcal{O}_D)_v^\times$ . Let  $\mathbb{T}_{S,A}^{\text{univ}} = A[T_v, S_v]_{v \notin S}$  be the commutative polynomial ring in the formal variables  $T_v, S_v$ . Consider the left action of  $(D \otimes_F \mathbb{A}_F^f)^\times$  on  $W_\sigma$ -valued functions on  $(D \otimes_F \mathbb{A}_F^f)^\times$  given by  $(gf)(z) = f(zg)$ . Then we make  $S_{\sigma,\psi}(U, A)$  a  $\mathbb{T}_{S,A}^{\text{univ}}$ -module by letting  $S_v$  act via the double coset  $U \begin{pmatrix} \pi_v & 0 \\ 0 & \pi_v \end{pmatrix} U$  and  $T_v$  via  $U \begin{pmatrix} \pi_v & 0 \\ 0 & 1 \end{pmatrix} U$ . These are independent of the choices of  $\pi_v$ . We will write  $\mathbb{T}_{\sigma,\psi}(U, A)$  or  $\mathbb{T}_{\sigma,\psi}(U)$  for the image of  $\mathbb{T}_{S,A}^{\text{univ}}$  in  $\text{End } S_{\sigma,\psi}(U, A)$ .

Note that if  $\sigma$  is trivial, then we may also define Hecke operators at  $\mathfrak{p}$ . We let  $U_{\pi_{\mathfrak{p}}}$  be the Hecke operator given by the double coset  $U \begin{pmatrix} \pi_{\mathfrak{p}} & 0 \\ 0 & 1 \end{pmatrix} U$  and let  $V_{\pi_{\mathfrak{p}}}$  be given by  $U \begin{pmatrix} 1 & 0 \\ 0 & \pi_{\mathfrak{p}} \end{pmatrix} U$ . Note that these may depend on the choice of  $\pi_{\mathfrak{p}}$ .

Let  $\mathfrak{m}$  be a maximal ideal of  $\mathbb{T}_{S,A}^{\text{univ}}$ . We say that  $\mathfrak{m}$  is in the support of  $(\sigma, \psi)$  if  $S_{\sigma,\psi}(U, A)_{\mathfrak{m}} \neq 0$ . Now let  $\mathcal{O}$  be the ring of integers in  $\overline{\mathbb{Q}}_p$ , with residue field  $\mathbb{F} = \overline{\mathbb{F}}_p$ , and suppose that  $A = \mathcal{O}$  in the above discussion and that  $\sigma$  has open kernel. Consider a maximal ideal  $\mathfrak{m} \subset \mathbb{T}_{S,\mathcal{O}}^{\text{univ}}$  which is induced by a maximal ideal of  $\mathbb{T}_{\sigma,\psi}(U, \mathcal{O})$ . Then (see for example section 3.4.1 of [Kis09b]) there is a semisimple Galois representation  $\overline{\rho}_{\mathfrak{m}} : G_F \rightarrow \text{GL}_2(\mathbb{F})$  associated to  $\mathfrak{m}$  which is characterised up to conjugacy by the property that if  $v \notin S$  then  $\overline{\rho}_{\mathfrak{m}}|_{G_{F_v}}$  is unramified, and if  $\text{Frob}_v$  is an arithmetic Frobenius at  $v$  then the trace of  $\overline{\rho}_{\mathfrak{m}}(\text{Frob}_v)$  is the image of  $T_v$  in  $\mathbb{F}$ .

We are now in a position to define what it means for a representation to be modular of some weight. Let  $F_{\mathfrak{p}}$  have ring of integers  $\mathcal{O}_{F_{\mathfrak{p}}}$ , and let  $\sigma$  be an irreducible  $\mathbb{F}$ -representation of  $\text{GL}_2(\mathbb{F}_{\mathfrak{p}})$ , so  $\sigma$  is isomorphic to  $\sigma_{m,n} := \det^m \otimes \text{Sym}^n \mathbb{F}^2$  for some  $0 \leq m < p-1$ ,  $0 \leq n \leq p-1$ . Throughout the paper we allow  $m, n$  to vary over these ranges. We also denote by  $\sigma$  the representation of  $\text{GL}_2(\mathcal{O}_{F_{\mathfrak{p}}})$  induced by the surjection  $\mathcal{O}_{F_{\mathfrak{p}}} \twoheadrightarrow \mathbb{F}_{\mathfrak{p}}$ .

**Definition 2.1.** We say that  $\overline{\rho}$  is modular of weight  $\sigma$  if for some  $D, S, U, \psi$ , and  $\mathfrak{m}$  as above, with  $U_{\mathfrak{p}} = \text{GL}_2(\mathcal{O}_{F_{\mathfrak{p}}})$ , we have  $S_{\sigma,\psi}(U, \mathbb{F})_{\mathfrak{m}} \neq 0$  and  $\overline{\rho}_{\mathfrak{m}} \cong \overline{\rho}$ .

Assume from now on that  $\overline{\rho}$  is modular of some weight, and fix  $D, S, U, \psi, \mathfrak{m}$  as in the definition. Write  $W(\overline{\rho})$  for the set of weights  $\sigma$  for which  $\overline{\rho}$  is modular of weight  $\sigma$ .

One can gain information about the weights associated to a particular Galois representation by considering lifts to characteristic zero. The key is the following basic lemma.

**Lemma 2.2.** *Let  $\psi : F^\times \backslash (\mathbb{A}_F^f)^\times \rightarrow \mathcal{O}^\times$  be a continuous character, and write  $\overline{\psi}$  for the composite of  $\psi$  with the projection  $\mathcal{O}^\times \rightarrow \mathbb{F}^\times$ . Fix a representation  $\sigma$  of  $U_{\mathfrak{p}}$  on a finite free  $\mathcal{O}$ -module  $W_\sigma$ , and an irreducible representation  $\sigma'$  of  $U_{\mathfrak{p}}$  on a finite free  $\mathbb{F}$ -module  $W_{\sigma'}$ . Suppose that we have  $\sigma|_{U_v \cap \mathcal{O}_{F_v}^\times} = \psi^{-1}|_{U_v \cap \mathcal{O}_{F_v}^\times}$  and  $\sigma'|_{U_v \cap \mathcal{O}_{F_v}^\times} = \overline{\psi}^{-1}|_{U_v \cap \mathcal{O}_{F_v}^\times}$  for all finite places  $v$ .*

*Let  $\mathfrak{m}$  be a maximal ideal of  $\mathbb{T}_{S,\mathcal{O}}^{\text{univ}}$ .*

*Suppose that  $W_{\sigma'}$  occurs as a  $U_{\mathfrak{p}}$ -module subquotient of  $W_{\overline{\sigma}} := W_\sigma \otimes \mathbb{F}$ . If  $\mathfrak{m}$  is in the support of  $(\sigma', \overline{\psi})$ , then  $\mathfrak{m}$  is in the support of  $(\sigma, \psi)$ .*

*Conversely, if  $\mathfrak{m}$  is in the support of  $(\sigma, \psi)$ , then  $\mathfrak{m}$  is in the support of  $(\sigma', \overline{\psi})$  for some irreducible  $U_{\mathfrak{p}}$ -module subquotient  $W_{\sigma'}$  of  $W_{\overline{\sigma}}$ .*

*Proof.* The first part is proved just as in Lemma 3.1.4 of [Kis09b], and the second part follows from Proposition 1.2.3 of [AS86].  $\square$

## 3. TAME LIFTS

We recall some group-theoretic results from section 3 of [CDT99]. First, recall the irreducible finite-dimensional representations of  $\mathrm{GL}_2(\mathbb{F}_p)$  over  $\overline{\mathbb{Q}}_p$ . Once one fixes an embedding  $\mathbb{F}_{p^2} \hookrightarrow M_2(\mathbb{F}_p)$ , any such representation is equivalent to one in the following list:

- For any character  $\chi : \mathbb{F}_p^\times \rightarrow \overline{\mathbb{Q}}_p^\times$ , the representation  $\chi \circ \det$ .
- For any  $\chi : \mathbb{F}_p^\times \rightarrow \overline{\mathbb{Q}}_p^\times$ , the representation  $\mathrm{sp}_\chi = \mathrm{sp} \otimes (\chi \circ \det)$ , where  $\mathrm{sp}$  is the representation of  $\mathrm{GL}_2(\mathbb{F}_p)$  on the space of functions  $\mathbb{P}^1(\mathbb{F}_p) \rightarrow \overline{\mathbb{Q}}_p$  with average value zero.
- For any pair  $\chi_1 \neq \chi_2 : \mathbb{F}_p^\times \rightarrow \overline{\mathbb{Q}}_p^\times$ , the representation

$$I(\chi_1, \chi_2) = \mathrm{Ind}_{B(\mathbb{F}_p)}^{\mathrm{GL}_2(\mathbb{F}_p)} (\chi_1 \otimes \chi_2),$$

where  $B(\mathbb{F}_p)$  is the Borel subgroup of upper-triangular matrices in  $\mathrm{GL}_2(\mathbb{F}_p)$ , and  $\chi_1 \otimes \chi_2$  is the character

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mapsto \chi_1(a)\chi_2(d).$$

- For any character  $\chi : \mathbb{F}_{p^2}^\times \rightarrow \overline{\mathbb{Q}}_p^\times$  with  $\chi \neq \chi^p$ , the cuspidal representation  $\Theta(\chi)$  characterised by

$$\Theta(\chi) \otimes \mathrm{sp} \cong \mathrm{Ind}_{\mathbb{F}_{p^2}^\times}^{\mathrm{GL}_2(\mathbb{F}_p)} \chi.$$

We now recall the reductions mod  $p$  of these representations. Let  $\sigma_{m,n}$  be the irreducible  $\overline{\mathbb{F}}_p$ -representation  $\det^m \otimes \mathrm{Sym}^n \mathbb{F}^2$ , with  $0 \leq m < p-1$ ,  $0 \leq n \leq p-1$ . Then we have:

**Lemma 3.1.** *Let  $L$  be a finite free  $\mathcal{O}$ -module with an action of  $\mathrm{GL}_2(\mathbb{F}_p)$  such that  $V = L \otimes_{\mathcal{O}} \overline{\mathbb{Q}}_p$  is irreducible. Let  $\tilde{a}$  denote the Teichmüller lift of  $a$ .*

- (1) *If  $V \cong \chi \circ \det$  with  $\chi(a) = \tilde{a}^m$ , then  $L \otimes_{\mathcal{O}} \mathbb{F} \cong \sigma_{m,0}$ .*
- (2) *If  $V \cong \mathrm{sp}_\chi$  with  $\chi(a) = \tilde{a}^m$ , then  $L \otimes_{\mathcal{O}} \mathbb{F} \cong \sigma_{m,p-1}$ .*
- (3) *If  $V \cong I(\chi_1, \chi_2)$  with  $\chi_i(a) = \tilde{a}^{m_i}$  for distinct  $m_i \in \mathbb{Z}/(p-1)\mathbb{Z}$ , then  $L \otimes_{\mathcal{O}} \mathbb{F}$  has two Jordan-Hölder subquotients:  $\sigma_{m_2, \{m_1 - m_2\}}$  and  $\sigma_{m_1, \{m_2 - m_1\}}$  where  $0 < \{m\} < p-1$  and  $\{m\} \equiv m \pmod{p-1}$ .*
- (4) *If  $V \cong \Theta(\chi)$  with  $\chi(c) = \tilde{c}^{i+(p+1)j}$  where  $1 \leq i \leq p$  and  $j \in \mathbb{Z}/(p-1)\mathbb{Z}$ , then  $L \otimes_{\mathcal{O}} \mathbb{F}$  has two Jordan-Hölder subquotients:  $\sigma_{1+j, i-2}$  and  $\sigma_{i+j, p-1-i}$ . Both occur unless  $i = p$  (when only the first occurs), or  $i = 1$  (when only the second one occurs), and in either of these cases  $L \otimes_{\mathcal{O}} \mathbb{F} \cong \sigma_{1+j, p-2}$ .*

*Proof.* This is Lemma 3.1.1 of [CDT99].  $\square$

In what follows, we will sometimes consider the above representations as representations of  $\mathrm{GL}_2(\mathcal{O}_{F_p})$  via the natural projection map.

We now recall some definitions relating to potentially semistable lifts of particular type. We use the conventions of [Sav05].

**Definition 3.2.** Let  $\tau$  be an inertial type. We say that a lift  $\rho$  of  $\bar{\rho}|_{G_{F_p}}$  is parallel potentially Barsotti-Tate (respectively parallel potentially semistable) of type  $\tau$  if  $\rho$  is potentially Barsotti-Tate (respectively potentially semistable with all Hodge-Tate weights equal to 0 or 1) with  $\det \rho = \epsilon\psi$ , where  $\epsilon$  denotes the cyclotomic

character and  $\psi$  is some finite order character of order prime to  $p$ , and moreover the corresponding Weil-Deligne representation, when restricted to  $I_{F_p}$ , is isomorphic to  $\tau$ .

