POLYNOMIALS, MEANDERS, AND PATHS IN THE LATTICE OF NONCROSSING PARTITIONS

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Abstract. For every polynomial f of degree n with no double roots, there is an associated family $\mathcal{C}(f)$ of harmonic algebraic curves, fibred over the circle, with at most $n-1$ singular fibres. We study the combinatorial topology of $\mathcal{C}(f)$ in the generic case when there are exactly $n-1$ singular fibres. In this case, the topology of $\mathcal{C}(f)$ is determined by the data of an n-tuple of noncrossing matchings on the set $\{0, 1, \ldots, 2n-1\}$ with certain extra properties. We prove that there are $2(2n)^{n-2}$ such n-tuples, and that all of them arise from the topology of $\mathcal{C}(f)$ for some polynomial f.

1. Introduction

Let $f(z)$ be a monic polynomial of degree n with no double roots. For each $\theta \in \mathbb{R}/\pi\mathbb{Z}$, we can associate to $f(z)$ a plane algebraic curve

$$C_\theta(f) = \{z \ : \ \mathrm{Im}(e^{-i\theta} f(z)) = 0\}.$$

For instance, $C_{\pi/2}(f) = \{z \ : \ \mathrm{Re}(f(z)) = 0\}$; see Figure 1 for a pictorial example. Let $\mathcal{C}(f)$ denote the family of curves $C_\theta(f)$, fibered over the base $\mathbb{R}/\pi\mathbb{Z}$. Motivated by Gauss's first proof of the fundamental theorem of algebra, Martin, Singer, and the author [MSS] initiated a study of the combinatorial topology of the families $\mathcal{C}(f)$, which we continue in this article.

We begin by recalling the basic properties of the family $\mathcal{C}(f)$; see [MSS] for details. Since $\mathrm{Im}(e^{-i\theta} f(x+iy))$ is a harmonic function of the variables x, y, the maximum principle dictates that the curves $C_\theta(f)$ do not have any bounded connected components. In fact, the curve $C_\theta(f)$ has $2n$ asymptotes, at angles $\frac{\pi k + \theta}{n}$ for integers $0 \le k \le 2n-1$.

The fiber $C_\theta(f)$ is singular if and only if there is a root r of the derivative $f'(z)$ lying on $C_\theta(f)$. By hypothesis $f(r) \neq 0$ when r is a root of $f'(z)$, so each root of $f'(z)$ lies on a unique $C_\theta(f)$; therefore $C_\theta(f)$ is singular for at most $n-1$ values

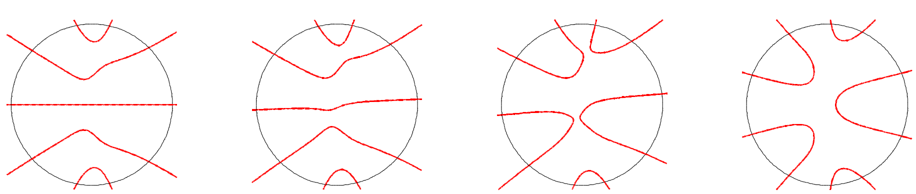

Figure 1. The curves $C_\theta(f)$, where $f(z)$ is the quintic $z^5 + 6z^3 + 3z^2 + 5z - 2$, and $\theta = 0, \frac{\pi}{12}, \frac{\pi}{6}, \frac{\pi}{2}$ respectively.

of θ (modulo π). If $C_\theta(f)$ is nonsingular, it has n connected components, each homeomorphic to $\mathbb{R}$, and each containing two of the $2n$ asymptotes.

Recall that a *matching* on a set of size $2n$ is a partition of the set into n subsets of size 2. Let $\langle a, b\rangle$ denote the set of integers in the interval $[a, b]$. If $C_\theta(f)$ is nonsingular, then the preceding remarks show that $C_\theta(f)$ induces a matching $M_\theta(f)$ on the set $\langle 0, 2n-1\rangle$, in which k, k' are matched if and only if the asymptotes at angles $\frac{\pi k+\theta}{n}$ and $\frac{\pi k'+\theta}{n}$ lie on the same component of $C_\theta(f)$; to make this definition we must choose coset representatives for $\mathbb{R}/\pi\mathbb{Z}$, and we take $\theta \in [0, \pi)$. For instance, the matching $M_0(f)$ for the quintic polynomial f of Figure 1 is $\{\{0, 5\}, \{1, 4\}, \{2, 3\}, \{6, 9\}, \{7, 8\}\}$.

Since the components of $C_\theta(f)$ do not cross, $M_\theta(f)$ is not just a matching—it is even a noncrossing matching:

Definition 1.1. Let $\mathcal{P}$ be a partition of a subset of $\mathbb{R}$. Two blocks of $\mathcal{P}$ are said to *cross* if there are integers $i < j < k < \ell$ such that i, k belong to one block and j, ℓ belong to the other block. If no two blocks cross, then the partition is said to be *noncrossing*. A *noncrossing matching (of order n)* is a noncrossing partition (of a set of size $2n$) into blocks of size 2.

This definition has a simple geometric interpretation. Given a partition of the set $\langle 0, N\rangle$ one can place the integers from 0 to N cyclically around a circle, and take the convex hulls of each of the blocks; the partition is noncrossing if and only if the convex hulls do not meet. See Figure 2 for examples.

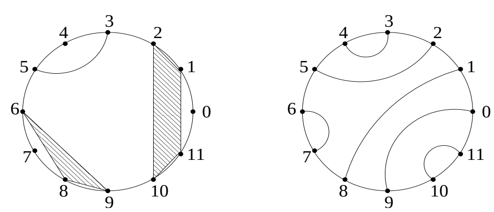

FIGURE 2. *Left*: The noncrossing partition whose non-singleton blocks are $\{1, 2, 10, 11\}$, $\{3, 5\}$ and $\{6, 8, 9\}$. *Right*: A noncrossing matching.

Suppose that the fibre $C_\theta(f)$ has exactly one singular point r, and suppose furthermore that $f''(r) \neq 0$, so that the singularity at r is ordinary (nodal). As t approaches θ from below, two components of $C_t(f)$ unite at the point r. As t increases away from θ, the two components separate in a perpendicular direction, as illustrated in Figure 3. (We will prove in Proposition 2.5 that this picture is correct.) Thus, for ϵ sufficiently small, the matching $M_{\theta+\epsilon}(f)$ is a *flip* of the matching $M_{\theta-\epsilon}(f)$, in the following sense.

Definition 1.2. Let M and M' be two matchings on the same underlying set. We say that M' is a *flip* of M (equivalently, M is a flip of M') if all but two of the pairs in M are also paired in M'; that is, if M' can be obtained from M by taking two pairs $\{i, j\}, \{k, \ell\}$ in M and replacing them with $\{i, \ell\}, \{j, k\}$ in M'.

Suppose now that f is generic in the sense that the family $\mathcal{C}(f)$ has the maximum $n-1$ singular fibres, which are necessarily ordinary as above. As θ varies from 0 to π, each time we cross a singular fibre the matching $M_\theta(f)$ changes by a flip. In

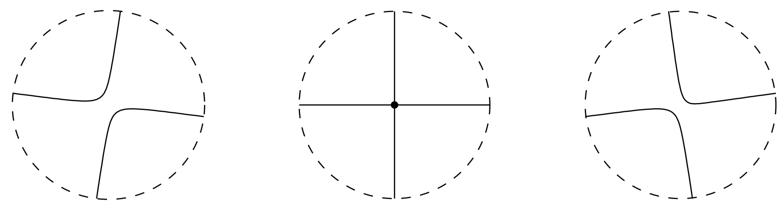

FIGURE 3. The curves $C_t(f)$ near $z = r$ for $t < \theta$, $t = \theta$, and $t > \theta$, where $C_\theta(f)$ has an ordinary singularity at r.

this manner we obtain an n-tuple $\mathcal{N}(f) = (M_1, \ldots, M_n)$ of noncrossing matchings such that M_{s+1} is a flip of M_s for $1 \leq s < n$; moreover, we will see that M_n is the matching $M_1(-1)$, in which $i-1, j-1$ are matched if and only if i, j are matched in M_1 (with subtraction taken modulo $2n$). The details of this construction can be found in Section 2. Observe that $\mathcal{N}(f)$ is a combinatorial invariant which classifies the topology of the family $\mathcal{C}(f)$.

Definition 1.3. A *necklace (of order n)* is an n-tuple $\mathcal{N} = (M_1, \ldots, M_n)$ of noncrossing matchings on the set $\langle 0, 2n-1\rangle$, such that M_{s+1} is a flip of M_s for $1 \leq s < n$, and $M_n = M_1(-1)$.

Example 1.4. Let $M_1 = \{\{0,1\},\{2,3\},\{4,5\}\}, M_2 = \{\{0,3\},\{1,2\},\{4,5\}\}, M_3 = \{\{0,5\},\{1,2\},\{3,4\}\}$. Then (M_1, M_2, M_3) is a necklace of order 3.

In this article we will prove the following two theorems, answering the questions raised in Section 4 of [MSS], and classifying the possibilities for the topology of the family $\mathcal{C}(f)$ when f is generic as above.

Theorem 1.5. *Let $\mathcal{N}$ be a necklace. Then there exists a polynomial f such that $\mathcal{N} = \mathcal{N}(f)$.*

Theorem 1.6. *The number of necklaces of order n is $2(2n)^{n-2}$.*

We outline our proof of these theorems. To begin, recall the main result of [MSS]: if M is a noncrossing matching of order n, then there exists a polynomial f such that $M = M_\theta(f)$. This is proved by induction, as follows. The noncrossing matching M contains at least one adjacent pair $\{i, i+1\}$, which we can assume without loss of generality is $\{2n-2, 2n-1\}$. Let M' be the noncrossing matching on $\langle 0, 2n-3\rangle$ obtained from M by deleting this pair. By the induction assumption $M' = M_\theta(g)$ for some polynomial g, and we show that $M = M_\theta(g(z)(z - z_0))$ for a suitably chosen z_0.