Note that for a two-dimensional de Rham representation with all Hodge-Tate weights equal to 0 or 1, the condition that all pairs of labeled Hodge-Tate weights are  $\{0, 1\}$  is equivalent to the condition that the determinant is the product of the cyclotomic character, a finite order character, and an unramified character; the condition of being parallel is slightly stronger than this.

**Definition 3.3.** We say that a representation  $\rho : G_{F_p} \rightarrow \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$  is ordinary if  $\rho|_{I_{F_p}}$  is an extension of a finite order character by a finite order character times the cyclotomic character. We say that a representation  $\rho : G_F \rightarrow \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$  is ordinary if  $\rho|_{G_{F_p}}$  is ordinary.

We now need some special cases of the inertial local Langlands correspondence of Henniart (see the appendix to [BM02]). If  $\chi_1 \neq \chi_2 : \mathbb{F}_p^\times \rightarrow \mathcal{O}^\times$ , let  $\tau_{\chi_1, \chi_2}$  be the inertial type  $\chi_1 \oplus \chi_2$  (considered as a representation of  $I_{F_p}$  via local class field theory). Then we let  $\sigma(\tau_{\chi_1, \chi_2})$  be a representation on a finite  $\mathcal{O}$ -module given by taking a lattice in  $I(\chi_1, \chi_2)$ . If  $\chi : \mathbb{F}_p^\times \rightarrow \mathcal{O}^\times$ , we let  $\tau_\chi = \chi \oplus \chi$ , and  $\sigma(\tau_\chi)$  be  $\chi \circ \det$ .

If  $\chi : \mathbb{F}_{p^2}^\times \rightarrow \mathcal{O}^\times$  with  $\chi \neq \chi^p$ , we let  $\tau_{\chi, \chi^p} = \chi \oplus \chi^p$  (again, regarded as a representation of  $I_{F_p}$  via local class field theory), and we let  $\sigma(\tau_{\chi, \chi^p})$  be a representation on a finite  $\mathcal{O}$ -module given by taking a lattice in  $\Theta(\chi)$ .

**Lemma 3.4.** *Fix a type  $\tau$  as above (i.e.,  $\tau = \tau_{\chi_1, \chi_2}$ ,  $\tau_\chi$ , or  $\tau_{\chi, \chi^p}$ ). Suppose that  $\bar{\rho}$  is modular of weight  $\sigma$ , and that  $\sigma$  is a  $\mathrm{GL}_2(\mathbb{F}_p)$ -module subquotient of  $\sigma(\tau) \otimes_{\mathcal{O}} \mathbb{F}$ . Then  $\bar{\rho}$  lifts to a modular Galois representation which is parallel potentially Barsotti-Tate of type  $\tau$  at  $\mathfrak{p}$ . Similarly, if  $\bar{\rho}$  is modular of weight  $\sigma_{m, p-1}$ , then  $\bar{\rho}$  lifts to a modular Galois representation which is parallel potentially semistable of type  $\tilde{\omega}^m \oplus \tilde{\omega}^m$  (see the beginning of section 5 for the definition of the fundamental character  $\omega$ ). Conversely, if  $\bar{\rho}$  lifts to a modular Galois representation which is parallel potentially Barsotti-Tate of type  $\tau$  at  $\mathfrak{p}$ , then  $\bar{\rho}$  is modular of weight  $\sigma$  for some  $\mathrm{GL}_2(\mathbb{F}_p)$ -module subquotient  $\sigma$  of  $\sigma(\tau) \otimes_{\mathcal{O}} \mathbb{F}$ .*

*Proof.* By Lemma 2.2, the Jacquet-Langlands correspondence, and the discussion in section 3.1.14 of [Kis09b],  $\bar{\rho}$  is modular of weight  $\sigma$  for some  $\mathrm{GL}_2(\mathbb{F}_p)$ -module subquotient  $\sigma$  of  $\sigma(\tau) \otimes_{\mathcal{O}} \mathbb{F}$  if and only if there is an automorphic representation  $\pi$  of  $\mathrm{GL}_2(\mathbb{A}_F)$  corresponding to a Hilbert modular form of parallel weight 2, such that

- the Galois representation  $\rho_\pi : G_F \rightarrow \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$  associated to  $\pi$  lifts  $\bar{\rho}$ , and
- $\pi_{\mathfrak{p}}$  contains  $\sigma(\tau)^*$  as a  $\mathrm{GL}_2(\mathcal{O}_{\mathfrak{p}})$ -representation.

Since  $\pi$  corresponds to a Hilbert modular form of parallel weight 2,  $\rho_\pi|_{G_{F_p}}$  is potentially semistable with Hodge-Tate weights 0 and 1, and is potentially Barsotti-Tate provided that  $\pi_{\mathfrak{p}}$  is not a twist of the Steinberg representation. By the compatibility of the local and global Langlands correspondences at places dividing  $p$  (see [Kis08]), and the results of sections A2 and A3 of Henniart's appendix to [BM02], it follows that the conditions above on  $\pi$  are equivalent to

- the Galois representation  $\rho_\pi : G_F \rightarrow \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$  associated to  $\pi$  lifts  $\bar{\rho}$ , and
- $\rho_\pi|_{G_{F_p}}$  has type  $\tau$ .

Furthermore, in each of these cases it follows from the discussion in section A2 of Henniart's appendix to [BM02] that  $\pi_{\mathfrak{p}}$  is not a twist of the Steinberg representation.

This proves everything apart from the assertion that if  $\bar{\rho}$  is modular of weight  $\sigma_{m,p-1}$ , then  $\bar{\rho}$  lifts to a modular Galois representation which is parallel potentially semistable of type  $\tilde{\omega}^m \oplus \tilde{\omega}^m$ . This may be proved by a very similar argument; in this case, the argument above provides an automorphic representation  $\pi$  of  $\mathrm{GL}_2(\mathbb{A}_F)$  corresponding to a Hilbert modular form of parallel weight 2, such that

- the Galois representation  $\rho_{\pi} : G_F \rightarrow \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$  associated to  $\pi$  lifts  $\bar{\rho}$ , and
- $\pi_{\mathfrak{p}}$  contains  $\mathrm{sp}_{\tilde{\omega}^m \circ \det}^*$  as a  $\mathrm{GL}_2(\mathcal{O}_{\mathfrak{p}})$ -representation.

It then follows from section A2 of Henniart's appendix to [BM02] that  $\pi_{\mathfrak{p}}$  is a twist of the Steinberg representation, from which the result follows easily.  $\square$

#### 4. WEIGHT CYCLING

We now explain the weight cycling argument due to Kevin Buzzard which proves modularity in an additional weight in the non-ordinary case. There is an exposition of this argument in section 5 of [Tay06] in the case that  $p$  splits completely in  $F$ , and the argument goes over essentially unchanged in our setting. Since our notation and assumptions differ from those of [Tay06], we sketch a proof here.

Fix a character  $\psi : F^{\times} \backslash (\mathbb{A}_F^f)^{\times} \rightarrow \mathcal{O}^{\times}$ . If  $A$  is an  $\mathcal{O}$ -module we will regard  $\psi$  as an  $A$ -valued character via the structure homomorphism. Suppose that  $A$  is a field and that  $\sigma^{\vee}$  denotes the dual of  $\sigma$ . Then (*cf.* the discussion on page 742 of [Tay06], recalling that by assumption we have  $(U(\mathbb{A}_F^f)^{\times} \cap t^{-1}D^{\times}t)/F^{\times} = 1$  for all  $t \in (D \otimes_F \mathbb{A}_F^f)^{\times}$ ) there is a perfect pairing

$$\langle \cdot, \cdot \rangle : S_{\sigma, \psi}(U, A) \times S_{\sigma^{\vee}, \psi^{-1}}(U, A) \rightarrow A$$

given by

$$\langle f_1, f_2 \rangle = \sum_i \langle f_1(t_i), f_2(t_i) \rangle$$

where

$$(D \otimes_F \mathbb{A}_F^f)^{\times} = \prod_i D^{\times} t_i U(\mathbb{A}_F^f)^{\times}$$

and the pairing between  $f_1(t_i)$  and  $f_2(t_i)$  is the usual pairing between a representation and its dual. A standard calculation shows that under this pairing the adjoint of  $S_x$  is  $S_x^{-1}$ , the adjoint of  $T_x$  is  $T_x S_x^{-1}$ , and when it is defined the adjoint of  $U_{\pi_{\mathfrak{p}}}$  is  $V_{\pi_{\mathfrak{p}}} S_{\mathfrak{p}}^{-1}$ .

Let

$$U_0 = \prod_{v \nmid p} U_v \times I_{\mathfrak{p}}$$

where  $I_{\mathfrak{p}}$  is the Iwahori subgroup of  $\mathrm{GL}_2(\mathcal{O}_{F_{\mathfrak{p}}})$  consisting of matrices which are upper-triangular modulo  $\mathfrak{p}$ , and let

$$U_1 = \prod_{v \nmid p} U_v \times I_{\mathfrak{p}}^1$$

where  $I_{\mathfrak{p}}^1$  is the subgroup of  $I_{\mathfrak{p}}$  whose entries are congruent to  $\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}$  modulo  $\mathfrak{p}$ . Let  $\sigma = 1$ , the trivial representation, so that the operators  $U_{\pi_{\mathfrak{p}}}$  and  $V_{\pi_{\mathfrak{p}}}$  are defined on  $S_{1, \psi}(U_1, A)$  and  $S_{1, \psi^{-1}}(U_1, A)$  for any  $\mathcal{O}$ -algebra  $A$ . Let

$$\delta^n : \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mapsto d^n,$$



an  $\mathbb{F}^\times$ -character of the standard Borel subgroup  $B(\mathbb{F}_p)$  of  $\mathrm{GL}_2(\mathbb{F}_p)$ . Then there is a natural embedding

$$S_{\delta^n, \psi}(U_0, \mathbb{F}) \hookrightarrow S_{1, \psi}(U_1, \mathbb{F})$$

which is equivariant for the actions of  $\mathbb{T}_{S, \mathbb{F}}^{\mathrm{univ}}$ , and the image of  $S_{\delta^n, \psi}(U_0, \mathbb{F})$  is stable under the actions of  $U_{\pi_p}$  and  $V_{\pi_p}$ . We use this action as the definition of  $U_{\pi_p}$  and  $V_{\pi_p}$  on  $S_{\delta^n, \psi}(U_0, \mathbb{F})$ .

There is also a natural isomorphism

$$S_{\delta^n, \psi}(U_0, \mathbb{F}) \cong S_{\mathrm{Ind}(\delta^n), \psi}(U, \mathbb{F})$$

where  $\mathrm{Ind}(\delta^n)$  is obtained as the induction from  $B(\mathbb{F}_p)$  to  $\mathrm{GL}_2(\mathbb{F}_p)$  of  $\delta^n$ . We use this isomorphism to define actions of  $U_{\pi_p}$  and  $V_{\pi_p}$  on  $S_{\mathrm{Ind}(\delta^n), \psi}(U, \mathbb{F})$ . We can think of this induction as being the functions

$$\theta : \mathrm{GL}_2(\mathbb{F}_p) \rightarrow \mathbb{F}$$

with the property that for all  $b \in B(\mathbb{F}_p)$ ,  $g \in \mathrm{GL}_2(\mathbb{F}_p)$ ,

$$\theta(bg) = \delta^n(b)\theta(g).$$

The action of  $\mathrm{GL}_2(\mathbb{F}_p)$  is by

$$(g\theta)(x) = \theta(xg).$$

This isomorphism identifies  $f \in S_{\delta^n, \psi}(U_0, \mathbb{F})$  with  $F \in S_{\mathrm{Ind}(\delta^n), \psi}(U, \mathbb{F})$  where

$$f(x) = F(x)(1)$$

and

$$F(x)(g) = f(xg^{-1}).$$

Now, we have a short exact sequence

$$0 \rightarrow \sigma_{0, n} \rightarrow \mathrm{Ind}(\delta^n) \rightarrow \sigma_{n, p-1-n} \rightarrow 0$$

of  $\mathrm{GL}_2(\mathbb{F}_p)$ -modules and thus a short exact sequence

$$0 \rightarrow S_{\sigma_{0, n}, \psi}(U, \mathbb{F}) \xrightarrow{\alpha} S_{\mathrm{Ind}(\delta^n), \psi}(U, \mathbb{F}) \xrightarrow{\beta} S_{\sigma_{n, p-1-n}, \psi}(U, \mathbb{F}) \rightarrow 0$$

and, localising at  $\mathfrak{m}$ , a short exact sequence

$$0 \rightarrow S_{\sigma_{0, n}, \psi}(U, \mathbb{F})_{\mathfrak{m}} \xrightarrow{\alpha} S_{\mathrm{Ind}(\delta^n), \psi}(U, \mathbb{F})_{\mathfrak{m}} \xrightarrow{\beta} S_{\sigma_{n, p-1-n}, \psi}(U, \mathbb{F})_{\mathfrak{m}} \rightarrow 0.$$

**Proposition 4.1.** *If  $n < p - 1$  and  $S_{\sigma_{0, n}, \psi}(U, \mathbb{F})_{\mathfrak{m}} = 0$  then the map  $V_{\pi_p} : S_{\mathrm{Ind}(\delta^n), \psi}(U, \mathbb{F})_{\mathfrak{m}} \rightarrow S_{\mathrm{Ind}(\delta^n), \psi}(U, \mathbb{F})_{\mathfrak{m}}$  is an isomorphism.*

*Proof.* Under the assumption that  $S_{\sigma_{0, n}, \psi}(U, \mathbb{F})_{\mathfrak{m}} = 0$ , we have an isomorphism

$$S_{\mathrm{Ind}(\delta^n), \psi}(U, \mathbb{F})_{\mathfrak{m}} \xrightarrow{\beta} S_{\sigma_{n, p-1-n}, \psi}(U, \mathbb{F})_{\mathfrak{m}}.$$

We claim that there is an injection

$$\kappa : S_{\sigma_{n, p-1-n}, \psi}(U, \mathbb{F})_{\mathfrak{m}} \rightarrow S_{\mathrm{Ind}(\delta^n), \psi}(U, \mathbb{F})_{\mathfrak{m}}$$

such that  $\kappa \circ \beta = V_{\pi_p}$ . This would clearly establish the result. Of course, it is enough to construct an injection

$$\kappa : S_{\sigma_{n, p-1-n}, \psi}(U, \mathbb{F}) \rightarrow S_{\mathrm{Ind}(\delta^n), \psi}(U, \mathbb{F})$$

which commutes with the action of  $\mathbb{T}_{S, \mathbb{F}}^{\mathrm{univ}}$ , and satisfies  $\kappa \circ \beta = V_{\pi_p}$ .

The verification that this is possible is exactly as in [Tay06].  $\square$

**Proposition 4.2.** *If  $S_{\sigma_{0, p-1}, \psi}(U, \mathbb{F})_{\mathfrak{m}} = 0$  then the map  $U_{\pi_p} : S_{\sigma_{0, 0}, \psi}(U, \mathbb{F})_{\mathfrak{m}} \rightarrow S_{\sigma_{0, 0}, \psi}(U, \mathbb{F})_{\mathfrak{m}}$  is an isomorphism.*

*Proof.* Under the assumption that  $S_{\sigma_{0,p-1},\psi}(U, \mathbb{F})_{\mathfrak{m}} = 0$ , we have an isomorphism

$$S_{\text{Ind}(1),\psi}(U, \mathbb{F})_{\mathfrak{m}} \xrightarrow{\beta} S_{\sigma_{0,0},\psi}(U, \mathbb{F})_{\mathfrak{m}}.$$

We claim that there is an injection

$$\kappa : S_{\sigma_{0,0},\psi}(U, \mathbb{F}) \rightarrow S_{\text{Ind}(1),\psi}(U, \mathbb{F})$$

which commutes with the action of  $\mathbb{T}_{S, \mathbb{F}}^{\text{univ}}$ , and satisfies  $\beta \circ \kappa = U_{\pi_{\mathfrak{p}}}$ .

Again, the verification that this is possible is exactly as in [Tay06].  $\square$

Now, there is a maximal ideal  $\mathfrak{m}^*$  of  $\mathbb{T}_{S, \mathbb{F}}^{\text{univ}}$  with the property that for each  $x \notin S$ ,  $T_x - \alpha \in \mathfrak{m}^*$  if and only if  $T_x S_x^{-1} - \alpha \in \mathfrak{m}^*$ , and  $S_x - \beta \in \mathfrak{m}^*$  if and only if  $S_x^{-1} - \beta \in \mathfrak{m}^*$ . Thus

$$\bar{\rho}_{\mathfrak{m}^*} \cong \bar{\rho}_{\mathfrak{m}^*}^{\vee}(1).$$

Then from the duality explained above between  $S_{\sigma, \psi}$  and  $S_{\sigma^{\vee}, \psi^{-1}}$ , we obtain

**Corollary 4.3.** *If  $n < p - 1$  and  $S_{\sigma_{0,n},\psi}(U, \mathbb{F})_{\mathfrak{m}} = 0$  then the map*

$$U_{\pi_{\mathfrak{p}}} : S_{\text{Ind}(\delta^{-n}),\psi^{-1}}(U, \mathbb{F})_{\mathfrak{m}^*} \rightarrow S_{\text{Ind}(\delta^{-n}),\psi^{-1}}(U, \mathbb{F})_{\mathfrak{m}^*}$$

*is an isomorphism.*

*Proof.* This follows at once from Proposition 4.1.  $\square$

**Proposition 4.4.** *Suppose that  $0 \leq n \leq p - 1$ . If  $\bar{\rho}$  has a lift to a modular representation  $\rho : G_F \rightarrow \text{GL}_2(\overline{\mathbb{Q}}_p)$  which is parallel potentially Barsotti-Tate of type  $\tilde{\omega}^{m+n} \oplus \tilde{\omega}^m$  and is not ordinary, then  $\bar{\rho}$  is modular of weight  $\sigma_{m,n}$  and of weight  $\sigma_{m+n,p-1-n}$ .*

*Proof.* We know from Lemma 3.4 that  $\bar{\rho}$  is modular of weight at least one of  $\sigma_{m+n,p-1-n}$  and  $\sigma_{m,n}$ . Suppose for the sake of contradiction (and without loss of generality) that  $\bar{\rho}$  is modular of weight  $\sigma_{m+n,p-1-n}$  but not weight  $\sigma_{m,n}$ . Twisting, we may without loss of generality assume that  $m = 0$ .

Take  $D, S, U, \psi$  and  $\mathfrak{m}$  as in section 2, chosen so that if  $n \neq p - 1$  there is an eigenform in  $S_{\text{Ind}(\delta^n),\psi}(U, \overline{\mathbb{Q}}_p)$  corresponding to  $\rho$ , and if  $n = p - 1$  there is such a form in  $S_{1,\psi}(U, \overline{\mathbb{Q}}_p)$ . Note that by local-global compatibility and standard properties of the local Langlands correspondence, the assumption that  $\rho$  is not ordinary shows that  $U_{\mathfrak{p}}$  has a non-unit eigenvalue on  $S_{\text{Ind}(\delta^n),\psi}(U, \overline{\mathbb{Q}}_p)$  (respectively  $S_{1,\psi}(U, \overline{\mathbb{Q}}_p)$ ). [One needs to check that if  $U_{\mathfrak{p}}$  has only unit eigenvalues then  $\rho$  is necessarily ordinary. To do this, one uses local-global compatibility to show that the action of Frobenius on the associated weakly admissible representation has a unit eigenvalue; the corresponding eigenspace is then a sub-weakly-admissible module, from which ordinarity follows. See Proposition 5.3.1 of [Ger09] for a much more general result.]

Suppose first that  $n = p - 1$ . Then by Proposition 4.2,  $U_{\pi_{\mathfrak{p}}}$  is an isomorphism on  $S_{1,\psi}(U, \mathbb{F})$ ; but this is a contradiction.

Suppose now that  $n < p - 1$ . Then by Corollary 4.3,  $U_{\pi_{\mathfrak{p}}}$  is an isomorphism on  $S_{\text{Ind}(\delta^{-n}),\psi^{-1}}(U, \mathbb{F})_{\mathfrak{m}^*}$ . Again, this is a contradiction, as by the above duality there is an eigenform in  $S_{\text{Ind}(\delta^{-n}),\psi^{-1}}(U, \overline{\mathbb{Q}}_p)_{\mathfrak{m}^*}$  corresponding to the non-ordinary representation  $\rho^{\vee}(1)$ .  $\square$

## 5. NECESSARY CONDITIONS

Suppose that  $K$  is a finite extension of  $\mathbb{Q}_p$ , with residue field  $k$ . Let  $S_k = \{\tau : k \hookrightarrow \overline{\mathbb{F}}_p\}$ . For each  $\tau \in S_k$  we define the fundamental character  $\omega_{K,\tau}$  corresponding to  $\tau$  to be the composite

$$I_K \xrightarrow{\sim} \mathcal{O}_K^\times \longrightarrow k^\times \xrightarrow{\tau} \overline{\mathbb{F}}_p^\times,$$

where the first map is the isomorphism given by local class field theory, normalised so that a uniformiser corresponds to geometric Frobenius. We will generally suppress the subscript  $K$  and write simply  $\omega_\tau$ . If  $\chi$  is a character of  $G_K$  or  $I_K$ , we denote its reduction mod  $p$  by  $\overline{\chi}$ .

If  $K$  is totally ramified (e.g. if  $K = F_p$ ), we let  $\omega$  be the unique fundamental character. Note that  $\omega^{e(K/\mathbb{Q}_p)}$  is the (restriction to  $I_K$  of the) mod  $p$  cyclotomic character, e.g. by Lemma 6.2 below. Let  $\sigma_1, \sigma_2$  denote the two embeddings of the quadratic extension of  $k$  into  $\overline{\mathbb{F}}_p$ , and let  $\omega_{\sigma_1}, \omega_{\sigma_2}$  denote the two corresponding fundamental characters of  $I_K$ . Write  $\tilde{\omega}, \tilde{\omega}_{\sigma_1}$ , and  $\tilde{\omega}_{\sigma_2}$  for the Teichmüller lifts of  $\omega, \omega_{\sigma_1}$ , and  $\omega_{\sigma_2}$  respectively.

If  $\overline{\rho}|_{G_{F_p}}$  is semisimple, then Schein defines a set of predicted weights for  $\overline{\rho}$  as follows.

**Definition 5.1** ([Sch08a]). The set  $W^?( \overline{\rho} )$  is the set of weights  $\sigma_{m,n}$  such that there exists  $1 \leq x \leq e$  with either  $\overline{\rho}|_{I_{F_p}} \cong \omega_{\sigma_1}^{m+n+x} \omega_{\sigma_2}^{m+e-x} \oplus \omega_{\sigma_1}^{m+e-x} \omega_{\sigma_2}^{m+n+x}$  or  $\overline{\rho}|_{I_{F_p}} \cong \omega^{m+n+x} \oplus \omega^{m+e-x}$ .

Let  $W(\overline{\rho})$  be the set of weights  $\sigma$  such that  $\overline{\rho}$  is modular of weight  $\sigma$ . Our aim in this section is to prove that  $W(\overline{\rho}) \subset W^?( \overline{\rho} )$ .

**5.1. Breuil modules with descent data.** Let  $k$  be a finite extension of  $\mathbb{F}_p$ , define  $K_0 = W(k)[1/p]$ , and let  $K$  be a finite totally ramified extension of  $K_0$  of degree  $e'$ . Suppose that  $L$  is a  $p$ -adic subfield of  $K$  such that  $K/L$  is Galois and tamely ramified. Assume further that there is a uniformiser  $\pi$  of  $\mathcal{O}_K$  such that  $\pi^{e(K/L)} \in L$ , where  $e(K/L)$  is the ramification degree of  $K/L$ , and fix such a  $\pi$ . Since  $K/L$  is tamely ramified, the category of *Breuil modules with coefficients and descent data from  $K$  to  $L$*  is defined as follows (see [Sav08]). Let  $k_E$  be a finite extension of  $\mathbb{F}_p$ . The category  $\text{BrMod}_{\text{dd},L}$  consists of quadruples  $(\mathcal{M}, \text{Fil}^1 \mathcal{M}, \phi_1, \{\widehat{g}\})$  where:

- $\mathcal{M}$  is a finitely generated  $(k \otimes_{\mathbb{F}_p} k_E)[u]/u^{e'p}$ -module, free over  $k[u]/u^{e'p}$ .
- $\text{Fil}^1 \mathcal{M}$  is a  $(k \otimes_{\mathbb{F}_p} k_E)[u]/u^{e'p}$ -submodule of  $\mathcal{M}$  containing  $u^{e'} \mathcal{M}$ .
- $\phi_1 : \text{Fil}^1 \mathcal{M} \rightarrow \mathcal{M}$  is  $k_E$ -linear and  $\phi$ -semilinear (where  $\phi : k[u]/u^{e'p} \rightarrow k[u]/u^{e'p}$  is the  $p$ -th power map) with image generating  $\mathcal{M}$  as a  $(k \otimes_{\mathbb{F}_p} k_E)[u]/u^{e'p}$ -module.
- $\widehat{g} : \mathcal{M} \rightarrow \mathcal{M}$  are additive bijections for each  $g \in \text{Gal}(K/L)$ , preserving  $\text{Fil}^1 \mathcal{M}$ , commuting with the  $\phi_1$ -, and  $k_E$ -actions, and satisfying the following axioms: if  $a \in k \otimes_{\mathbb{F}_p} k_E$ ,  $m \in \mathcal{M}$  then  $\widehat{g}(au^i m) = g(a)((g(\pi)/\pi)^i \otimes 1)u^i \widehat{g}(m)$ ; for all  $g_1, g_2 \in \text{Gal}(K/L)$  we have  $\widehat{g}_1 \circ \widehat{g}_2 = \widehat{g_1 \circ g_2}$ ; and the map  $\widehat{1}$  is the identity.