Let $\mathcal{N} = (M_1, \ldots, M_n)$ be the necklace arising from a polynomial f of degree n. We will see in Section 3 that $\mathcal{N}$ must satisfy certain conditions deriving from the fact that f has exactly n roots. For instance, if $r < s$, then in the terminology of [MSS, Definition 2.4], the bimatching corresponding to the pair (M_r, M_s) must be a basketball. Any necklace satisfying these necessary conditions will be called a *strong necklace*. We will see in Section 3 that the inductive argument of [MSS] can be applied to strong necklaces, so that every strong necklace arises from a polynomial. Moreover, we will show that the set of strong necklaces has size $2(2n)^{n-2}$; although this is a purely combinatorial statement, it is worth remarking that we will actually invoke the geometry of polynomials in the proof (more precisely, in the proof of

Corollary 3.11). One of the themes of this article is that the geometry of polynomials can be used to prove nontrivial statements about noncrossing matchings, and vice-versa.

It remains to prove the combinatorial fact that every necklace is actually a strong necklace. We can reinterpret this statement as follows.

Definition 1.7. Let $G_{ncm,n}$ be the graph whose vertices are the noncrossing matchings of order n, and whose edges are the pairs (M, M') such that M' is a flip of M. (Since n will always be fixed, we will generally omit the subscript n from $G_{ncm,n}$.)

Then a necklace is a path of length $n-1$ from M to $M(-1)$ in G_{ncm}, and we wish to show that every such path actually gives rise to a strong necklace. (As we will see in Section 4, there is an equivalent reinterpretation in terms of the lattice $NC(n)$ of noncrossing partitions on the set $\langle 0, n-1\rangle$.) The distance from M to $M(-1)$ in G_{ncm} turns out to be exactly $n-1$: in fact in Section 5 we will give a formula for the distance from M to M' in G_{ncm} in terms of the number of connected components of the *system of meanders* associated to the pair M, M'.

This formula in hand, we return in Section 6 to the proof of our main result. Using the machinery we have developed in the rest of the article, we construct a map from the set of necklaces to the set of *maximal chains of* 2*-divisible noncrossing partitions of* $\langle 0, 2n-1\rangle$ (see Definition 6.9). This map is seen to be injective. However, Edelman [Ede] has shown that the set of maximal chains of 2-divisible noncrossing partitions of $\langle 0, 2n-1\rangle$ has size $2(2n)^{n-2}$. By our earlier enumeration of strong necklaces, it follows that this map is a bijection and that every necklace is a strong necklace, completing the proof. In the final section, we use work of Armstrong [Arm] to construct a map from the set of maximal chains of 2-divisible noncrossing partitions of $\langle 0, 2n-1\rangle$ to the set of necklaces; we are grateful to the anonymous referee for suggesting this argument to us. As a consequence we obtain some further enumerative properties of necklaces and basketballs.

2. Variation of $M_\theta(f)$

In this section we describe how $M_\theta(f)$ varies with θ. The two main technical results (Lemma 2.1 and Proposition 2.5) are intuitively transparent; their proofs are simply a matter of getting the details in order, and can safely be ignored by the reader.

If $R > 0$, let D_R denote the disk $\{|z| \le R\}$, and let S_R denote its boundary circle $\{|z| = R\}$.

Lemma 2.1. *For $\theta \in [0, \pi)$, suppose that $C_\theta(f)$ has a smooth connected component (necessarily homeomorphic to $\mathbb{R}$) which contains the asymptotes at angles $\frac{\pi k+\theta}{n}$ and $\frac{\pi k'+\theta}{n}$.*

1. *If $\theta \neq 0$, then $C_t(f)$ has a smooth connected component which contains the asymptotes at angles $\frac{\pi k+t}{n}$ and $\frac{\pi k'+t}{n}$ for all t sufficiently close to θ.*
2. *If $\theta = 0$, then $C_t(f)$ has a smooth connected component which contains the asymptotes at angles $\frac{\pi k+t}{n}$ and $\frac{\pi k'+t}{n}$ for all $t > 0$ sufficiently close to 0, and which contains the asymptotes at angles $\frac{\pi(k-1)+t}{n}$ and $\frac{\pi(k'-1)+t}{n}$ for all $t < \pi$ sufficiently close to π.*

Proof. Choose $R \gg 0$ such that

- $C_t(f) \cap S_R$ contains exactly $2n$ points for all t,
- for all integers $j \in \langle 0, 2n-1\rangle$, the point $P_j(t)$ of $C_t(f) \cap S_R$ which is closest in argument to $\frac{\pi j+t}{n}$ lies on the asymptote at angle $\frac{\pi j+t}{n}$, and
- $\arg(P_j(t))$ is within $\frac{\pi}{6n}$ of $\frac{\pi j+t}{n}$.

To see that it is possible to choose R satisfying the first condition, observe by [MSS, Remark 1.7] that it suffices to show that the set

$$\{r \ : \ S_r \text{ is tangent to } C_t(f) \text{ for some } t\}$$

is bounded, and to take R larger than this bound. But $C_t(f)$ is tangent to S_r if and only if the function $h_{t,r}(\lambda) = \mathrm{Im}(e^{-it} f(re^{i\lambda})) : \mathbb{R} \to \mathbb{R}$ has a double root. This is easily seen to be possible only if r is bounded in terms of the coefficients of f (and independently of t), since $h_{t,r}(\lambda)$ is the sum of $r^n \sin(n\lambda - t)$ and a trigonometric polynomial of degree at most $n-1$ whose coefficients are bounded by r^{n-1} times the coefficients of f. Now as $R \to \infty$ the roots of $h_{t,R}(\lambda)$ approach $\frac{\pi j+t}{n}$ uniformly in t, so the second and third conditions can also be satisfied.

Now let C denote the component of $C_\theta(f)$ containing the asymptotes at angles $\frac{\pi k+\theta}{n}$ and $\frac{\pi k'+\theta}{n}$, and suppose that δ is less than the distance between $C \cap D_R$ and $(C_\theta(f) \setminus C) \cap D_R$, and also that $\delta < \frac{2R}{3n}$. Set $h_t(z) = \mathrm{Im}(e^{-it} f(z))$. By item (1) of [MSS, Lemma 3.7] (and in the notation of that lemma), $C_t(f) \cap D_R \subset N_{\delta/2}(C_\theta(f) \cap D_R)$ for t sufficiently close to θ; moreover, by item (2) of *ibid.*, $C_t(f) \cap N_{\delta/2}(C) \cap D_R$ is non-empty. Assume also that t is sufficiently close to θ so that for $t \neq \theta$, $C_t(f)$ is smooth. It follows from the definition of δ that $C_t(f)$ has a component C_t such that $C_t \cap D_R$ has endpoints Q_t, Q'_t where $\arg(Q_t), \arg(Q'_t)$ differ from $\arg P_k(\theta), \arg P_{k'}(\theta)$ respectively by at most $\frac{\pi}{2R}\frac{\delta}{2} < \frac{\pi}{6n}$.

Choose $0 < \epsilon < \min(\frac{\pi}{6}, \pi - \theta)$ such that all $t \in (\theta, \theta + \epsilon)$ are sufficiently close to θ as above. By construction, $\arg(Q_t)$ differs from $\frac{\pi k+t}{n}$ by at most $3 \cdot \frac{\pi}{6n} = \frac{\pi}{2n}$. It follows that $Q_t = P_k(t)$, and similarly that $Q'_t = P_{k'}(t)$. Therefore C_t contains the asymptotes at angles $\frac{\pi k+t}{n}$ and $\frac{\pi k'+t}{n}$, as desired.

An identical argument works for an interval of the form $(\theta - \epsilon, \theta)$ with $0 < \epsilon < \min(\theta, \frac{\pi}{6})$, except that when $\theta = 0$ we must consider t lying in an interval $(\pi - \epsilon, \pi)$ instead. In this exceptional case we see that $\arg(Q_t)$ lies within $\frac{\pi}{2n}$ of $\frac{\pi(k-1)+t}{n} = \frac{\pi k+(t-\pi)}{n}$, and similarly for $\arg(Q'_t)$.

∎

Definition 2.2. Let M be a matching on a subset S of the interval $[0, 2n)$. If ϵ is any real number, let $S(\epsilon)$ denote the set $\{i + \epsilon \ : \ i \in S\}$, with addition taken modulo $2n$. Define the *rotation of M by ϵ*, denoted $M(\epsilon)$, to be the matching on $S(\epsilon)$ such that $i + \epsilon, j + \epsilon$ are matched in $M(\epsilon)$ if and only if i, j are matched in M.

Proposition 2.3. (1) *If U is an open interval in $(0, \pi)$ such that $C_\theta(f)$ is nonsingular for all $\theta \in U$, then $M_\theta(f)$ (as a function of θ) is locally constant on U.*

(2) *If $C_0(f)$ is nonsingular, then for all $\theta > 0$ sufficiently close to 0 the matching $M_\theta(f)$ is $M_0(f)$, while for all $\theta < 0$ sufficiently close to 0 the matching $M_\theta(f)$ is $M_0(f)(-1)$.*

Proof. To prove (1), apply Lemma 2.1(1) to each component of $C_\theta(f)$ in turn, for each $\theta \in U$. Similarly, to prove (2), apply Lemma 2.1(2) to each component of $C_0(f)$ in turn. ∎

Definition 2.4. (1) We say that $C_\theta(f)$ is *completely ordinary* if $C_\theta(f)$ has exactly one singular point, and it is ordinary; that is, if there is exactly one root r of $f'(z)$ such that $r \in C_\theta(f)$, and $f''(r) \neq 0$.

(2) We say that a monic polynomial $f(z)$ of degree n is *completely generic* if $f(z)$ has no double roots, the family $\mathcal{C}(f)$ has $n-1$ singular fibres (which are necessarily completely ordinary), and $C_0(f)$ is nonsingular.

The requirement that $C_0(f)$ be nonsingular stems from our choice of the interval $[0,\pi)$ as coset representatives for $\mathbb{R}/\pi\mathbb{Z}$ in the definition of $M_\theta(f)$.

Proposition 2.5. *Suppose that $\theta \in (0,\pi)$ and $C_\theta(f)$ is completely ordinary. Let M be the matching $M_t(f)$ for t approaching θ from below, and let M' be the matching $M_t(f)$ for t approaching θ from above. Then M' is a flip of M.*

Proof. Since $C_\theta(f)$ is completely ordinary, it has one singular connected component, and $n-2$ smooth components. Applying Lemma 2.1(1) to each of the $n-2$ smooth components in turn, we see that the matchings M and M' have (at least) $n-2$ pairs in common. To conclude that M' is a flip of M, it remains only to dispense with the possibility that $M' = M$.