The category  $\text{BrMod}_{\text{dd},L}$  is equivalent to the category of finite flat group schemes over  $\mathcal{O}_K$  together with a  $k_E$ -action and descent data on the generic fibre from  $K$  to  $L$  (this equivalence depends on  $\pi$ ).

In particular, to each object of  $\text{BrMod}_{\text{dd},L}$  we can associate a  $G_L$ -representation that extends the  $G_K$ -action on the geometric points of the corresponding group scheme. However, we choose in this paper to adopt the conventions of [BM02] and [Sav05], rather than those of [BCDT01]; thus rather than working with the usual contravariant equivalence of categories, we work with a covariant version of it, so that our formulae for generic fibres will differ by duality and a twist from those following the conventions of [BCDT01]. To be precise, we obtain the associated  $G_L$ -representation (which we will refer to as the generic fibre) of an object of  $\text{BrMod}_{\text{dd},L}$  via the functor  $T_{\text{st},2}^L$  defined immediately before [Sav05, Lem. 4.9].

Let  $E$  be a finite extension of  $\mathbb{Q}_p$  with integers  $\mathcal{O}_E$ , maximal ideal  $\mathfrak{m}_E$ , and residue field  $k_E$ . Recall from [Sav05, Sec. 2] that the functor  $D_{\text{st},2}^K$  is an equivalence of categories between the category of  $E$ -representations of  $G_L$  which are semistable when restricted to  $G_K$  and have Hodge-Tate weights in  $\{0, 1\}$ , and the category of weakly admissible filtered  $(\phi, N)$ -modules  $D$  with descent data and  $E$ -coefficients such that  $\text{Fil}^0(K \otimes_{K_0} D) = K \otimes_{K_0} D$  and  $\text{Fil}^2(K \otimes_{K_0} D) = 0$ .

Suppose that  $\rho$  is a representation in the source of  $D_{\text{st},2}^K$ . In what follows we must assume that the reader has some familiarity with the notation and terminology of [Sav05, Sec. 4], but we will endeavour to give precise references to what we use.

Write  $S$  for the ring  $S_{K,\mathcal{O}_E}$  defined in the second paragraph of *loc. cit.* Then [Sav05, Prop. 4.13] gives an essentially surjective functor, again denoted  $T_{\text{st},2}^L$ , from the category [Sav05, Def. 4.1] of *strongly divisible modules  $\mathcal{M}$  with  $\mathcal{O}_E$ -coefficients and descent data* in  $S[1/p] \otimes_{K_0 \otimes E} D_{\text{st},2}^K(\rho)$  to the category of Galois-stable  $\mathcal{O}_E$ -lattices in  $\rho$ . By [Sav05, Cor. 4.12] this functor is compatible with reduction mod  $\mathfrak{m}_E$ , so that applying  $T_{\text{st},2}^L$  to the object  $(k \otimes_{\mathbb{F}_p} k_E)[u]/(u^{e'p}) \otimes_{S/\mathfrak{m}_E S} (\mathcal{M}/\mathfrak{m}_E \mathcal{M})$  of  $\text{BrMod}_{\text{dd},L}$  yields a reduction mod  $p$  of  $\rho$ .

To simplify notation, for the remainder of the paper we write simply  $\mathcal{M}/\mathfrak{m}_E \mathcal{M}$  for the above reduction mod  $\mathfrak{m}_E$  of  $\mathcal{M}$  in  $\text{BrMod}_{\text{dd},L}$  (we will never mean the literal  $S/\mathfrak{m}_E S$ -module). Let  $\ell$  denote the residue field of  $L$ .

**Lemma 5.2.** *Let  $\bar{\chi} : \text{Gal}(K/L) \rightarrow k_E^\times$  be a character, and let  $c$  be an element of  $(\ell \otimes_{\mathbb{F}_p} k_E)^\times$ . Define  $\mathcal{M}(\bar{\chi}, c)$ , a free of rank one Breuil module with  $k_E$ -coefficients and descent data from  $K$  to  $L$ , as follows: let  $\mathcal{M}(\bar{\chi}, c)$  have generator  $v$ , and set*

$$\text{Fil}^1 \mathcal{M}(\bar{\chi}, c) = \mathcal{M}(\bar{\chi}, c), \quad \phi_1(v) = cv, \quad \hat{g}(v) = (1 \otimes \bar{\chi}(g))v$$

for  $g \in \text{Gal}(K/L)$ . Then  $T_{\text{st},2}^L(\mathcal{M}(\bar{\chi}, 1)) = \bar{\chi}$ , and  $T_{\text{st},2}^L(\mathcal{M}(\bar{\chi}, c))$  is an unramified twist of  $\bar{\chi}$ .

*Proof.* Let  $\chi : \text{Gal}(K/L) \rightarrow E^\times$  be the Teichmüller lift of  $\bar{\chi}$ . For  $\tilde{c} \in (W(\ell) \otimes_{\mathbb{Z}_p} \mathcal{O}_E)^\times$  lifting  $c$ , define a weakly admissible filtered  $(\phi, N)$ -module with descent data  $D(\chi, \tilde{c})$  over  $K_0 \otimes_{\mathbb{Q}_p} E$  with generator  $\mathbf{v}$ ,

$$\text{Fil}^i(K \otimes_{K_0} D(\chi, \tilde{c})) = \begin{cases} K \otimes_{K_0} D(\chi, \tilde{c}) & \text{if } i \leq 1 \\ 0 & \text{if } i > 1 \end{cases}$$

and

$$\phi(\mathbf{v}) = p\tilde{c}\mathbf{v}, \quad N = 0, \quad \hat{g} \cdot \mathbf{v} = (1 \otimes \chi(g))\mathbf{v}.$$

Now one checks directly from the definition that there is a strongly divisible  $\mathcal{O}_E$ -module  $\mathcal{M}(\chi, \tilde{c})$  contained in  $S \otimes_{W(k)} D(\chi, \tilde{c})$  generated by  $\mathbf{v}$  and satisfying

$$\text{Fil}^1 \mathcal{M}(\chi, \tilde{c}) = \mathcal{M}(\chi, \tilde{c}), \quad \phi_1(\mathbf{v}) = \tilde{c}\mathbf{v}, \quad \hat{g} \cdot \mathbf{v} = (1 \otimes \chi(g))\mathbf{v}.$$

In particular  $\mathcal{M}(\chi, \tilde{c})/\mathfrak{m}_E \mathcal{M}(\chi, \tilde{c}) = \mathcal{M}(\bar{\chi}, c)$ . One checks (exactly as in [Sav05, Ex. 2.14]) that  $D_{\text{st},2}^K(\chi) = D(\chi, 1)$ , and it is then standard that the representation giving rise to  $D(\chi, \tilde{c})$  is an unramified twist of  $\chi$ . The result now follows from the discussion of the functor  $T_{\text{st},2}^L$  immediately before the statement of the lemma.  $\square$

**5.2. Actual weights are predicted weights.** In this section we make use of the results of [Sav08] to prove results on the possible forms of Galois representations which are modular of a specified weight. Suppose that  $\bar{\rho}$  is modular of weight  $\sigma_{m,n}$ . Then by Lemma 3.4,  $\bar{\rho}|_{G_{F_p}}$  has a parallel potentially semistable lift of type  $\tilde{\omega}^{m+n} \oplus \tilde{\omega}^m$ .

**Lemma 5.3.** *If  $\bar{\rho}$  is modular of weight  $\sigma_{m,n}$ , then  $\det \bar{\rho}|_{I_{F_p}} = \omega^{2m+n+e}$ .*

*Proof.* As remarked above,  $\bar{\rho}|_{G_{F_p}}$  has a parallel potentially semistable lift of type  $\tilde{\omega}^{m+n} \oplus \tilde{\omega}^m$ , say  $\rho$ . It suffices to check that  $\det \rho|_{I_{F_p}} = \varepsilon \tilde{\omega}^{2m+n}$ , where  $\varepsilon$  is the  $p$ -adic cyclotomic character (recalling again that the reduction mod  $p$  of  $\varepsilon$  is  $\omega^e$ ). This follows at once from Definition 3.2 and the results of section B.2 of [CDT99].  $\square$

We begin by addressing the case when  $\bar{\rho}|_{G_{F_p}}$  is reducible.

**Lemma 5.4.** *Suppose that  $\bar{\rho}$  is modular of weight  $\sigma_{m,n}$ , and that  $\bar{\rho}|_{G_{F_p}}$  is reducible.*

*Then  $\bar{\rho}|_{I_{F_p}} \cong \begin{pmatrix} \omega^{m+n+x} & * \\ 0 & \omega^{m+e-x} \end{pmatrix}$  or  $\bar{\rho}|_{I_{F_p}} \cong \begin{pmatrix} \omega^{m+e-x} & * \\ 0 & \omega^{m+n+x} \end{pmatrix}$  for some  $1 \leq x \leq e$ .*

*Proof.* We consider first the case  $n = p - 1$ . By Lemma 3.4,  $\bar{\rho}|_{G_{F_p}}$  has a parallel potentially semistable lift  $\rho$  of type  $\tilde{\omega}^m \oplus \tilde{\omega}^m$ .

If  $\rho$  is not potentially crystalline then it is automatically ordinary in the sense of Definition 3.3. (We briefly recall the argument: twisting by a character, one reduces to the case where  $\rho$  is semistable, i.e., where  $m = 0$ . Then it follows from the relation  $N\varphi = p\varphi N$  that the kernel of  $N$  on the weakly admissible filtered  $(\varphi, N)$ -module associated to  $\rho$  must be a free of rank one submodule with Newton polygon of constant slope  $-1$ , hence itself is weakly admissible. Therefore  $\rho$  has a subcharacter that is an unramified twist of the cyclotomic character.) So the result in this case follows (with  $x = e$ ).

If the lift  $\rho$  is in fact potentially crystalline then without loss of generality we may twist and suppose that  $m = 0$ , and thus that  $\bar{\rho}$  has a crystalline lift of Hodge-Tate weights 0 and 1. Then  $\bar{\rho}|_{G_{F_p}}$  is flat, and by Theorem 3.4.3 of [Ray74] it has a subcharacter of the form  $\omega^i$  for some  $0 \leq i \leq e$ . Since  $m = 0$  and  $n = p - 1$ , this subcharacter is of the form  $\omega^{m+n+x}$  with  $1 \leq x \leq e$  whenever  $i > 0$ , and is of the form  $\omega^{m+e-x}$  whenever  $i < e$ . The result now follows from Lemma 5.3.

For the remainder of the proof suppose that  $n < p - 1$ . Let  $K$  be the splitting field of  $u^{p^2-1} - \pi_p$  over  $F_p$ , and let  $\varpi$  be a choice of  $\pi_p^{1/(p^2-1)}$  in  $K$ . Let  $k_2$  denote the residue field of  $K$ , and if  $g \in \text{Gal}(K/F_p)$  define  $\bar{\eta}(g)$  to be the image of  $g(\varpi)/\varpi$  in  $k_2$ . The field  $F_p$  will play the role of  $L$  in the argument.

Set

$$\tau = \tilde{\omega}_{\sigma_1}^{m+n+1} \tilde{\omega}_{\sigma_2}^{m-1} \oplus \tilde{\omega}_{\sigma_1}^{m-1} \tilde{\omega}_{\sigma_2}^{m+n+1} = \tilde{\omega}_{\sigma_1}^{(n+2)+(p+1)(m-1)} \oplus \tilde{\omega}_{\sigma_1}^{p((n+2)+(p+1)(m-1))}.$$

By Lemma 3.1(4) and the description of the inertial local Langlands correspondence that precedes Lemma 3.4, the weight  $\sigma_{m,n}$  is a Jordan-Hölder factor of  $\sigma(\tau) \otimes_{\mathcal{O}} \mathbb{F}$ . It follows from Lemma 3.4 that  $\bar{\rho}|_{G_{F_p}}$  has a lift to a parallel potentially Barsotti-Tate

representation of type  $\tau$ , and we may suppose this lift to be valued in  $\mathcal{O}_E$  for some finite extension of  $\mathbb{Q}_p$  with residue field  $k_E$  into which  $k_2$  embeds, and with uniformiser  $\pi_E$ . The lift becomes Barsotti-Tate over  $K$ , and the  $\pi_E$ -torsion in the corresponding  $p$ -divisible group gives rise to a finite flat  $k_E$ -module scheme  $\mathcal{G}$  over  $\mathcal{O}_K$  with descent data to  $F_{\mathfrak{p}}$ , with generic fibre  $\bar{\rho}|_{G_{F_{\mathfrak{p}}}}$ , such that the descent data to  $F_{\mathfrak{p}}$  is  $\omega_{\sigma_1}^{m+n+1}\omega_{\sigma_2}^{m-1} \oplus \omega_{\sigma_1}^{m-1}\omega_{\sigma_2}^{m+n+1}$ . We claim that this implies the lemma.

Suppose that  $\bar{\rho}|_{G_{F_{\mathfrak{p}}}} \cong \begin{pmatrix} \psi_1 & * \\ 0 & \psi_2 \end{pmatrix}$ . Then by a scheme-theoretic closure argument [BCDT01, Lem. 4.1.3],  $\mathcal{G}$  must contain a finite flat subscheme  $\mathcal{G}_1$  with descent data which has generic fibre  $\psi_1$ . Twisting by a suitable power of  $\omega$ , we may assume  $m = 0$ . By Theorem 3.5 of [Sav08], we may write the Breuil module  $\mathcal{M}$  corresponding to  $\mathcal{G}_1$  in the form

- $\mathcal{M} = ((k_2 \otimes_{\mathbb{F}_p} k_E)[u]/u^{e(p^2-1)p}) \cdot w$
- $\text{Fil}^1 \mathcal{M} = u^r \mathcal{M}$
- $\phi_1(u^r w) = cw$  for some  $c \in k_E^\times$
- $\hat{g}(w) = (\bar{\eta}(g)^\kappa \otimes 1)w$  for  $g \in \text{Gal}(K/F_{\mathfrak{p}})$ .

Here  $\kappa, r$  are integers with  $\kappa \in [0, p^2 - 1]$  and  $r \in [0, e(p^2 - 1)]$  satisfying  $\kappa \equiv p(\kappa + r) \pmod{p^2 - 1}$  or equivalently  $r \equiv (p - 1)\kappa \pmod{p^2 - 1}$ . Since the descent data on  $\mathcal{G}$  is  $\omega_{\sigma_1}^{n+1-p} \oplus \omega_{\sigma_2}^{n+1-p}$  it follows that  $\kappa$  must be congruent to one of  $n + 1 - p$  or  $p(n + 1 - p) \pmod{p^2 - 1}$ ; this follows also from [GS, Cor 5.2]. Hence  $\kappa = n + p^2 - p$  or  $pn + (p - 1)$ .

In the first case we find  $r \equiv (p - 1)(n + 2) \pmod{p^2 - 1}$ , and therefore  $r = (p - 1)(n + 2) + y(p^2 - 1)$  for some  $0 \leq y < e$ . One checks that there is a map  $f : \mathcal{M}(\omega^{n+y+1}, c) \rightarrow \mathcal{M}$  mapping  $v \mapsto u^{pr/(p-1)}w$ , where  $\mathcal{M}(\omega^{n+y+1}, c)$  is as defined in Lemma 5.2. The kernel of this map does not contain any free  $k_2[u]/u^{e(p^2-1)p}$ -submodules, and so by [Sav04, Prop 8.3] the map  $f$  induces an isomorphism on generic fibres. By Lemma 5.2 we deduce that  $\psi_1|_{I_{F_{\mathfrak{p}}}} = \omega^{n+x}$  with  $x = y + 1 \in [1, e]$ .

In the second case we find  $r \equiv (p - 1)(p - n - 1) \pmod{p^2 - 1}$ , and therefore  $r = (p - 1)(p - n - 1) + y(p^2 - 1)$  for some  $0 \leq y < e$ . One checks that there is a map  $\mathcal{M}(\omega^y, c) \rightarrow \mathcal{M}$  mapping  $v \mapsto u^{pr/(p-1)}w$ . As in the previous case this map induces an isomorphism on generic fibres, and by Lemma 5.2 we deduce that  $\psi_1|_{I_{F_{\mathfrak{p}}}} = \omega^{e-x}$  with  $x = e - y \in [1, e]$ .

Now in either of the two cases the result follows from Lemma 5.3.  $\square$

We now consider the irreducible case.

**Lemma 5.5.** *Suppose that  $\bar{\rho}$  is modular of weight  $\sigma_{m,n}$ , and that  $\bar{\rho}|_{G_{F_{\mathfrak{p}}}}$  is irreducible. Then  $\bar{\rho}|_{I_{F_{\mathfrak{p}}}} \cong \omega_{\sigma_1}^{m+n+x}\omega_{\sigma_2}^{m+e-x} \oplus \omega_{\sigma_1}^{m+e-x}\omega_{\sigma_2}^{m+n+x}$  for some  $1 \leq x \leq e$ .*

*Proof.* This may be proved in essentially the same way as Lemma 5.4. However, the result follows easily from the results of [Sch08a]. Note that  $\sigma_{m,n}$  is a Jordan-Hölder factor of  $\det^m \otimes \text{Ind}_{B(\mathbb{F}_p)}^{\text{GL}_2(\mathbb{F}_p)}(\delta^n)$ . Then in the case  $n \neq 0$ ,  $p - 1$  the result follows at once from (the proof of) Proposition 3.3 of [Sch08a]; while Schein works in the case of an indefinite quaternion algebra, his arguments are ultimately purely local, using Raynaud's classification of finite flat group schemes of type  $(p, \dots, p)$ .

In the case  $n = 0$  or  $p - 1$  a very similar but rather easier analysis applies. In this case, by Lemma 3.4,  $\bar{\rho}|_{G_{F_{\mathfrak{p}}}}$  has a parallel potentially semistable lift of type  $\tilde{\omega}^m \oplus \tilde{\omega}^m$ . If it is not potentially crystalline then it is automatically ordinary, a contradiction. Thus the lift must in fact be potentially crystalline, and after twisting,

one needs only to consider the case  $m = 0$ , and one is reduced to determining the possible generic fibres of finite flat group schemes over  $F_p$ , which is immediate from Raynaud's analysis (Theorem 3.4.3 of [Ray74]), together with Lemma 5.3.  $\square$

Putting Lemma 5.4 and Lemma 5.5 together, we obtain the following.

**Corollary 5.6.** *If  $\bar{\rho}|_{G_{F_p}}$  is semisimple, then  $W(\bar{\rho}) \subset W^?(\bar{\rho})$ .*

If  $e \geq p - 1$  then one can check that  $W^?(\bar{\rho})$  is the whole set of weights  $\sigma_{m,n}$  with  $\det \bar{\rho}|_{I_{F_p}} = \omega^{2m+n+e}$ , so in that case Corollary 5.6 follows from Lemma 5.3 alone.

## 6. LOCAL LIFTS

If  $\bar{\rho}$  is to be modular of weight  $\sigma \in W^?(\bar{\rho})$ , then Lemma 3.4 entails that  $\bar{\rho}|_{G_{F_p}}$  must have a parallel potentially Barsotti-Tate lift of some particular type or types. The existence of certain of these local lifts will be a key ingredient in the proof that  $\bar{\rho}$  is indeed modular of weight  $\sigma$ . Our aim in this section is to produce these lifts.

**6.1. The niveau 1 case.** Let  $K$  be a finite extension of  $\mathbb{Q}_p$ , and let  $S_K$  denote the set of embeddings  $\tau : K \hookrightarrow \overline{\mathbb{Q}_p}$ .

**Definition 6.1.** If  $\rho$  is a  $\overline{\mathbb{Q}_p}$ -valued crystalline representation of  $G_K$  and  $\tau \in S_K$  we say that the Hodge-Tate weights of  $\rho$  with respect to  $\tau$  are the  $i$  for which

$$gr_{\tau}^{-i}(\rho) := gr^{-i}((\rho \otimes_{\mathbb{Q}_p} \mathbf{B}_{\text{dR}})^{G_K} \otimes_{\overline{\mathbb{Q}_p} \otimes_{\mathbb{Q}_p} K, 1 \otimes \tau} \overline{\mathbb{Q}_p}) \neq 0,$$

counted with multiplicity  $\dim gr_{\tau}^{-i}(\rho)$ . We denote the multiset of Hodge-Tate weights of  $\rho$  with respect to  $\tau$  by  $\text{HT}_{\tau}(\rho)$ ; it has cardinality  $\dim \rho$ .

If  $\sigma \in S_K$ , we let  $\bar{\sigma}$  be the induced element of  $S_k$ .

**Lemma 6.2.** *Let  $A = \{a_{\tau}\}_{\tau \in S_K}$  be a set of integers. Then there is a crystalline character  $\varepsilon_A^K$  of  $G_K$  such that  $\text{HT}_{\tau}(\varepsilon_A^K) = a_{\tau}$  for all  $\tau \in S_K$ , and  $\varepsilon_A^K$  is unique up to unramified twist. Furthermore,  $\varepsilon_A^K|_{I_K} = \prod_{\tau \in S_k} \omega_{\tau}^{b_{\tau}}$ , where*

$$b_{\tau} = \sum_{\sigma \in S_K: \bar{\sigma} = \tau} a_{\sigma}.$$

*Proof.* We only sketch a proof; the full details will appear in [GHS]. The existence of  $\varepsilon_A^K$  is easy, as one has only to write down the corresponding weakly admissible filtered module. Uniqueness up to unramified twist is clear, because a crystalline character all of whose Hodge-Tate weights are 0 is automatically unramified. Finally, to compute the reduction modulo  $p$ , it suffices to treat the case where all but one element of  $A$  is 0, and the remaining element is 1, as the general case then follows by taking a product of such characters. In this case (with some effort) one can construct a strongly divisible module corresponding to a lattice in  $\varepsilon_A^K$ , and from the reduction mod  $p$  of that strongly divisible module one determines  $\varepsilon_A^K|_{I_K}$  by applying [Sav08, Cor. 2.7].  $\square$

**Lemma 6.3.** *Suppose that*

$$\bar{\rho}|_{I_{F_p}} \cong \begin{pmatrix} \omega^{m+n+x} & 0 \\ 0 & \omega^{m+e-x} \end{pmatrix}$$

*with  $1 \leq x \leq e$ , and that  $\bar{\rho}|_{G_{F_p}}$  itself is decomposable (which is automatic if  $\omega^{m+n+x} \neq \omega^{m+e-x}$ ). Then  $\bar{\rho}|_{G_{F_p}}$  has a parallel potentially Barsotti-Tate lift  $\rho$*

of type  $\tilde{\omega}^{m+n} \oplus \tilde{\omega}^m$ . If  $x \neq e$  then there is a non-ordinary such lift; if  $x = e$  then there is a non-ordinary such lift provided that  $n + e > p - 1$ , unless  $e \leq p - 1$  and  $n = p - 1$ .

*Proof.* Let  $A = \{a_\tau\}_{\tau \in S_{F_p}}$  have exactly  $x$  elements equal to 1, and the remaining  $e - x$  elements equal to 0. Let  $B = \{1 - a_\tau\}_{\tau \in S_{F_p}}$ . Then by Lemma 6.2 we may take  $\rho$  to be given by an unramified twist of  $\tilde{\omega}^{m+n} \varepsilon_A^{F_p}$  plus an unramified twist of  $\tilde{\omega}^m \varepsilon_B^{F_p}$  (with the unramified twists chosen so that this is indeed a lift of  $\bar{\rho}$ , and also so that  $\rho$  is parallel). This lift is ordinary precisely if one of  $\varepsilon_A^{F_p}$  or  $\varepsilon_B^{F_p}$  is an unramified twist of the cyclotomic character, which occurs if and only if  $x = e$ .

Now suppose that  $x = e$ . If  $e > p - 1$ , then because  $\omega^{p-1} = 1$  we may instead take  $A$  to have exactly  $x - (p - 1)$  elements equal to 1, and the rest equal to 0, and produce a non-ordinary lift with  $B$  and  $\rho$  defined as above. If  $e \leq p - 1$  but  $n + e > p - 1$ , we take  $A$  to have exactly  $n + e - (p - 1)$  elements equal to 1 and the rest equal to zero. Set  $B = \{1 - a_\tau\}_{\tau \in S_{F_p}}$  and take  $\rho$  to be given by an unramified twist of  $\tilde{\omega}^m \varepsilon_A^{F_p}$  plus an unramified twist of  $\tilde{\omega}^{m+n} \varepsilon_B^{F_p}$ . This is non-ordinary provided that  $n \neq p - 1$ .  $\square$

**6.2. The niveau 2 case: some strongly divisible modules.** In the remainder of the section we wish to prove the following.