Recall our standing hypothesis that $f(z)$ has no double roots. Without loss of generality, let us suppose that $C_\theta(f)$ is singular at $z=0$, with $f(0) = e^{i\theta}$, $f'(0) = 0$, and $f''(0) \neq 0$. By the existence of uniformizers, we can choose a holomorphic function $g(w)$ on a disk D'_r of radius r centered at 0 (in the complex w-plane) such that $(f \circ g)(w) = h(w) = e^{i\theta}(1+w^2)$. Let S'_r denote the boundary of D'_r. Choose r sufficiently small that $g : D'_r \to g(D'_r)$ and $g\,|_{S'_r}$ are both homeomorphisms.

Observe that $C_\theta(h)$ is completely ordinary, consisting of the real and imaginary w-axes (note that h is not monic). Let $A_0, \dots, A_3$ be (short) closed arcs of S'_r such that the interior of A_i contains $re^{\pi i/2}$ for $i = 0,\dots,3$. One checks directly that for t approaching θ from above, $C_t(h)$ connects a point on A_0 to a point on A_1 and a point on A_2 to a point on A_3, whereas for t approaching θ from below, A_0 and A_3 are paired, as are A_1 and A_2.

Suppose that the singular component of $C_\theta(f)$ has a smooth half-arc (that is, a subset homeomorphic to a real half-line) connecting a point on $g(A_i)$ to the asymptote at angle $\frac{\pi k_i + \theta}{n}$. By an argument identical to the proof of Lemma 2.1, the same is true for all t sufficiently close to θ. It follows that M contains the pairs $\{k_0,k_3\},\{k_1,k_2\}$, while M' contains the pairs $\{k_0,k_1\},\{k_2,k_3\}$. $\square$

Let f be a completely generic polynomial. Let $\theta_1 < \cdots < \theta_{n-1}$ be the values of θ such that $C_\theta(f)$ is singular, and consider the intervals $I_1 = (0,\theta_1), I_2 = (\theta_1,\theta_2), \dots, I_n = (\theta_{n-1},\pi)$. From Proposition 2.3(1) we know that $M_\theta(f)$ is constant on each of the intervals I_s; let $M(f)_s$ denote this matching. By Proposition 2.5 we see that $M(f)_{s+1}$ is a flip of $M(f)_s$ for $1 \le s < n$. By Proposition 2.3(2) we see that $M(f)_n = M(f)_1(-1)$.

Definition 2.6. If f is a completely generic polynomial, then

$$\mathcal{N}(f) = (M(f)_1, \dots, M(f)_n)$$

is a necklace, which we call the necklace arising from f.

Theorem 1.5 can now be restated more precisely as follows.

Theorem 2.7. *Let $\mathcal{N}$ be a necklace. Then there exists a completely generic polynomial f such that $\mathcal{N} = \mathcal{N}(f)$.*

Remark 2.8. We can extend the definition of $\mathcal{N}(f)$ to polynomials f which have $n-1$ distinct singular fibres, but for which $C_0(f)$ is singular, as follows. We obtain matchings $M_1, \ldots, M_{n-1}$ just as in the completely generic case, using the nonsingular intervals $(0, \theta_1), \ldots, (\theta_{n-2}, \pi)$. Now define $\mathcal{N}(f) = (M_1, \ldots, M_{n-1}, M_1(-1))$. To check that this is actually a necklace, note that if we define $g(z)$ via $f(z) = e^{-i\delta} g(ze^{i\delta/n})$, we will soon see in (3.9) that $(M_1, \ldots, M_{n-1}, M_1(-1))$ is the necklace of $g(z)$ for $\delta < 0$ sufficiently small .

Let $\overline{G}_{ncm}$ be the quotient of the graph G_{ncm} in which M and $M(-1)$ are identified for all noncrossing matchings M. Let $\overline{M}$ denote the image of M in the quotient. Given a polynomial f such that $\mathcal{C}(f)$ has $n-1$ distinct singular fibres, it seems to be more intrinsic to associate to f the $(n-1)$-cycle $\overline{\mathcal{N}}(f)$ containing the vertices $\overline{M(f)}_1, \ldots, \overline{M(f)}_{n-1}$ in $\overline{G}_{ncm}$, instead of the necklace $\mathcal{N}(f)$: this is free of the choice of coset representatives for $\mathbb{R}/\pi\mathbb{Z}$, and does not require any ad hoc correction when $C_0(f)$ is singular. On the other hand, this does lose information, since $\overline{\mathcal{N}}(f) = \overline{\mathcal{N}}(g)$ whenever $f(z) = e^{-i\delta} g(ze^{i\delta/n})$. In any event, from a combinatorial viewpoint necklaces seem to be good objects to study, as we shall amply see.

3. Necklaces arising from polynomials

Part of our aim in [MSS] was to determine which ordered pairs (M, M') of noncrossing matchings can arise as a pair $(M_\alpha(f), M_\beta(f))$ for $\alpha, \beta \in [0, \pi)$ with $\alpha < \beta$. By an elementary parity argument, each connected component of $C_\alpha(f)$ must cross at least one connected component of $C_\beta(f)$, and the point where they meet is a root of the polynomial $f(z)$ (see the discussion in [MSS, Section 1]). Since $f(z)$ has exactly n roots, it follows that each connected component of $C_\alpha(f)$ crosses exactly one component of $C_\beta(f)$, and vice-versa. It follows that the matching $M_\alpha(f) \cup M_\beta(f)(\frac{1}{2})$ on the set $\{0, \frac{1}{2}, 1, \ldots, 2n - \frac{1}{2}\}$ has exactly n crossings.

Definition 3.1. We say that the ordered pair of noncrossing matchings (M, M') is a *basketball* if the matching $M \cup M'(\frac{1}{2})$ has exactly n crossings. (This differs slightly from the terminology of [MSS], where a pair (M, M') as above would have been called an ordered pair of matchings corresponding to a basketball.)

Evidently, for an ordered pair (M, M') of noncrossing matchings to arise as a pair $(M_\alpha(f), M_\beta(f))$ as above, it is necessary that (M, M') be a basketball; [MSS, Theorem 3.1] states that this is sufficient.

Let $\mathcal{N} = \mathcal{N}(f) = (M_1, \ldots, M_n)$ be a necklace arising from a polynomial $f(z)$. By the above discussion, the pair of noncrossing matchings (M_r, M_s) must be a basketball for all $1 \le r < s \le n$. There is another, slightly less obvious, property of $\mathcal{N}$.

Definition 3.2. If (M, M') is an ordered pair of matchings on the set $\langle 0, 2n-1 \rangle$ and the blocks $\{i, j\} \in M$ and $\{i' + \frac{1}{2}, j' + \frac{1}{2}\} \in M'(\frac{1}{2})$ cross in $M \cup M'(\frac{1}{2})$, we say that $\{i, j\} \in M$ crosses $\{i', j'\} \in M'$.

Lemma 3.3. *Let $\mathcal{N} = \mathcal{N}(f)$ be as above. Suppose $r < s < t$, and let $\{i_\star, j_\star\}$ be a pair in $M_\star$ for each $\star \in \{r, s, t\}$. If $\{i_r, j_r\} \in M_r$ crosses $\{i_s, j_s\} \in M_s$, and $\{i_r, j_r\} \in M_r$ crosses $\{i_t, j_t\} \in M_t$, then $\{i_s, j_s\} \in M_s$ crosses $\{i_t, j_t\} \in M_t$.*

Proof. Let $\lambda_r < \lambda_s < \lambda_t$ be angles in $[0, \pi)$ such that $M_\star = C_{\lambda_\star}(f)$ for $\star \in \{r, s, t\}$. Let $C_\star$ be the connected component of $C_{\lambda_\star}$ corresponding to the matched pair $\{i_\star, j_\star\}$. The claim of the lemma is equivalent to: if C_r meets C_s and C_t, then C_s

meets C_t. But the intersection of C_r with both C_s and C_t must be the unique root of $f(z)$ lying on C_r, and the lemma follows. □

Definition 3.4. Let $(M_1, \ldots, M_k)$ be a k-tuple of noncrossing matchings on the set $\langle 0, 2n-1\rangle$. We call $(M_1, \ldots, M_k)$ a *strong pseudonecklace (of length k and order n)* (for lack of a better term!) if it satisfies:

(1) The pair of noncrossing matchings (M_r, M_s) is a basketball for all $1 \le r < s \le k$;
(2) If $r < s < t$ and $\{i_r, j_r\} \in M_r$ crosses $\{i_s, j_s\} \in M_s$, and $\{i_r, j_r\} \in M_r$ crosses $\{i_t, j_t\} \in M_t$, then $\{i_s, j_s\} \in M_s$ crosses $\{i_t, j_t\} \in M_t$.

Note that we make no additional hypotheses about how M_r relates to M_{r+1}, or M_k relates to M_1.

Example 3.5. The reader may verify that if $(M_1, \ldots, M_k)$ is a strong pseudonecklace, then so is $(M_k(1), M_1, \ldots, M_{k-1})$.

Definition 3.6. A strong pseudonecklace $(M_1, \ldots, M_k)$ is a *strong necklace of length k (and order n)* if $M_{r+1} \neq M_r$ for $1 \le r < k$. A *strong necklace* (without reference to its length) will always mean a strong necklace of length equal to its order n.

It is not immediately clear that a strong necklace is actually a necklace (i.e., that M_{r+1} is a flip of M_r, and $M_n = M_1(-1)$); this will be a consequence of the fact that every strong pseudonecklace arises from a polynomial (Theorem 3.8), whose proof begins with the following observation.

Let $(M_1, \ldots, M_k)$ be a strong pseudonecklace, and let $\{i_{1,m}, j_{1,m}\}$ be an enumeration of the pairs in M_1, with $1 \le m \le n$. Let $\{i_{r,m}, j_{r,m}\}$ denote the (unique) pair in M_r which crosses $\{i_{1,m}, j_{1,m}\}$. For each m, let P_m be the subset of $\langle 0, 2nk-1\rangle$ consisting of the integers $ki_{r,m} + (r-1)$ and $kj_{r,m} + (r-1)$ for $1 \le r \le k$. It is clear that the P_m are disjoint and exhaust the set $\langle 0, 2nk-1\rangle$, i.e., together they form a partition $\mathcal{P}$ of $\langle 0, 2nk-1\rangle$.

Lemma 3.7. *The map $(M_1, \ldots, M_k) \mapsto \mathcal{P}$ defines a bijection between the set of strong pseudonecklaces of length k and order n, and the set of noncrossing partitions of $\langle 0, 2nk-1\rangle$ into n blocks of size $2k$.*

Proof. The lemma is clear for $n = 1$, so we suppose $n \ge 2$. First we check that the partition $\mathcal{P}$ is noncrossing. This is clear from the geometric picture, but we give a formal proof. Observe first that the pairs $\{ki_{r,m} + (r-1), kj_{r,m} + (r-1)\}$ and $\{ki_{s,m'} + (s-1), kj_{s,m'} + (s-1)\}$ cross if and only if $m = m'$ (and $r \neq s$). We refer to the numbers $ki_{r,m} + (r-1)$ and $kj_{r,m} + (r-1)$ as *counterparts*.

Let α, α' and β, β' be distinct pairs of counterparts in the block P_m, and suppose without loss of generality that $\alpha < \beta < \alpha' < \beta'$. If $m' \neq m$ and $\gamma, \delta \in P_{m'}$, then we need to prove that γ, δ are both lie in the same one of the following four sets: (α, β), (β, α'), (α', β'), or $[0, \alpha) \cup (\beta', 2nk-1]$.

Suppose, for instance, that $\gamma \in (\alpha, \beta)$. Let γ' be the counterpart of γ. Then $\gamma' \in (\alpha, \alpha')$ since $\{\gamma, \gamma'\}$ does not cross $\{\alpha, \alpha'\}$. Similarly $\gamma' \in [0, \beta) \cup (\beta', 2nk-1]$ since $\{\gamma, \gamma'\}$ does not cross $\{\beta, \beta'\}$. Hence γ' is in (α, β) as well. An analogous argument works for γ in each of the other three sets. Now we are done if $\delta = \gamma'$. If not, the same argument shows that δ and its counterpart δ' both lie in the same one of the four sets above. Since $\{\gamma,\gamma'\}$ crosses $\{\delta,\delta'\}$, all four of $\gamma,\gamma',\delta,\delta'$ must lie in the same set, and the partition $\mathcal{P}$ is indeed noncrossing.

To show that the map $(M_1,\ldots,M_k)\mapsto\mathcal{P}$ is a bijection, we construct an inverse. Let $\mathcal{P}$ be a noncrossing partition of $\langle 0,2nk-1\rangle$ into n blocks of size $2k$. Let P be a block of $\mathcal{P}$, containing elements $\alpha_1<\cdots<\alpha_{2k}$. Observe that $\alpha_{i+1}\equiv\alpha_i+1 \pmod{2k}$. Indeed, if $\alpha_i+1<\alpha_{i+1}$, then the set $\langle\alpha_i+1,\alpha_{i+1}-1\rangle$ must be a union of blocks of $\mathcal{P}$ since $\mathcal{P}$ is noncrossing, hence it has size divisible by $2k$. It follows that each block P_m contains two numbers of the form $kx+(r-1)$ for each $1\le r\le k$; let these be $ki_{r,m}+(r-1)$ and $kj_{r,m}+(r-1)$. For each $1\le r\le k$ define M_r to be the collection of pairs $\{\{i_{r,m},j_{r,m}\}\ :\ 1\le m\le n\}$. It is easy to see that $(M_1,\ldots,M_k)$ is a strong pseudonecklace, and that this construction is a two-sided inverse to the map $(M_1,\ldots,M_k)\mapsto\mathcal{P}$. □

Theorem 3.8. *Suppose $0\le\lambda_1<\cdots<\lambda_k<\pi$, and let $(M_1,\ldots,M_k)$ be a strong pseudonecklace of length k and order n. Then there exists a monic polynomial $f(z)$ of degree n such that $M_i=M_{\lambda_i}(f)$ for $1\le i\le k$.*

Proof. The statement is trivial for $n=1$. Assume that the theorem is true for pseudonecklaces of order $n-1$.

The induction step proceeds in two stages. First, we show that if the strong pseudonecklace $(M_1,\ldots,M_k)$ occurs as $(M_{\lambda_1}(f),\ldots,M_{\lambda_k}(f))$ for all choices of $0\le\lambda_1<\cdots<\lambda_k<\pi$, then so does the strong pseudonecklace $(M_k(1),M_1,\ldots,M_{k-1})$.

Let $f(z)$ be a monic polynomial of degree n, and define another monic polynomial $g(z)$ by the formula $f(z)=e^{-i\delta}g(ze^{i\delta/n})$. Put $w=ze^{i\delta/n}$. Then $\mathrm{Im}(e^{-i\theta}f(z))=0$ if and only if $\mathrm{Im}(e^{-i(\delta+\theta)}g(w))=0$, and so the asymptotes at angles $\frac{\pi k+\theta}{n},\frac{\pi k'+\theta}{n}$ are matched in $C_\theta(f)$ if and only if the asymptotes at angles $\frac{\pi k+(\theta+\delta)}{n},\frac{\pi k'+(\theta+\delta)}{n}$ are matched in $C_{\theta+\delta}(g)$. It follows that

$$M_{\theta+\delta}(g)=\begin{cases}M_\theta(f) & \text{if } 0\le\theta+\delta<\pi\\ M_\theta(f)(1) & \text{if } \pi\le\theta+\delta<2\pi\,.\end{cases}\tag{3.9}$$

Given $0\le\lambda_1<\cdots<\lambda_k<\pi$, choose $\lambda_1<\delta<\lambda_2$. By assumption we can find a monic polynomial f so that

$$(M_{\lambda_2-\delta}(f),\ldots,M_{\lambda_k-\delta}(f),M_{\pi+\lambda_1-\delta}(f))=(M_1,\ldots,M_k)\,.$$

If $g(z)$ is defined as above, the preceding argument shows that

$$(M_{\lambda_1}(g),\ldots,M_{\lambda_k}(g))=(M_k(1),M_1,\ldots,M_{k-1})$$

as desired. Note that we can use a similar argument to show that if $(M_1,\ldots,M_k)$ occurs as $(M_{\lambda_1}(f),\ldots,M_{\lambda_k}(f))$ for all choices of λ_i with $\lambda_1\ne 0$, then it also occurs for all choices of λ_i with $\lambda_1=0$.

For the second stage, note that if $(M_1,\ldots,M_k)$ corresponds as in Lemma 3.7 to the partition $\mathcal{P}$ of $\langle 0,2nk-1\rangle$ into n blocks of size $2k$, then $(M_k(1),M_1,\ldots,M_{k-1})$ corresponds to the partition $\mathcal{P}(1)$. Observe that any partition $\mathcal{P}$ of $\langle 0,2nk-1\rangle$ into n blocks of size $2k$ contains at least one block of the form $\{x,x+1,\ldots,x+2k-1\}$; this is geometrically clear, and can be proved in an identical manner to [MSS, Proposition 2.11]. By the first stage of the proof, it suffices to prove the statement of the theorem for strong pseudonecklaces $(M_1,\ldots,M_k)$ corresponding (via Lemma 3.7) to partitions $\mathcal{P}$ containing the block $\{0,1,\ldots,k-1,2nk-k,\ldots,2nk-1\}$, i.e., for strong pseudonecklaces such that $\{0,2n-1\}$ is a pair in M_i for all i. Moreover, we have seen that we may assume $\lambda_1>0$.

But then the result follows directly from [MSS, Theorem 3.3]. Indeed, let $\check{M}_i$ be the matching on $\langle 0, 2n-3\rangle$ obtained by omitting the pair $\{2n-2, 2n-1\}$ from $M_i(-1)$. By the induction assumption $(\check{M}_1, \dots, \check{M}_k) = (M_{\lambda_1}(\check{f}), \dots, M_{\lambda_k}(\check{f}))$ for some polynomial $\check{f}$ of degree $n-1$. Then $(M_1, \dots, M_k) = (M_{\lambda_1}(f), \dots, M_{\lambda_k}(f))$ for $f(z) = \check{f}(z)(z-R)$ for $R \gg 0$. □

Remark 3.10. The proof of Theorem 3.8 can be modified slightly to ensure that the polynomial f that one constructs is completely generic. Indeed, it suffices to ensure that f has $n-1$ distinct singular fibres, for one can always ensure that any particular fibre is nonsingular (in our case, $C_0(f)$) by replacing $f(z)$ with $g(z)$ given by $f(z) = e^{-i\delta} g(ze^{i\delta/n})$ for δ sufficiently small that $M_{\lambda_i}(g) = M_{\lambda_i - \delta}(f) = M_{\lambda_i}(f)$, invoking Proposition 2.3(1) (take $\delta < 0$ if $\lambda_i = 0$). Now by induction $\check{f}(z)$ can be assumed to be completely generic, and f has distinct singular fibres for $R \gg 0$ by [MSS, Lemma 3.5(3),(4)].

Corollary 3.11. *Every strong necklace $\mathcal{N} = (M_1, \dots, M_n)$ is a necklace; that is, M_{r+1} is a flip of M_r for $1 \le r < n$, and $M_n = M_1(-1)$. Moreover, $\mathcal{N} = \mathcal{N}(f)$ for a completely generic polynomial $f(z)$.*

Proof. Pick any $0 < \lambda_1 < \cdots < \lambda_n < \pi$, and by Theorem 3.8 choose $f(z)$ monic of degree n such that $M_r = M_{\lambda_r}(f)$. Since $M_r \neq M_{r+1}$ for $1 \le r < n$, the family $\mathcal{C}(f)$ must be singular for some $\theta_r \in (\lambda_r, \lambda_{r+1})$. Since the family has at most $n-1$ singularities, the $C_{\theta_r}(f)$ for $1 \le r \le n-1$ must be all of the singular fibres. Hence f is completely generic and $(M_1, \dots, M_n) = \mathcal{N}(f)$. □

We remark that while the first statement of Corollary 3.11 is purely combinatorial, our proof uses the geometric input of Theorem 3.8. The following two corollaries have proofs which are similar to that of Corollary 3.11; for Corollary 3.13, invoke Remark 3.10.

Corollary 3.12. *There do not exist strong necklaces of length k and order n with $k > n$.*

Corollary 3.13. *For any strong necklace $(M_1, \dots, M_k)$ of length $k < n$, there exists a (strong) necklace $\mathcal{N}$ such that $(M_1, \dots, M_k)$ is obtained by omitting $n-k$ matchings from $\mathcal{N}$.*

We conclude this section with the following enumerative result.