**Lemma 6.4.** *Suppose that  $\bar{\rho}|_{G_{F_p}}$  is irreducible and  $\bar{\rho}|_{I_{F_p}} \cong \omega_{\sigma_1}^{m+n+x} \omega_{\sigma_2}^{m+e-x} \oplus \omega_{\sigma_1}^{m+e-x} \omega_{\sigma_2}^{m+n+x}$  for some  $1 \leq x \leq e$ . Then  $\bar{\rho}|_{G_{F_p}}$  has a parallel potentially Barsotti-Tate lift of type  $\tilde{\omega}^{m+n} \oplus \tilde{\omega}^m$ .*

We prepare for the proof of Lemma 6.4 by constructing certain strongly divisible modules.

Let  $K$  be a totally ramified finite extension of  $\mathbb{Q}_p$  with ramification index  $e$  and residue field  $k$ , and fix a uniformiser  $\pi$  of  $K$ . Let  $K_2$  be the splitting field of  $u^{p^2-1} - \pi$ , and write  $e_2 = (p^2 - 1)e$  and  $k_2$  respectively for the ramification index and residue field of  $K_2$ . Similarly write  $e_1 = (p - 1)e$ . Choose  $\varpi \in K_2$  a uniformiser with  $\varpi^{p^2-1} = \pi$ . Let  $E(u)$  be an Eisenstein polynomial for  $\varpi$ , and write  $E(u) = u^{e_2} + pF(u)$ , so that  $F(u)$  is a polynomial in  $u^{p^2-1}$  over  $W(k)$  whose constant term is a unit.

If  $g \in G_K$  write  $\tilde{\omega}_2(g) = g\varpi/\varpi \in \mu_{p^2-1}(K_2)$  and write  $\omega_2(g)$  for the image of  $\tilde{\omega}_2(g)$  in  $k_2$ . Then  $\tilde{\omega} = \tilde{\omega}_2^{p+1}|_{I_K}$  and  $\omega = \omega_2^{p+1}|_{I_K}$ , but we will abuse notation and also write  $\tilde{\omega}, \omega$  for  $\tilde{\omega}_2^{p+1}, \omega_2^{p+1}$  respectively. These characters may all equally well be regarded as functions on  $\text{Gal}(K_2/K)$ . Note that if  $\sigma : k_2 \hookrightarrow \overline{\mathbb{F}_p}$  then in the notation of previous sections we have  $\omega_\sigma = \sigma \circ \omega_2|_{I_{K_2}}$ .

Let  $E$  denote the coefficient field for our representations, with integer ring  $\mathcal{O}_E$  and maximal ideal  $\mathfrak{m}_E$ . Assume that  $E$  is ramified over  $\mathbb{Q}_p$  and that  $W(k_2)$  embeds into  $E$ . Let  $k_E$  denote the residue field of  $E$ ; to simplify notation we fix an identification of  $k_E$  with a subfield of  $\overline{\mathbb{F}_p}$ . Write  $S = S_{K_2, \mathcal{O}_E}$  (notation as in [Sav05, Sec. 4]). Recall that  $\phi : S \rightarrow S$  is the  $W(k_2)$ -semilinear,  $\mathcal{O}_E$ -linear map sending  $u \mapsto u^p$ . The group  $\text{Gal}(K_2/K)$  acts  $W(k_2)$ -semilinearly on  $S$  via  $g \cdot u = (\tilde{\omega}_2(g) \otimes 1)u$ . Set  $c = \frac{1}{p}\phi(E(u)) \in S^\times$ .

**Theorem 6.5.** *Let  $0 \leq j \leq e_1$  be an integer and set  $J = (p + 1)j$ . There exists a strongly divisible  $\mathcal{O}_E$ -module  $\mathcal{M} = \mathcal{M}_j$  with tame descent data from  $K_2$  to  $K$  and*



generators  $g_1, g_2$  such that  $\overline{\mathcal{M}} = \mathcal{M}/\mathfrak{m}_E\mathcal{M}$  has the form

$$\begin{aligned} \mathrm{Fil}^1\mathcal{M} &= \langle u^J\overline{g}_1, u^{e_2-J}\overline{g}_2 \rangle, \\ \phi_1(u^J\overline{g}_1) &= \overline{g}_2, \quad \phi_1(u^{e_2-J}\overline{g}_2) = \overline{g}_1, \\ \widehat{g}(\overline{g}_1) &= \overline{g}_1, \quad \widehat{g}(\overline{g}_2) = \omega^n(g)\overline{g}_2 \end{aligned}$$

for  $g \in \mathrm{Gal}(K_2/K)$ , where  $n$  is the least nonnegative residue of  $j$  modulo  $p-1$ .

*Proof.* Let  $\mathcal{M}$  be the free  $S$ -module generated by  $g_1, g_2$ . To give  $\mathcal{M}$  the structure of a strongly divisible module with descent data and coefficients, we must specify  $\mathrm{Fil}^1\mathcal{M}$ ,  $\phi$ ,  $N$ , and descent data and check that these satisfy the conditions (1)-(12) of [Sav05, Def. 4.1]. This is what we now do.

Choose any  $x_1, x_2 \in \mathfrak{m}_E$  with  $x_1x_2 = p$  (this is where we use the hypothesis that  $E/\mathbb{Q}_p$  is ramified) and let  $\mathrm{Fil}^1\mathcal{M}$  be the submodule of  $\mathcal{M}$  generated by

$$h_1 := u^Jg_1 + x_1g_2, \quad h_2 := -x_2F(u)g_1 + u^{e_2-J}g_2, \quad (\mathrm{Fil}^1S)\mathcal{M}.$$

We would like to define a map  $\phi : \mathcal{M} \rightarrow \mathcal{M}$ , semilinear with respect to  $\phi$  on  $S$ , so that  $\phi_1 = \frac{1}{p}\phi|_{\mathrm{Fil}^1\mathcal{M}}$  is well-defined and satisfies

$$(6.2.1) \quad \phi_1(u^Jg_1 + x_1g_2) = g_2$$

$$(6.2.2) \quad \phi_1(-x_2F(u)g_1 + u^{e_2-J}g_2) = g_1.$$

This entails  $\phi_1(E(u)g_1) = \phi_1(u^{e_2-J}h_1 - x_1h_2) = u^{p(e_2-J)}g_2 - x_1g_1$ , suggesting that we should define

$$\phi(g_1) = c^{-1}(u^{p(e_2-J)}g_2 - x_1g_1)$$

and similarly

$$\phi(g_2) = c^{-1}(x_2\phi(F(u))g_2 + u^{pJ}g_1).$$

Extending this map  $\phi$ -semilinearly to all of  $\mathcal{M}$ , one checks that equations (6.2.1) and (6.2.2) hold, so that  $\phi(\mathrm{Fil}^1\mathcal{M})$  is contained in  $p\mathcal{M}$  and generates it over  $S$ .

We will now check that this choice of  $\mathrm{Fil}^1\mathcal{M}$  satisfies condition (2) of [Sav05, Def. 4.1]. Recall from the discussion preceding [Sav05, Def. 4.1] that each element of  $S$  can be written uniquely as  $\sum_{i \geq 0} r_i(u)E(u)^i/i!$  where each  $r_i$  is a polynomial of degree less than  $e_2$ , and such an element lies in  $\mathrm{Fil}^1S$  if and only if  $r_0 = 0$ . We claim that each coset in  $\mathrm{Fil}^1\mathcal{M}/(\mathrm{Fil}^1S)\mathcal{M}$  has a representative  $ah_1 + bh_2$  with  $a, b$  polynomials of degree less than  $e_2 - J$  and  $J$  respectively. Indeed, given a coset  $Ah_1 + Bh_2 + (\mathrm{Fil}^1S)\mathcal{M}$  with  $A, B \in S$ , we can alter the coset representative as follows: write  $B$  as the sum of a polynomial of degree less than  $e_2$  and an element of  $\mathrm{Fil}^1S$ , and absorb  $h_2$  times the latter into  $(\mathrm{Fil}^1S)\mathcal{M}$ ; use the relation  $u^Jh_2 = E(u)g_2 - x_2F(u)h_1$  to eliminate the terms in  $B$  of degree at least  $J$  (thus altering the coefficient of  $h_1$ ); write the new coefficient of  $h_1$  as the sum of a polynomial of degree less than  $e_2$  and an element of  $\mathrm{Fil}^1S$ , and absorb the latter into  $(\mathrm{Fil}^1S)\mathcal{M}$ ; finally, use the relation  $u^{e_2-J}h_1 = E(u)g_1 + x_1h_2$  to eliminate the terms of degree at least  $e_2 - J$  in the coefficient of  $h_1$ , noting that in this last step one does not re-introduce terms of degree at least  $u^J$  into the coefficient of  $h_2$ .

If  $I$  is any ideal of  $\mathcal{O}_E$ , we wish to see that  $\mathrm{Fil}^1\mathcal{M} \cap I\mathcal{M} = I\mathrm{Fil}^1\mathcal{M}$ . We have seen that an arbitrary element  $m$  of  $\mathrm{Fil}^1\mathcal{M}$  has the form  $m = ah_1 + bh_2 + s_1g_1 + s_2g_2$  with  $s_1, s_2 \in \mathrm{Fil}^1S$  and  $a, b$  polynomials of degree less than  $e_2 - J$  and  $J$  respectively. Suppose such an element lies in  $I\mathcal{M}$ . The coefficient of  $g_2$  for this element is  $x_1a + u^{e_2-J}b + s_2$ . This must lie in  $IS$ ; but because an element  $\sum_i r_i(u)E(u)^i/i!$  (with  $\deg(r_i) < e_2$  for all  $i$ ) lies in  $IS$  if and only if all the coefficients of the

polynomials  $r_i$  lie in  $W(k_2) \otimes I$ , and because  $x_1 a$ ,  $u^{e_2 - J} b$  have no terms in common of the same degree, it follows that  $s_2 \in I(\text{Fil}^1 S)$  and the coefficients of  $b$  lie in  $W(k_2) \otimes I$ . Then  $ah_1 + s_1 g_1$  still lies in  $I\mathcal{M}$ , and now we can see that  $s_1 \in I(\text{Fil}^1 S)$  and the coefficients of  $a$  lie in  $W(k_2) \otimes I$ . We conclude that  $m \in I(\text{Fil}^1 \mathcal{M})$ , as desired.

Next we turn to descent data. If  $g \in \text{Gal}(K_1/K)$ , set  $\widehat{g}(g_1) = g_1$  and  $\widehat{g}(g_2) = \tilde{\omega}^n(g)g_2$ , and extend  $\widehat{g}$  to  $\mathcal{M}$  semilinearly with respect to the usual action of  $g$  on  $S$ . One sees that  $\widehat{g}$  preserves  $\text{Fil}^1 \mathcal{M}$  (remember that  $F(u)$  is a polynomial in  $u^{p^2-1}$  over  $W(k)$ ) and commutes with  $\phi$ . So, summarizing all our work so far, we have shown that the tuple  $(\mathcal{M}, \text{Fil}^1 \mathcal{M}, \phi, \{\widehat{g}\})$  satisfies all the axioms of a strongly divisible  $\mathcal{O}_E$ -module with tame descent data [Sav05, Def. 4.1] other than the axioms involving the monodromy operator  $N$ .

Ignoring the action of  $\mathcal{O}_E$  and the descent data and regarding  $(\mathcal{M}, \text{Fil}^1 \mathcal{M}, \phi)$  simply as a strongly divisible  $\mathbb{Z}_p$ -module over  $K_2$ , it follows from [Bre00, Prop 5.1.3(1)] that there exists a *unique*  $W(k_2) \otimes \mathbb{Z}_p$ -endomorphism  $N : \mathcal{M} \rightarrow \mathcal{M}$  satisfying axioms (5)–(8) of [Sav05, Def. 4.1], except that we have axiom (5) only with respect to  $s \in S_{K_2, \mathbb{Z}_p}$  until we know that  $N$  commutes with the action of  $\mathcal{O}_E$ . For the latter, if  $z \in \mathcal{O}_E^\times$  we observe that  $zNz^{-1}$  satisfies the same list of axioms that determines  $N$  uniquely, so  $z$  and  $N$  commute; since  $\mathcal{O}_E^\times$  generates  $\mathcal{O}_E$  as a  $\mathbb{Z}_p$ -module we conclude that  $N$  is an  $\mathcal{O}_E$ -endomorphism. To conclude that  $\mathcal{M}$  (with its associated structures) is a strongly divisible  $\mathcal{O}_E$ -module, the only thing left is to confirm the remainder of axiom (12), that  $N$  commutes with  $\widehat{g}$  for each  $g \in \text{Gal}(K_2/K)$ ; for this, use the same argument as in the previous sentence, this time applied to  $\widehat{g}N\widehat{g}^{-1}$ .