Proposition 3.14. *The number of strong necklaces is $2(2n)^{n-2}$.*

Proof. Let $\#S$ denote the size of a set S, let $SPN(k,n)$ denote the set of strong pseudonecklaces of length k and order n, and let SN denote the set of strong necklaces. Given a strong pseudonecklace $\mathcal{N} = (M_1, \dots, M_k)$ of length k and order n, define $S(\mathcal{N}) = \{r < k \ : \ M_r = M_{r+1}\}$. Let S be any subset of $\langle 1, k-1\rangle$. There is a bijection between the set $\{\mathcal{N} \in SPN(k,n) \ : \ S \subset S(\mathcal{N})\}$ and the set $SPN(k - \#S, n)$, obtained by omitting M_{r_i+1} from $\mathcal{N}$ for each $r_i \in S$. Note that the strong necklaces are exactly the strong pseudonecklaces of length n and order

n such that $S(\mathcal{N}) = \emptyset$. By the inclusion-exclusion principle, we have

$$\begin{aligned}
\#SN &= \sum_{S \subset \langle 1, n-1 \rangle} (-1)^{\#S} \cdot \#\{\mathcal{N} \in SPN(n,n) \ : \ S \subset S(\mathcal{N})\} \\
&= \sum_{S \subset \langle 1, n-1 \rangle} (-1)^{\#S} \cdot \#SPN(n - \#S, n) \\
&= \sum_{j=0}^{n-1} (-1)^j \binom{n-1}{j} \cdot \#SPN(n-j, n)\,.
\end{aligned}$$

But by Lemma 3.7 the strong pseudonecklaces of length $n-j$ and order n are in bijection with the noncrossing partitions of $\langle 0, 2n(n-j)-1 \rangle$ into n blocks of size $2(n-j)$, and according to [Ede, Lemma 4.1] there are exactly $\frac{1}{n}\binom{2(n-j)n}{n-1}$ of these, so that

$$\#SN = \frac{1}{n} \sum_{j=0}^{n-1} (-1)^j \binom{n-1}{j} \binom{2(n-j)n}{n-1}\,.$$

This binomial sum can be evaluated with ease as follows. Observe that $\binom{2(n-j)n}{n-1}$ is a polynomial in j of degree $n-1$ with leading coefficient $\frac{1}{(n-1)!} \cdot (-2n)^{n-1}$. We have the following general binomial identity [GKP, Equation (5.52)]:

$$\sum_{j=0}^{n-1} (-1)^j \binom{n-1}{j} (a_0 + a_1 j + \cdots + a_{n-1} j^{n-1}) = (-1)^{n-1}(n-1)! \cdot a_{n-1}\,.$$

Applying this identity to our sum for $\#SN$ yields

$$\#SN = \frac{1}{n}(-1)^{n-1}(n-1)!\,\frac{(-2n)^{n-1}}{(n-1)!} = 2(2n)^{n-2}$$

as desired. □

4. The dictionary between G_{ncm} and $NC(n)$

Before proceeding to our proof that every necklace is a strong necklace, we pause for a moment to show that the graph G_{ncm} is isomorphic to the graph $NC(n)$ whose definition is as follows.

Definition 4.1. The graph $NC(n)$ is the graph whose vertices are the noncrossing partitions of the set $\langle 0, n-1 \rangle$, and whose edges are pairs $(\mathcal{P}, \mathcal{P}')$ of partitions such that $\mathcal{P}'$ is obtained from $\mathcal{P}$ by replacing two blocks of $\mathcal{P}$ with their union, or vice-versa.

In fact $NC(n)$ is a lattice, in which $\mathcal{P}$ lies below $\mathcal{P}'$ if the blocks of $\mathcal{P}'$ are unions of blocks $\mathcal{P}$. It is of course well-known that the vertices of G_{ncm} and $NC(n)$ are in bijection, but we wish to record the stronger statement that the graphs are isomorphic. This observation is not logically necessary for our proof that every necklace is a strong necklace. However, the graph $NC(n)$ (or more precisely, the lattice it underlies) has been well-studied (see e.g. [Kre, ES, Sim]) and so it will be of interest that the properties we eventually establish for G_{ncm} apply equally well to $NC(n)$.

We recall the standard bijection between the vertices of G_{ncm} and $NC(n)$ (but we omit the proof that the construction is well-defined and bijective). Let M be a noncrossing matching on the set $\langle 0, 2n-1 \rangle$; it is especially useful here to think of M as a matching on $2n$ points on a circle, labelled cyclically from 0 to $2n-1$. Beginning from $0, 1$, etc., relabel these points $0, 0', 1, 1', \ldots, n-1, (n-1)'$. Now each edge of M joins an unprimed point a to a primed point b', and from the equivalence relation generated by the relations $a \sim b$ we obtain a partition $\mathcal{P}(M)$.

Conversely, suppose we begin with a noncrossing partition $\mathcal{P}$ on the set $\langle 0, n-1\rangle$. We wish to define a matching $M(\mathcal{P})$. Suppose that $\{a_{i1}, \ldots, a_{ij_i}\}$ is a block of $\mathcal{P}$, with $a_{i1} < \cdots < a_{ij_i}$. We match a'_{i1} with a_{i2}, a'_{i2} with a_{i3}, and so forth, as well as a'_{ij_i} with a_{i1}. Having done this for all the blocks, we obtain a matching on $\{0, 0', 1, 1', \ldots, n-1, (n-1)'\}$. Relabel these from 0 to $2n-1$, and we obtain $M(\mathcal{P})$. The maps $M \mapsto \mathcal{P}(M)$ and $\mathcal{P} \mapsto M(\mathcal{P})$ are inverse to one another.

Next, we recall that the graph $NC(n)$ is self-dual, as follows. If $\mathcal{P}$ is a vertex of $NC(n)$, we associate to $\mathcal{P}$ an element $\sigma(\mathcal{P})$ of the symmetric group S_n: write each of the blocks of $\mathcal{P}$ as $\{a_{i1}, \ldots, a_{ij_i}\}$ with $a_{i1} < \cdots < a_{ij_i}$, and define $\sigma(\mathcal{P})$ to be the product of the cycles $(a_{i1} \ \cdots \ a_{ij_i})$. The dual of $\mathcal{P}$ is the vertex $\mathcal{P}\check{}$ satisfying

$$\sigma(\mathcal{P}) \cdot \sigma(\mathcal{P}\check{}) = (0 \ 1 \ \cdots \ n-1)$$

with multiplication taken from right to left. (Caution: it is not true that $(\mathcal{P}\check{})\check{} = \mathcal{P}$.) We will return to this point of view in Section 7. For details see, e.g., [McC] or the memoir [Arm].

The full statement that we wish to establish is as follows.

Proposition 4.2. *The map $M \mapsto \mathcal{P}(M)$ defines a graph isomorphism from G_{ncm} to $NC(n)$, under which $\mathcal{P}(M(-1)) = \mathcal{P}(M)\check{}$.*

Proof. If M is a matching on $\langle 0, 2n-1\rangle$, let $\overline{M}$ denote the corresponding matching on $\{0, 0', 1, 1', \ldots, n-1, (n-1)'\}$. We prove the duality statement first. Note that if $\overline{M}$ has an edge from a' to b, then $\overline{M(-1)}$ has an edge from $(b-1)'$ to a. Then $\sigma(\mathcal{P}(M(-1)))$ sends $b-1$ to a, while $\sigma(\mathcal{P}(M))$ sends a to b, and so

$$\sigma(\mathcal{P}(M)) \cdot \sigma(\mathcal{P}(M(-1))) = (0 \ 1 \ \cdots \ n-1);$$

the duality claim follows.

Next, suppose that $\mathcal{P}'$ is obtained from $\mathcal{P}$ by replacing two blocks $\{a_1, \ldots, a_j\}$ and $\{b_1, \ldots, b_k\}$ with their union, so that $(\mathcal{P}, \mathcal{P}')$ is an edge in $NC(n)$. We wish to show that $(M(\mathcal{P}), M(\mathcal{P}'))$ is an edge in G_{ncm}, i.e., that $M(\mathcal{P}')$ is a flip of $M(\mathcal{P})$ (equivalently, that $\overline{M(\mathcal{P}')}$ is a flip of $\overline{M(\mathcal{P})}$). Suppose without loss of generality that $a_1 < \cdots < a_j$ and $b_1 < \cdots < b_k$, so that a'_i and a_{i+1} are joined by an edge in $\overline{M(\mathcal{P})}$ (with the subscript $i+1$ taken to be 1 if $i = j$), and similarly for b'_i and b_{i+1}. Without loss of generality assume $a_1 < b_1$. Since $\mathcal{P}$ is a noncrossing partition, the block $\{a_1, \ldots, a_j, b_1, \ldots, b_k\} \in \mathcal{P}'$ must have the form

$$a_1 < \ldots < a_\ell < b_1 < \ldots < b_k < a_{\ell+1} < \cdots < a_j$$

for some ℓ (possibly $\ell = j$). Then $\overline{M(\mathcal{P}')}$ is obtained from $\overline{M(\mathcal{P})}$ by flipping the edges $\{a'_\ell, a_{\ell+1}\}$ and $\{b'_k, b_1\}$ to edges $\{a'_\ell, b_1\}$ and $\{b'_k, a_{\ell+1}\}$ (with $\ell + 1$ taken to be 1 if $\ell = j$).

Conversely, suppose that M is a flip of M', with pairs $\{a', b\}$ and $\{c', d\}$ in $\overline{M}$ replaced by $\{a', d\}$ and $\{c', b\}$ in $\overline{M'}$. The only blocks of $\mathcal{P}(M)$, $\mathcal{P}(M')$ affected by this flip are the one or two blocks containing a, b, c, d. The block of $\mathcal{P}(M)$ containing a, b corresponds to a sequence of pairs $\{a', b\}, \{b', x_1\}, \{x'_1, x_2\}, \ldots, \{x'_j, a\}$ in $\overline{M}$. If one of these pairs is $\{c', d\} = \{x'_i, x_{i+1}\}$, then the flip breaks this single block into the two blocks $\{b, x_1, \ldots, x_i = c\}$ and $\{x_{i+1} = d, \ldots, x_j, a\}$ in $\mathcal{P}(M')$. If not, then

c, d lie in another block corresponding to pairs $\{c', d\}, \{d', y_1\}, \ldots, \{y'_k, c\}$ in $\overline{M}$, and the flip joins these two blocks into the block $\{b, x_1, \ldots, x_j, a, d, y_1, \ldots, y_k, c\}$. In both cases $(\mathcal{P}, \mathcal{P}')$ is an edge in $NC(n)$. □

Example 4.3. We will see shortly that the diameter of the graph G_{ncm} is $n-1$, and that the distance from M to $M(-1)$ is $n-1$. Therefore, under the dictionary described in this section, we see that the diameter of $NC(n)$ is $n-1$, and that the distance from $\mathcal{P}$ to $\mathcal{P}^{\vee}$ is $n-1$. Moreover, necklaces correspond to diameters of the form $\mathcal{P} \rightsquigarrow \mathcal{P}^{\vee}$ in $NC(n)$. In particular, there are exactly $2(2n)^{n-2}$ diameters of this form.

5. Meanders and distances in G_{ncm}

Given two noncrossing matchings M and M' on the set $\langle 0, 2n-1\rangle$, we would like to compute the distance between M and M' in the graph G_{ncm}. The key is to consider the system of meanders associated to the pair (M, M'). Meanders have been studied previously by Lando and Zvonkin [LZ], Arnol'd, and many others. Systems of meanders are often defined geometrically, but in order to remain precise, we give the following combinatorial definition, which applies equally well to matchings which are not noncrossing.

Definition 5.1. Let M, M' be (not necessarily noncrossing) matchings. Let $\sim$ be the equivalence relation on the set $\langle 0, 2n-1\rangle$ generated by the equivalences $i \sim j$ for every pair $\{i, j\} \in M$, and $i' \sim j'$ for every $\{i', j'\} \in M'$. The *system of meanders* $\Pi_0(M, M')$ is the set of equivalence classes of $\langle 0, 2n-1\rangle$ under this relation. Let $\pi_0(M, M')$ denote the number of equivalence classes. If $\pi_0(M, M') = 1$ then the pair (M, M') is called a *meander*.

Example 5.2. We have $\Pi_0(M, M) = M$, and $\pi_0(M, M) = n$. In fact, $\pi_0(M, M') = n$ if and only if $M = M'$.

Consider the multigraph $G(M, M')$ with $2n$ vertices labelled from 0 to $2n-1$, and with an edge (i, j) for each pair $\{i, j\}$ in M and M'. The graph is bivalent, and therefore it decomposes as a collection of loops; each loop alternates between edges coming from pairs in M and edges coming from pairs in M'. The blocks of the partition $\Pi_0(M, M')$ are exactly the labels of each loop, and $\pi_0(M, M')$ is the number of loops, i.e., the number of connected components of $G(M, M')$. In particular, each block of $\Pi_0(M, M')$ is equipped with a natural *unoriented* cyclic ordering: the order obtained by traversing the corresponding loop.

It is clear from this description that if M'' is a flip of M', then the only loops that are affected by replacing M' with M'' in $\Pi_0(M, M')$ are the one or two loops that contain edges corresponding to the two pairs of M' that are flipped. Symmetrically, either one or two blocks in $\Pi_0(M, M'')$ are affected. Therefore $\pi_0(M, M'') = \pi_0(M, M') + \delta$ with $\delta \in \{-1, 0, 1\}$.

Remark 5.3. If M, M' are noncrossing, then $\Pi_0(M, M')$ has even more structure: we can orient the loops of $\Pi_0(M, M')$ in a natural manner, as follows. Observe that every edge in $G(M, M')$ has one even endpoint and one odd endpoint. Since each loop alternates between edges coming from pairs in M and edges coming from pairs in M', we may orient every loop so that edges of M flow from even endpoints to odd endpoints, and edges of M' flow from odd endpoints to even endpoints. Let $x \to y$ denote an oriented edge from x to y.

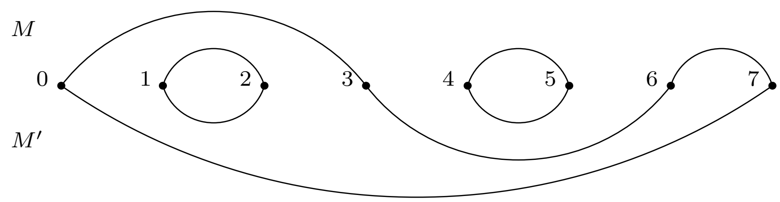

Figure 4. A realization of the system of meanders for the matchings M with pairs $\{0,3\}$, $\{1,2\}$, $\{4,5\}$, $\{6,7\}$ and M' with pairs $\{0,7\}$, $\{1,2\}$, $\{3,6\}$, $\{4,5\}$. In this example, $\pi_0(M,M')=3$.

Suppose M, M' are noncrossing, and that the noncrossing matching M'' is obtained from M' by flipping $\{x,y\},\{z,w\}$ to $\{x,z\}$,$\{y,w\}$. Then x,w have the same parity, and y,z have the same parity; suppose without loss of generality that x,w are odd. If x,y,z,w are in the same loop L in $G(M,M')$, then the oriented loop L consists of an edge $x\to y$, a path from y to w, an edge $w\to z$, and a path from z to x. Removing the edges $x\to y$ and $w\to z$, and replacing them with edges $x\to z$ and $w\to y$ yields two loops. Therefore $\pi_0(M,M'')=\pi_0(M,M')+1$. By a similar argument, if $\{x,y\}$ and $\{z,w\}$ were in different blocks of $\Pi_0(M,M')$, then the two blocks become one block in $\Pi_0(M,M'')$, and $\pi_0(M,M'')=\pi_0(M,M')-1$.

In summary, if M,M' are noncrossing matchings and the noncrossing matching M'' is a flip of M', then $\pi_0(M,M'')=\pi_0(M,M')\pm 1$.

Remark 5.4. Although we shall not make use of this fact, it may be worth noting that if we define $\{\{0,1\},\dots,\{2n-2,2n-1\}\}$ to be an even noncrossing matching, then by Remark 5.3 each noncrossing matching M obtains a well-defined parity, and the length of a sequence of flips from M to M' has parity equal to the difference of the parities of M and M'.

To study the system of meanders for a pair of noncrossing matchings, it is helpful to realize the system of meanders geometrically, as follows. Consider the points $\{0,1,\dots,2n-1\}$ along the real axis in the complex plane. For each pair $\{x,y\}\in M$, connect the point x on the real axis to the point y on the real axis via a smooth curve in the upper half-plane, in such a manner that none of the curves cross; then do the same in the lower half-plane for the matching M', so that the whole picture consists of some number of simple closed curves, as illustrated in Figure 4. The number of simple closed curves in the picture is $\pi_0(M,M')$, and the numbers on each curve are the same (and in the same order) as the numbers on each loop of $G(M,M')$. Such a collection of curves is called a *realization* of the system of meanders $\Pi_0(M,M')$.

We will make use of a preferred realization of $\Pi_0(M,M')$, denoted $C(M,M')$, in which the curve joining each pair is a semicircle. If $\{x,y\}\in M'$, let $\widehat{xy}$ denote the semicircle joining x and y in the lower half-plane. (We will use this notation exclusively for pairs in M', and not pairs in M.) It is not difficult to see that our semicircles do not cross: if $\{x,y\},\{z,w\}$ are two pairs in M' satisfying $x<y<z<w$, then no two points on $\widehat{xy}$ and $\widehat{zw}$ have the same horizontal coordinate. On the other hand, if $x<z<w<y$, then $\widehat{zw}$ lies above $\widehat{xy}$. Up to renaming the points, these are the only possibilities we need to consider; and similarly for M. Suppose

$x < y$ and $z < w$. Observe that for any point P on $\widehat{zw}$, the ray extending vertically downwards from P intersects $\widehat{xy}$ if and only if $x < z < w < y$.

Lemma 5.5. *If M, M' are noncrossing matchings and $\pi_0(M, M') < n$, then there exists a noncrossing matching M'' such that M'' is a flip of M', and $\pi_0(M, M'') = \pi_0(M, M') + 1$.*

Proof. Since $\pi_0(M, M') < n$, we can find two pairs $\{x, y\}$ and $\{z, w\}$ in M' which lie in the same block of $\Pi_0(M, M')$. Without loss of generality we may assume $x < z < w < y$. (Up to relabelling, the only other possibility is $x < y < z < w$; then replace M, M' by $M(2n-w), M'(2n-w)$ to reach the desired situation.) We may suppose further that the pairs $\{x, y\}$ and $\{z, w\}$ have been chosen to minimize the sum $(z-x)+(y-w)$. We will prove that the matching M'' obtained by flipping $\{x, y\}, \{z, w\}$ to $\{x, z\}, \{y, w\}$ is noncrossing. By Remark 5.3, this suffices to prove the lemma.

If $\{u, v\}$ crosses $\{x, z\}$ in M'', exactly one of the endpoints of $\widehat{uv}$ (let's say u) lies in $\langle x+1, z-1\rangle$; since $\{u, v\} \in M'$ as well, and M' is noncrossing, v must lie in $\langle w+1, y-1\rangle$. Therefore, we wish to show that M' has no pairs with one endpoint in $\langle x+1, z-1\rangle$ and another in $\langle w+1, y-1\rangle$. Certainly no such pair lies in in the block of $\Pi_0(M, M')$ containing x, y, z, w, by the minimality property of $\{x, y\}$ and $\{z, w\}$.

Let C denote the connected component of $C(M, M')$ containing x, y, z, w. Pick a point P on $\widehat{zw}$, and extend a vertical ray downwards from P until the ray meets $\widehat{xy}$; let Q be the point of intersection, and let $\overline{PQ}$ denote the line segment joining P and Q. The interior of $\overline{PQ}$ cannot meet C: by the sentence immediately preceding the statement of the lemma, any semicircle $\widehat{uv}$ with $u < v$ that meets the interior of $\overline{PQ}$ must have $u \in \langle x+1, z-1\rangle$ and $v \in \langle w+1, y-1\rangle$, which we have just seen cannot happen for a semicircle of M' in C. Our setup is illustrated in Figure 5.

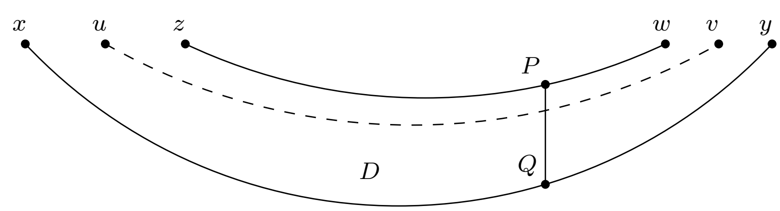

FIGURE 5. The setup for the proof of Lemma 5.5

We imagine that our entire picture is embedded in the sphere (the one-point compactification of the plane), so that the complement of C has two connected components, each of which is a topological open disk. Let $\mathring{D}$ denote the component containing the interior of $\overline{PQ}$, and let D denote its closure, the topological closed disk $\mathring{D} \cup C$. Then D contains $\overline{PQ}$, and since $\overline{PQ}$ is a path between two points on the boundary of D, it follows that $D \setminus \overline{PQ}$ has two connected components.