That  $\overline{\mathcal{M}} = \mathcal{M}/\mathfrak{m}_E \mathcal{M}$  has the desired form is obvious.  $\square$

Define  $\rho_j = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} T_{\text{st}, 2}^K(\mathcal{M}_j)$ , the potentially Barsotti-Tate Galois representation associated to  $\mathcal{M}_j$ .

**Proposition 6.6.** *The representation  $\rho_j$  has inertial type  $\tilde{\omega}^n \oplus 1$ , and  $\det(\rho_j)$  is the product of the cyclotomic character, a finite order character of order prime to  $p$ , and an unramified character.*

*Proof.* Let  $D$  denote the filtered module with descent data associated to  $\rho_j$ . We recall from the proof of [Sav05, Lem. 3.13] that  $D$  is equal to the kernel of  $N$  on  $\mathcal{M}_j[1/p]$ .

Write  $M = (W(k_2)[1/p] \otimes_{\mathbb{Q}_p} E) \otimes_{S[1/p]} \mathcal{M}_j[1/p]$ , the tensor product taken with respect to the map  $S[1/p] \rightarrow W(k_2)[1/p] \otimes_{\mathbb{Q}_p} E$  sending  $u$  to 0, and equip  $M$  with the maps  $N$  and  $\phi$  induced from  $\mathcal{M}_j$  (so in particular  $N = 0$  on  $M$ ), as well as induced descent data  $\widehat{g}$ . By [Bre97, Prop. 6.2.1.1] the canonical map  $\mathcal{M}_j[1/p] \rightarrow M$  has a unique  $W(k_2)[1/p]$ -linear section  $s : M \rightarrow \mathcal{M}_j[1/p]$  preserving  $\phi$  and  $N$ ; then the same uniqueness argument as in the last paragraph of the proof of Theorem 6.5 shows that  $s$  is an  $E$ -linear map and that  $s$  preserves descent data.

Recalling that  $N = 0$  on  $M$ , we see that  $D = \text{im}(s)$ , so in particular  $D$  has a  $W(k_2)[1/p] \otimes_{\mathbb{Q}_p} E$ -basis  $v_1, v_2$  with  $v_i = s(g_i)$ . Since  $s$  preserves descent data we have  $\widehat{g} \cdot v_1 = v_1$  and  $\widehat{g} \cdot v_2 = \tilde{\omega}^n(g)v_2$ . The first part of the proposition follows.

For the second part, if we regard  $K_2 \otimes_{\mathbb{Q}_p} E$  as an  $S[1/p]$ -algebra via the map  $u \mapsto \varpi$  then by [Bre97, Prop. 6.2.2.1] there is an isomorphism

$$f_\varpi : (K_2 \otimes_{\mathbb{Q}_p} E) \otimes_{S[1/p]} \mathcal{M}_j[1/p] \cong D_{K_2} := K_2 \otimes_{W(k_2)[1/p]} D$$

that identifies the filtrations on both sides. It follows that  $D_{K_2}$  is the free  $(K_2 \otimes_{\mathbb{Q}_p} E)$ -module generated by  $f_{\varpi}(g_1)$  and  $f_{\varpi}(g_2)$ , and that  $\text{Fil}^1 D_{K_2}$  is generated by  $f_{\varpi}(h_1) = \varpi^j f_{\varpi}(g_1) + x_1 f_{\varpi}(g_2)$ . (One checks that  $f_{\varpi}(h_1)$  and  $f_{\varpi}(h_2)$  are scalar multiples of one another in  $D_{K_2}$ .) In particular  $\text{Fil}^1 D_{K_2}$  is a free submodule of rank one in  $D_{K_2}$ , and so all pairs of labeled Hodge-Tate weights of  $\rho_j$  are  $\{0, 1\}$ . The claim follows from the note after Definition 3.2, together with the fact that  $K_2/K$  is tamely ramified (so that the finite order character has order prime to  $p$ ).  $\square$

**6.3. The niveau 2 case: conclusion of the proof.** We will now compute  $\bar{\rho}_j = T_{\text{st},2}^K(\mathcal{M}_j/\mathfrak{m}_E \mathcal{M}_j)$ , which by [Sav05, Prop 4.13] is the reduction mod  $p$  of  $\rho_j$ . More precisely we will compute  $\bar{\rho}_j|_{G_L}$  where  $L$  is the unramified quadratic extension of  $K$  contained in  $K_2$ .

As in Lemma 5.2 let  $\bar{\chi} : \text{Gal}(K_2/L) \rightarrow k_E^\times$  be a character and let  $\mathcal{M}(\bar{\chi})$  denote the rank one Breuil module with  $k_E$ -coefficients and descent data from  $K_2$  to  $L$  with generator  $v$  and

$$\text{Fil}^1 \mathcal{M}(\bar{\chi}) = \mathcal{M}(\bar{\chi}), \quad \phi_1(v) = v, \quad \widehat{g}(v) = (1 \otimes \bar{\chi}(g))v$$

for  $g \in \text{Gal}(K_2/L)$ . By Lemma 5.2 we have

$$(6.3.1) \quad T_{\text{st},2}^L(\mathcal{M}(\bar{\chi})) = \bar{\chi}.$$

Let  $\overline{\mathcal{M}}_j^2$  denote the Breuil module  $\overline{\mathcal{M}}_j = \mathcal{M}_j/\mathfrak{m}_E \mathcal{M}_j$  with its descent data restricted to  $\text{Gal}(K_2/L)$ , so that  $T_{\text{st},2}^L(\overline{\mathcal{M}}_j^2) = T_{\text{st},2}^K(\overline{\mathcal{M}}_j)|_{G_L}$ . We abuse notation and also let  $\omega_\sigma$  denote  $\sigma \circ \omega_2|_{G_L}$ .

**Proposition 6.7.** *We have  $\bar{\rho}_j|_{G_L} = T_{\text{st},2}^L(\overline{\mathcal{M}}_j^2) \cong \omega_{\sigma_1}^{j+e} \oplus \omega_{\sigma_2}^{j+e}$ .*

*Proof.* Let  $e_{\sigma_1}, e_{\sigma_2} \in k_2 \otimes k_E$  denote the idempotents corresponding to the embeddings  $\sigma_1, \sigma_2 : k_2 \hookrightarrow k_E$ , so that  $e_{\sigma_i}(a \otimes 1) = e_{\sigma_i}(1 \otimes \sigma_i(a))$ . Suppose  $\{\alpha, \beta\} = \{1, 2\}$ . If one ignores descent data, one checks that there is a map  $f_\beta : \mathcal{M}(\bar{\chi}) \rightarrow \overline{\mathcal{M}}_j^2$  obtained by sending

$$v \mapsto u^{p(j+e)} e_{\sigma_\alpha} \bar{g}_1 + u^{p(pe-j)} e_{\sigma_\beta} \bar{g}_2.$$

In order that this map be compatible with descent data, one checks that it is necessary and sufficient that  $e_{\sigma_\alpha}(1 \otimes \bar{\chi}) = e_{\sigma_\alpha}(\omega_2^{p(j+e)} \otimes 1)$ , i.e.,  $\bar{\chi} = \omega_{\sigma_\alpha}^{p(j+e)} = \omega_{\sigma_\beta}^{j+e}$ . We therefore have a map

$$f_1 \oplus f_2 : \mathcal{M}(\omega_{\sigma_1}^{j+e}) \oplus \mathcal{M}(\omega_{\sigma_2}^{j+e}) \rightarrow \overline{\mathcal{M}}_j^2.$$

Note that  $\ker(f_1 \oplus f_2)$  does not contain any free  $k_2[u]/u^{e_2 p}$ -submodules (this amounts to the fact that  $p(j+e)$  and  $p(pe-j)$  are both smaller than  $pe_2$ ); by [Sav04, Prop 8.3] we deduce that  $f_1 \oplus f_2$  induces an isomorphism on generic fibres, and the proposition follows from (6.3.1).  $\square$

Finally we have the following.

*Proof of Lemma 6.4.* Take  $K = F_{\mathfrak{p}}$  in the discussion of this and the previous subsection. Twisting by a suitable power of the (Teichmüller lift of) a fundamental character of level one, we may assume  $m = 0$ . Setting  $j = n + (p-1)(e-x)$ , one checks that  $\omega_{\sigma_1}^{j+e} = \omega_{\sigma_1}^{n+x} \omega_{\sigma_2}^{e-x}$  and similarly for  $\omega_{\sigma_2}^{j+e}$ . It follows from Proposition 6.7 that  $\bar{\rho}$  is an unramified twist of  $\bar{\rho}_j$ . Note that a twist of  $\rho_j$  by a suitable unramified character will be parallel; twisting this by a suitable finite-order unramified

character of order prime to  $p$  gives a parallel potentially Barsotti-Tate lift of  $\bar{\rho}$  that is an unramified twist of  $\rho_j$ . Since  $\rho_j$  has type  $\tilde{\omega}^n \oplus 1$  we are done.  $\square$

*Remark 6.8.* The reader may find it unnatural that although  $\rho_j$  becomes Barsotti-Tate over  $K_1 = K(\pi^{1/(p-1)})$ , we instead work with a strongly divisible module over  $K_2$  for  $\rho_j$  (because our method for computing  $\bar{\rho}_j|_{G_L}$  requires it). One can certainly write down the strongly divisible module over  $K_1$  instead (just replace  $J$  and  $e_2$  with  $j$  and  $e_1$  throughout the construction of  $\mathcal{M}_j$ ), whose reduction mod  $p$  corresponds to a group scheme  $\mathcal{G}$  over  $\mathcal{O}_{K_1}$  with generic fibre descent data from  $K_1$  to  $K$ . One can then hope to show directly, by extending the methods of [BCDT01, Sec. 5.4], that  $\mathcal{G} \times_{\mathcal{O}_{K_1}} \mathcal{O}_{K_2}$  (with generic fibre descent data from  $K_2$  to  $K$ ) corresponds to our  $\bar{\mathcal{M}}_j$ . However, this last step would require at least several extra pages of rather technical work, so we prefer to proceed as above instead.

## 7. THE MAIN THEOREMS

Recall that we are assuming that  $F$  is a totally real field in which the prime  $p$  is totally ramified. We now prove the main results of this paper, by combining the techniques of earlier sections with the lifting machinery of Khare-Wintenberger, as interpreted by Kisin. In particular, we use the following result.

**Theorem 7.1.** *Suppose that  $p > 2$  and that  $\bar{\rho} : G_F \rightarrow \mathrm{GL}_2(\bar{\mathbb{F}}_p)$  is modular. Assume that  $\bar{\rho}|_{G_{F(\zeta_p)}}$  is irreducible. If  $p = 5$  and the projective image of  $\bar{\rho}$  is isomorphic to  $\mathrm{PGL}_2(\mathbb{F}_5)$ , assume further that  $[F(\zeta_p) : F] = 4$ .*

- *Suppose that  $\bar{\rho}|_{G_{F_p}}$  has a non-ordinary parallel potentially Barsotti-Tate lift of type  $\tilde{\omega}^{m+n} \oplus \tilde{\omega}^m$ . Then  $\bar{\rho}$  has a modular lift which is parallel potentially Barsotti-Tate of type  $\tilde{\omega}^{m+n} \oplus \tilde{\omega}^m$  and non-ordinary.*
- *Suppose that  $\bar{\rho}$  has an ordinary modular lift. Suppose also that  $\bar{\rho}|_{G_{F_p}}$  has an ordinary parallel potentially Barsotti-Tate lift of type  $\tilde{\omega}^{m+n} \oplus \tilde{\omega}^m$ . Then  $\bar{\rho}$  has a modular lift which is parallel potentially Barsotti-Tate of type  $\tilde{\omega}^{m+n} \oplus \tilde{\omega}^m$  and ordinary.*

*Proof.* This is a special case of Corollary 3.1.7 of [Gee].  $\square$

In combination with the local computations of section 6, this shows us that in most cases if  $\bar{\rho}|_{G_{F_p}}$  is semisimple, and  $\sigma_{m,n} \in W^?(\bar{\rho})$ , then  $\bar{\rho}$  has a modular lift which is parallel potentially Barsotti-Tate of type  $\tilde{\omega}^{m+n} \oplus \tilde{\omega}^n$  (the exceptional cases being those where the only local lifts of this type are ordinary, and  $\bar{\rho}$  is not known to have an ordinary lift). By Lemma 3.4, this means that  $\bar{\rho}$  is modular of weight  $\sigma_{m,n}$  or  $\sigma_{m+n,p-1-n}$ . However, we can frequently guarantee that this lift is non-ordinary, and the weight cycling techniques of section 4 then give the following far more useful result.