Suppose there exists a pair $\{u, v\} \in M'$ with $u \in \langle x+1, z-1\rangle$ and $v \in \langle w+1, y-1\rangle$, and let C' be the connected component of $C(M, M')$ containing u, v. The semicircle $\widehat{uv}$ crosses the interior of $\overline{PQ}$; since C' does not cross C but does meet the interior of PQ, it follows that C' is contained in $\mathring{D}$. Since $D \setminus \overline{PQ}$ has two connected components and C' is a simple closed curve, it follows that C' meets $\overline{PQ}$ an even number of times. (Note that all intersections between $\overline{PQ}$ and $C(M, M')$ are transverse.) Therefore C' contains a second semicircle $\widehat{rs}$ which meets $\overline{PQ}$, so that, taking $r < s$, we have $r \in \langle x+1, z-1\rangle$ and $s \in \langle w+1, y-1\rangle$. The pairs $\{r, s\}$,$\{u, v\}$ contradict the minimality of the choice of $\{x, y\}$,$\{z, w\}$, so u, v cannot exist. This completes the proof. □

Definition 5.6. Let G_{mat} be the graph whose vertices are the matchings (not necessarily noncrossing) on $\langle 0, 2n-1\rangle$, and whose edges are the pairs (M, M') such that M' is a flip of M. Let d_{ncm} and d_{mat} denote the distance functions on G_{ncm} and G_{mat} respectively.

We are ready to give our formula for distances in the graph G_{ncm}.

Theorem 5.7. *If M and M' are noncrossing matchings on $\langle 0, 2n-1\rangle$, then*

$$d_{ncm}(M, M') = d_{mat}(M, M') = n - \pi_0(M, M')\,.$$

Proof. Since G_{ncm} is a subgraph of G_{mat}, we know *a priori* that

$$d_{mat}(M, M') \le d_{ncm}(M, M')\,.$$

(It is striking that although G_{mat} is much larger than G_{ncm}, distances between noncrossing matchings will be the same in both graphs!)

Suppose that $M = M_0, M_1, \dots, M_k = M'$ is a sequence of (not necessarily noncrossing) matchings such that M_{r+1} is a flip of M_r for $r < k$. Since $\pi_0(M, M) = n$ and $\pi_0(M, M_{r+1}) \ge \pi_0(M, M_r) - 1$, we have $\pi_0(M, M') \ge n - k$. Hence

$$n - \pi_0(M, M') \le d_{mat}(M, M')\,.$$

But by Lemma 5.5, one can produce a path $M' = M^0, M^1, \dots, M^j$ of length j in G_{ncm} such that

- M^{r+1} is a flip of M^r for $r < j$,
- $\pi_0(M, M^{r+1}) = \pi_0(M, M^r) + 1$ for $r < j$, and
- $M^j = M$.

Hence $n = \pi_0(M, M) = \pi_0(M, M') + j$, and $j = n - \pi_0(M, M')$. Therefore

$$d_{ncm}(M, M') \le n - \pi_0(M, M')\,,$$

and the theorem follows. □

The following corollaries are immediate.

Corollary 5.8. *The graphs G_{ncm} and $NC(n)$ have diameter $n-1$.*

Corollary 5.9. *The pair of noncrossing matchings (M, M') is a meander if and only if M, M' are a diameter in G_{ncm}.*

Remark 5.10. Theorem 5.7 and Corollaries 5.8 and 5.9 have been proved independently by H. Tracy Hall [Hal].

Remark 5.11. The proof of Theorem 5.7 shows that $d_{mat}(M, M') = n - \pi_0(M, M')$ for arbitrary matchings M, M' on $\langle 0, 2n-1\rangle$. Note that the analogue of Lemma 5.5 for arbitrary matchings is trivial: the difficulty in the proof of Lemma 5.5 was showing that the matching created by the desired flip was noncrossing.

We close this section with the following important observation.

Proposition 5.12. *For any noncrossing matching M, we have $\pi_0(M, M(-1)) = 1$, and $d_{ncm}(M, M(-1)) = n - 1$.*

Proof. We proceed by induction on n; check the claim directly for $n = 1$.

The noncrossing matching M contains at least one pair of the form $\{i, i+1\}$. Since $\pi_0(M, M') = \pi_0(M(k), M'(k))$ for any integer k, we can suppose without loss of generality that $i = 2n - 2$. Let $\widehat{M}$ be the noncrossing matching on $\langle 0, 2n-3\rangle$ obtained by omitting the pair $\{2n-2, 2n-1\}$ from M. Suppose that $\{0, j\} \in M$.

Note that the pairs in M are exactly the same as those in $\widehat{M}$, with the addition of $\{2n-2, 2n-1\}$. Moreover, the pairs in $M(-1)$ are exactly the same as those in $\widehat{M}(-1)$, except for the deletion of $\{j-1, 2n-3\}$ and the addition of $\{j-1, 2n-1\}$ and $\{2n-2, 2n-3\}$.

By the induction hypothesis, $\pi_0(\widehat{M}, \widehat{M}(-1)) = 1$, so that any $a, b \in \langle 0, 2n-3\rangle$ are joined by a sequence of pairs in $\widehat{M}$ and $\widehat{M}(-1)$. Precisely the same sequence of pairs in will join them in the system of meanders of M and $M(-1)$, except that the pair $j-1 \sim 2n-3$ must be replaced by $j-1 \sim 2n-1 \sim 2n-2 \sim 2n-3$ wherever it occurs. Hence $\langle 0, 2n-3\rangle$ is contained in a single block of $\Pi_0(M, M(-1))$, and since $2n-3 \sim 2n-2 \sim 2n-1$, we conclude that $\Pi_0(M, M(-1))$ has just one block, as desired. □

Remark 5.13. The same statement is false for arbitrary matchings (for instance, it fails for the matching $M = \{\{i, n+i\} \ : \ 0 \le i \le n-1\}$).

Proposition 5.12 has the following interesting geometric consequences. First, granting Lemma 2.1, we obtain a new proof of Proposition 2.5, at least in the case when f is completely generic: from the nonsingular locus of $\mathcal{C}(f)$ we obtain $\mathcal{N}(f) = (M_1, \ldots, M_n = M_1(-1))$ in which M_{r+1} is either a flip of M_r or equal to M_r. But if $M_{r+1} = M_r$, we obtain a path from M_1 to $M_1(-1)$ in G_{ncm} of length less than $n-1$, so M_{r+1} must be a flip of M_r.

Second, if $\mathcal{N} = (M_1, \ldots, M_n)$ is a necklace, then $M_r \neq M_s$ if $r < s$, or else the sequence of flips $M_1, \ldots, M_r, M_{s+1}, \ldots, M_n = M_1(-1)$ would be too short a path from M_1 to $M_1(-1)$. (Note that before Proposition 5.12 we only knew $M_r \neq M_s$ for $s = r+1$, not arbitrary $s > r$.) Therefore, if $f(z)$ is a polynomial of degree greater than two, it is not possible for $M_\theta(f)$ to flip from M to M' and then from M' back to M, as θ increases. This does not seem to be obvious from the geometry alone.

6. Basketballs and Meanders

Throughout this section, we will realize basketballs (M, M') geometrically, as follows. Draw a circle with $4n$ marked points, labelled $0, \frac{1}{2}, 1, \ldots, 2n - \frac{1}{2}$ counterclockwise around the circle. For each pair $\{i, j\} \in M$, draw an arc joining the points marked i, j, and for each pair $\{i, j\} \in M'$, draw an arc joining the points marked $i + \frac{1}{2}, j + \frac{1}{2}$, in such a manner that each arc of M crosses no arcs of M and exactly one arc of M', and vice-versa.

To prove that all necklaces are strong necklaces, we must first begin to understand the systems of meanders associated to basketballs. We start with some basic properties.

Proposition 6.1. *Suppose that the pair of noncrossing matchings (M, M') is a basketball. Then:*

(1) $\Pi_0(M, M')$ *is a noncrossing partition;*
(2) *If the pair* $\{s,t\} \in M$ *crosses the pair* $\{u,v\} \in M'$*, then* s,t,u,v *lie in the same block of* $\Pi_0(M,M')$*;*
(3) *If the pair* $\{s,t\} \in M$ *crosses the pair* $\{u,v\} \in M'$ *and* $s < u+\frac{1}{2} < t < v+\frac{1}{2}$*, then the three sets* $\langle s,u\rangle \cup \langle t,v\rangle$*,* $\langle u+1,t-1\rangle$*, and* $\langle v+1,2n-1\rangle \cup \langle 0,s-1\rangle$ *are unions of blocks of* $\Pi_0(M,M')$*.*

Remark 6.2. Under the hypotheses of Proposition 6.1(3), note that 6.1(2) and (3) imply that the block of $\Pi_0(M,M')$ that contains s,t,u,v is contained in $\langle s,u\rangle \cup \langle t,v\rangle$.

Remark 6.3. Note that Proposition 6.1(3) can be restated in a more uniform manner as follows. In our geometric realization of the basketball (M,M'), suppose the pair $\{s,t\} \in M$ crosses the pair $\{u,v\} \in M'$, and the points $s, u+\frac{1}{2}, t, v+\frac{1}{2}$ appear in counterclockwise order around the circle. Then the following three sets are unions of blocks of $\Pi_0(M,M')$: the integers which appear counterclockwise between $u+1$ and $t-1$; the integers which appear counterclockwise between $v+1$ and $s-1$; and the integers which appear counterclockwise between s and u or t and v.

Proof of Proposition 6.1. (2) If one of s,t is equal to one of u,v then the statement is clear, so we may assume that s,t,u,v are all distinct. Replacing M,M' by $M(k),M'(k)$ for a suitable integer k if necessary, we may suppose without loss of generality that $s < u+\frac{1}{2} < t < v+\frac{1}{2}$. Suppose that t is paired with x in M'. Then $x \notin \langle s,t\rangle$, or else $\{s,t\} \in M$ would cross $\{x,t\} \in M'$, and we would have $v = t$. Similarly, if s is paired with y in M', we must have $y \in \langle s+1,t-1\rangle$. The loop in $G(M,M')$ containing s,t contains edges $y-s-t-x$, with $x \notin \langle s,t\rangle$ and $y \in \langle s+1,t-1\rangle$. In order for the loop to close, it must contain another edge $z-w$ with $z \notin \langle s,t\rangle$ and $w \in \langle s+1,t-1\rangle$ (since the labels s,t cannot occur twice in the loop). The pair $\{z,w\}$ cannot be a pair in M, for it would cross $\{s,t\}$. But then $\{s,t\} \in M$ crosses $\{z,w\} \in M'$, and since (M,M') is a basketball, we must have $\{z,w\} = \{u,v\}$.