**Theorem 7.2.** *Suppose that  $p > 2$  and that  $\bar{\rho} : G_F \rightarrow \mathrm{GL}_2(\bar{\mathbb{F}}_p)$  is modular. Assume that  $\bar{\rho}|_{G_{F(\zeta_p)}}$  is irreducible. If  $p = 5$  and the projective image of  $\bar{\rho}$  is isomorphic to  $\mathrm{PGL}_2(\mathbb{F}_5)$ , assume further that  $[F(\zeta_p) : F] = 4$ . If  $\bar{\rho}|_{G_{F_p}}$  has a non-ordinary parallel potentially Barsotti-Tate lift of type  $\tilde{\omega}^{m+n} \oplus \tilde{\omega}^m$ ,  $0 \leq n \leq p-1$ , then  $\bar{\rho}$  is modular both of weight  $\sigma_{m,n}$  and of weight  $\sigma_{m+n,p-1-n}$ .*

*Proof.* By Theorem 7.1, there is a modular lift of  $\bar{\rho}$  which is parallel potentially Barsotti-Tate of type  $\tilde{\omega}^{m+n} \oplus \tilde{\omega}^m$ , and which is non-ordinary. The result follows from Proposition 4.4.  $\square$

We now extract some consequences from this result. Suppose that  $\bar{\rho}|_{G_{F_p}}$  is semisimple. Then we have already proved that  $W(\bar{\rho})$ , the set of weights  $\sigma$  for which  $\bar{\rho}$  is modular of weight  $\sigma$ , is contained in  $W^?(\bar{\rho})$  (this is Corollary 5.6). We can frequently deduce the converse implication, showing that if  $\sigma_{m,n} \in W^?(\bar{\rho})$  then  $\sigma_{m,n} \in W(\bar{\rho})$ . In particular we have the following.

**Corollary 7.3.** *Suppose that  $p > 2$  and that  $\bar{\rho} : G_F \rightarrow \mathrm{GL}_2(\overline{\mathbb{F}}_p)$  is modular. Assume that  $\bar{\rho}|_{G_{F(\zeta_p)}}$  is irreducible. If  $p = 5$  and the projective image of  $\bar{\rho}$  is isomorphic to  $\mathrm{PGL}_2(\mathbb{F}_5)$ , assume further that  $[F(\zeta_p) : F] = 4$ . Suppose that  $\bar{\rho}|_{G_{F_p}}$  is semisimple.*

- (1) *If  $\bar{\rho}|_{G_{F_p}}$  is irreducible or  $e \geq p$ , then  $\bar{\rho}$  is modular of weight  $\sigma$  if and only if  $\sigma \in W^?(\bar{\rho})$ .*
- (2) *If  $e \leq p - 1$ , then  $\bar{\rho}$  is modular of weight  $\sigma$  if and only if  $\sigma \in W^?(\bar{\rho})$  except possibly if  $\sigma = \sigma_{m,n}$  and*

$$\bar{\rho}|_{I_{F_p}} \cong \begin{pmatrix} \omega^{m+n+e} & 0 \\ 0 & \omega^m \end{pmatrix}$$

*with  $n + e \leq p - 1$  or  $n = p - 1$ .*

*Proof.* Write  $\sigma = \sigma_{m,n}$ . As already remarked, the ‘‘only if’’ direction is Corollary 5.6. For the ‘‘if’’ direction, by Theorem 7.2 it suffices to be able to produce a non-ordinary parallel potentially Barsotti-Tate lift of  $\bar{\rho}|_{G_{F_p}}$  of type  $\tilde{\omega}^{m+n} \oplus \tilde{\omega}^m$ . Since we assume  $\sigma_{m,n} \in W^?(\bar{\rho})$ , the representation  $\bar{\rho}|_{I_{F_p}}$  is as in Definition 5.1. Then existence of the desired lift follows from Lemma 6.3 when  $\bar{\rho}|_{G_{F_p}}$  is reducible, and from Lemma 6.4 when  $\bar{\rho}|_{G_{F_p}}$  is irreducible. Note that in part (2), the exceptional cases are precisely the ones where none of the lifts provided by Lemma 6.3 are non-ordinary.  $\square$

Note that there are at most four exceptional cases in part (2) of Corollary 7.3: there are two ways of ordering the diagonal characters, and each ordering will correspond either to one or two values of  $n$  (if  $n \not\equiv 0$  or  $n \equiv 0 \pmod{p-1}$  respectively).

In fact, if we assume in addition that  $\bar{\rho}$  has an ordinary modular lift, then we are able to dispose of most of these exceptional cases. This relies on something of a combinatorial coincidence; it turns out that in most cases where  $\sigma_{m,n} \in W^?(\bar{\rho})$  but  $\bar{\rho}|_{G_{F_p}}$  has only ordinary lifts of type  $\tilde{\omega}^{m+n} \oplus \tilde{\omega}^m$ , then  $\sigma_{m+n,p-1-n} \notin W^?(\bar{\rho})$ , so the combination of Theorem 7.1, Lemma 3.4 and Corollary 5.6 shows that in fact  $\sigma_{m,n} \in W(\bar{\rho})$ .

**Corollary 7.4.** *Suppose that  $p > 2$  and that  $\bar{\rho} : G_F \rightarrow \mathrm{GL}_2(\overline{\mathbb{F}}_p)$  is modular. Assume that  $\bar{\rho}|_{G_{F(\zeta_p)}}$  is irreducible. Suppose that  $e \leq p - 1$  and that  $\bar{\rho}|_{G_{F_p}}$  is semisimple and reducible. Suppose further that  $\bar{\rho}$  has an ordinary modular lift. If  $p = 5$  and the projective image of  $\bar{\rho}$  is isomorphic to  $\mathrm{PGL}_2(\mathbb{F}_5)$ , assume further that  $[F(\zeta_p) : F] = 4$ . Suppose that  $\sigma_{m,n} \in W^?(\bar{\rho})$ . If  $n = p - 1$ , suppose that*

$$\bar{\rho}|_{I_{F_p}} \not\cong \begin{pmatrix} \omega^{m+e} & 0 \\ 0 & \omega^m \end{pmatrix}.$$

*Then  $\bar{\rho}$  is modular of weight  $\sigma_{m,n}$ .*

*Proof.* If  $\bar{\rho}|_{G_{F_p}}$  has a non-ordinary parallel potentially Barsotti-Tate lift of type  $\tilde{\omega}^{m+n} \oplus \tilde{\omega}^m$ , then by Corollary 7.3,  $\bar{\rho}$  is modular of weight  $\sigma_{m,n}$ .

Suppose now that  $\bar{\rho}|_{G_{F_p}}$  does not have a non-ordinary parallel potentially Barsotti-Tate lift of type  $\tilde{\omega}^{m+n} \oplus \tilde{\omega}^m$ . By Lemma 6.3, we must have

$$\bar{\rho}|_{I_{F_p}} \cong \begin{pmatrix} \omega^{m+n+e} & 0 \\ 0 & \omega^m \end{pmatrix}.$$

Furthermore, either  $n + e \leq p - 1$ , or  $n = p - 1$ . The second case is precisely the case excluded by the statement of this corollary.

Thus we must have  $n + e \leq p - 1$ . By Lemma 6.3,  $\bar{\rho}|_{G_{F_p}}$  has a parallel potentially Barsotti-Tate lift of type  $\tilde{\omega}^{m+n} \oplus \tilde{\omega}^m$ , so that by the assumption that  $\bar{\rho}$  has an ordinary modular lift, Theorem 7.1, and Lemma 3.4,  $\bar{\rho}$  is modular of weight  $\sigma_{m,n}$  or  $\sigma_{m+n,p-1-n}$ . If  $n = 0$  then we may conclude further that  $\bar{\rho}$  is modular of weight  $\sigma_{m,0}$ . Assume for the sake of contradiction that  $\bar{\rho}$  is not modular of weight  $\sigma_{m,n}$ , so that we may assume that  $n \neq 0$ ,  $n + e \leq p - 1$ , and  $\sigma_{m+n,p-1-n} \in W(\bar{\rho})$ . In particular, by Corollary 5.6 we have  $\sigma_{m+n,p-1-n} \in W^?(\bar{\rho})$ , and we also know that  $e < p - 1$  (because  $e \leq p - 1 - n < p - 1$ ). Now, examining the definition of  $W^?(\bar{\rho})$  (Definition 5.1), we see that we must have

$$\bar{\rho}|_{I_{F_p}} \cong \begin{pmatrix} \omega^{m+x} & 0 \\ 0 & \omega^{m+n+e-x} \end{pmatrix}$$

for some  $1 \leq x \leq e$ . Comparing with the expression above, we see that either  $\omega^{m+x} = \omega^m$  or  $\omega^{m+x} = \omega^{m+n+e}$ . The first possibility requires  $x \equiv 0 \pmod{p-1}$ , a contradiction as  $1 \leq x \leq e < p - 1$ . The second requires  $x \equiv n + e \pmod{p-1}$ , which is a contradiction because  $1 \leq x \leq e < n + e \leq p - 1$ . The result follows.  $\square$

## REFERENCES

- [AS86] Avner Ash and Glenn Stevens, *Cohomology of arithmetic groups and congruences between systems of Hecke eigenvalues*, J. Reine Angew. Math. **365** (1986), 192–220.
- [BCDT01] Christophe Breuil, Brian Conrad, Fred Diamond, and Richard Taylor, *On the modularity of elliptic curves over  $\mathbf{Q}$ : wild 3-adic exercises*, J. Amer. Math. Soc. **14** (2001), no. 4, 843–939 (electronic).
- [BDJ08] Kevin Buzzard, Fred Diamond, and Frazer Jarvis, *On Serre’s conjecture for mod  $l$  Galois representations over totally real fields*, to appear, Duke Math. J., 2008.
- [BM02] Christophe Breuil and Ariane Mézard, *Multiplicités modulaires et représentations de  $GL_2(\mathbf{Z}_p)$  et de  $\text{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$  en  $l = p$* , Duke Math. J. **115** (2002), no. 2, 205–310, with an appendix by Guy Henniart.
- [Bre97] Christophe Breuil, *Représentations  $p$ -adiques semi-stables et transversalité de Griffiths*, Math. Annalen **307** (1997), 191–224.
- [Bre00] ———, *Groupes  $p$ -divisibles, groupes finis et modules filtrés*, Ann. of Math. (2) **152** (2000), no. 2, 489–549.
- [CDT99] Brian Conrad, Fred Diamond, and Richard Taylor, *Modularity of certain potentially Barsotti-Tate Galois representations*, J. Amer. Math. Soc. **12** (1999), no. 2, 521–567.
- [Edi92] Bas Edixhoven, *The weight in Serre’s conjectures on modular forms*, Invent. Math. **109** (1992), no. 3, 563–594. MR MR1176206 (93h:11124)
- [Geea] Toby Gee, *Automorphic lifts of prescribed types*, to appear, Math. Annalen.
- [Geeb] ———, *On the weights of mod  $p$  Hilbert modular forms*, to appear, Invent. Math.
- [Ger09] David Geraghty, *Modularity lifting theorems for ordinary Galois representations*, preprint, 2009.
- [GHS] Toby Gee, Florian Herzig, and David Savitt, *General Serre weight conjectures*, in preparation.
- [GS] Toby Gee and David Savitt, *Serre weights for quaternion algebras*, to appear, Compositio Mathematica.
- [Kis08] Mark Kisin, *Potentially semi-stable deformation rings*, J. Amer. Math. Soc. **21** (2008), no. 2, 513–546.

- [Kis09a] ———, *Modularity of 2-adic Barsotti-Tate representations*, Invent. Math. **178** (2009), no. 3, 587–634.
- [Kis09b] ———, *Moduli of finite flat group schemes, and modularity*, Ann. of Math. (2) **170** (2009), no. 3, 1085–1180.
- [Ray74] Michel Raynaud, *Schémas en groupes de type  $(p, \dots, p)$* , Bull. Soc. Math. France **102** (1974), 241–280. MR MR0419467 (54 #7488)
- [Sav04] David Savitt, *Modularity of some potentially Barsotti-Tate Galois representations*, Compos. Math. **140** (2004), no. 1, 31–63.
- [Sav05] ———, *On a conjecture of Conrad, Diamond, and Taylor*, Duke Math. J. **128** (2005), no. 1, 141–197.
- [Sav08] ———, *Breuil modules for Raynaud schemes*, Journal of Number Theory **128** (2008), 2939–2950.
- [Sch08a] Michael Schein, *Weights in Serre’s conjecture for Hilbert modular forms: the ramified case*, Israel J. Math. **166** (2008), 369–391.
- [Sch08b] Michael M. Schein, *On modular weights of Galois representations*, Math. Res. Lett. **15** (2008), no. 3, 537–542. MR MR2407230
- [Ser87] Jean-Pierre Serre, *Sur les représentations modulaires de degré 2 de  $\text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$* , Duke Math. J. **54** (1987), no. 1, 179–230.
- [Tay06] Richard Taylor, *On the meromorphic continuation of degree two  $L$ -functions*, Doc. Math. (2006), no. Extra Vol., 729–779 (electronic). MR MR2290604 (2008c:11154)

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