(1) Let P,P' be distinct blocks of $\Pi_0(M,M')$ which cross, and suppose $s,t \in P$ and $u,v \in P'$ satisfy $s < u < t < v$. In the loop of $G(M,M')$ corresponding to the block P, there must be an edge $s'-t'$ such that $s' < u < t' < v$ or $u < t' < v < s'$. In the former case, in the loop corresponding to P' there must be an edge $u'-v'$ such that $s' < u' < t' < v'$ or $v' < s' < u' < t'$, and similarly in the latter case. Since M,M' are noncrossing, one of $\{s',t'\}$ and $\{u',v'\}$ must be a pair in M, and the other in M'. By (2), s',t',u',v' lie in the same block of $\Pi_0(M,M')$, so $P = P'$.

(3) By Remark 6.3, it suffices to prove that $\langle u+1,t-1\rangle$ is a union of blocks of $\Pi_0(M,M')$. Suppose $x \in \langle u+1,t-1\rangle$ and $y \notin \langle u+1,t-1\rangle$ are a pair in either M or M'. We will derive a contradiction in each case; the idea is shown in Figure 6.

If $\{x,y\}$ is a pair in M, then either $y \in \langle s+1,u\rangle$ or $y \in \langle t+1,2n-1\rangle \cup \langle 0,s-1\rangle$ (note that $\{x,y\} \neq \{s,t\}$). In the former case, we have $y < u+\frac{1}{2} < x < v+\frac{1}{2}$, contradicting the hypothesis that only one edge of M crosses $\{u,v\} \in M'$. In the latter case, we have $y < s < x < t$ or $s < x < t < y$, contradicting the hypothesis that M is noncrossing.

Similarly, if $\{x,y\}$ is a pair in M', then either $y \in \langle t,v-1\rangle$ or $y \in \langle v+1,2n-1\rangle \cup \langle 0,u-1\rangle$ (note that $\{x,y\} \neq \{u,v\}$). In the former case, we have $s < x+\frac{1}{2} < t < y+\frac{1}{2}$, contradicting the hypothesis that $\{s,t\} \in M$ crosses only one edge in

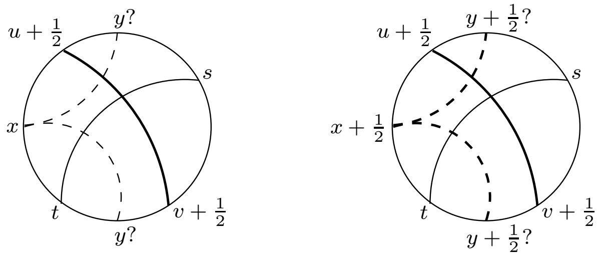

FIGURE 6. The proof of Proposition 6.1(3) for M (thin) and M' (**thick**). *Left:* the case $\{x, y\} \in M$. *Right:* the case $\{x, y\} \in M'$.

M'. In the latter case, we have $y < u < x < v$ or $u < x < v < y$, contradicting the hypothesis that M' is noncrossing. □

Remark 6.4. If M, M', and M'' are noncrossing matchings and M'' is a flip of M', then the discussion in Remark 5.3 shows that $\Pi_0(M, M'')$ is obtained by replacing two blocks $\Pi_0(M, M')$ with their union, or vice-versa. If (M, M') and (M, M'') are moreover both basketballs, then by Proposition 6.1(1), $(\Pi_0(M, M'), \Pi_0(M, M''))$ is an edge in $NC(2n)$.

We can now give the following characterization of basketballs in terms of systems of meanders.

Proposition 6.5. *Let (M, M') be a pair of noncrossing matchings. Then (M, M') is a basketball if and only if*

$$\pi_0(M, M') + \pi_0(M(-1), M') = n + 1\,, \tag{6.6}$$

or equivalently, if and only if M' lies on a path of length $n-1$ from M to $M(-1)$ in G_{ncm}.

Proof. To see that the two statements are indeed equivalent, note that (6.6) is equivalent to

$$d_{ncm}(M, M') + d_{ncm}(M(-1), M') = n - 1 \tag{6.7}$$

by Theorem 5.7. Since $d_{ncm}(M, M(-1)) = n - 1$, the only way for M' to lie on a path of length $n-1$ from M to $M(-1)$ in G_{ncm} is if (6.7) is satisfied.

In the 'only if' direction, if (M, M') is a basketball and $M' \neq M$, then we can apply Corollary 3.13 to obtain a path of length $n-1$ from M to $M(-1)$ in G_{ncm} that contains M'. If $M' = M$ the 'only if' direction is clear.

We will prove the 'if' direction by induction on $k = \pi_0(M, M')$. If $k = 1$, then (6.6) implies that $\pi_0(M(-1), M') = n$, i.e., $M' = M(-1)$. Since $(M, M(-1))$ is a basketball, the base case follows.

Now suppose that the statement is known for $k > 0$, and suppose M, M' satisfy (6.6) with $\pi_0(M, M') = k + 1$. Then $\pi_0(M(-1), M') = n - k < n$, and by Lemma 5.5 there exists a flip M'' of M' such that $\pi_0(M(-1), M'') = n - k + 1$. By the triangle inequality

$$d_{ncm}(M, M'') + d_{ncm}(M(-1), M'') \geq d_{ncm}(M, M(-1)) = n - 1\,,$$

and so we must have

$$\pi_0(M, M'') + \pi_0(M(-1), M'') \leq n + 1\,.$$

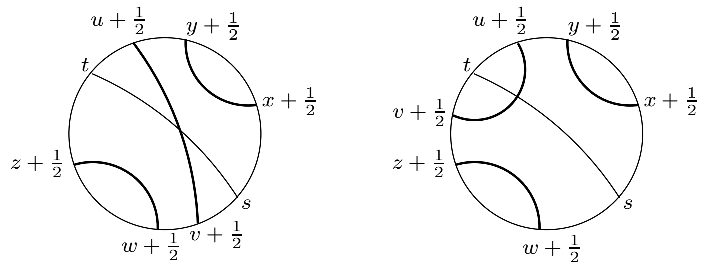

FIGURE 7. The two putative topological possibilities for $\{u, v\}$ in the proof of Proposition 6.8, with M (thin) and M' (**thick**).

Since $\pi_0(M, M'') \geq \pi_0(M, M')-1 = k$ by Remark 5.3, we conclude $\pi_0(M, M'') = k$. By the induction hypothesis (M, M'') is a basketball. We are therefore reduced to the following proposition. (Note that the roles of M' and M'' are reversed in the statement.) □

Proposition 6.8. *If (M, M') is a basketball and the noncrossing matching M'' is a flip of M' such that $\pi_0(M, M'') > \pi_0(M, M')$, then (M, M'') is also a basketball.*

Proof. Let $\{x, y\}, \{z, w\} \in M'$ be pairs that are flipped to form $\{x, w\}, \{y, z\} \in M''$, and without loss of generality suppose that $x < y < z < w$. Since $\pi_0(M, M'') > \pi_0(M, M')$, we know that x, y, z, w are contained in a single block of $\Pi_0(M, M')$. What we have to show is that our new pairs $\{x, w\}, \{y, z\} \in M''$ are each crossed by at most one (and hence exactly one) pair of M.

Suppose that M has a pair $\{s, t\}$ with $t \in \langle y+1, z\rangle$ and $s \in \langle w+1, 2n-1\rangle \cup \langle 0, x\rangle$. There are (up to easy equivalence) only two topological possibilities for the pair $\{u, v\}$ in M' that is crossed by $\{s, t\} \in M$, as shown in Figure 7. However, the first picture is impossible since $\{y, z\}$ and $\{x, w\}$ would both cross $\{u, v\}$ in M''; the second picture is impossible because z, w appear counterclockwise on the circle between $v + 1$ and $s - 1$, while x, y appear counterclockwise on the circle between s and u, and so by Proposition 6.1(3) and Remark 6.3 we could not have x, y, z, w in the same block of $\Pi_0(M, M')$. Therefore M has no such pair.

It follows that with the exception of the one pair in M which crosses $\{x, y\} \in M'$ and the one pair which crosses $\{z, w\} \in M'$, every other pair of M has both endpoints contained in the same one of the following four intervals: $\langle z + 1, w\rangle$, $\langle w + 1, 2n - 1\rangle \cup \langle 0, x\rangle$, $\langle x + 1, y\rangle$, and $\langle y + 1, z\rangle$. But no pair with both endpoints contained in the same one of those four intervals will cross either $\{y, z\} \in M''$ or $\{w, x\} \in M''$ (i.e., will cross $\{y + \frac{1}{2}, z + \frac{1}{2}\}$ or $\{w + \frac{1}{2}, x + \frac{1}{2}\}$). Thus $\{y, z\} \in M''$ and $\{x, w\} \in M''$ are each crossed by at most the two pairs of M that cross $\{x, y\}, \{z, w\} \in M'$; but since they must be crossed by an odd number of pairs of M, they are each crossed by exactly one pair of M. □

Definition 6.9. ([Ede, Section 4]) A noncrossing partition is called *k-divisible* if the cardinality of every block is divisible by k. On a set of size nk, a *maximal chain of k-divisible noncrossing partitions* is collection of k-divisible noncrossing partitions $(P_1, \ldots, P_n)$ such that P_i has $n + 1 - i$ blocks and P_i is a refinement of P_{i+1}.

Finally we arrive at our desired conclusion.

Theorem 6.10. *(1) The map which sends a necklace* $(M_1, \ldots, M_n)$ *to the sequence* $(\Pi_0(M_1, M_1), \Pi_0(M_1, M_2), \ldots, \Pi_0(M_1, M_n))$ *is a bijection between the set of necklaces of order* n *and the set of maximal chains of* 2*-divisible noncrossing partitions on* $\langle 0, 2n-1 \rangle$.

(2) Every necklace is a strong necklace.

Proof. First we must check that $(\Pi_0(M_1, M_1), \Pi_0(M_1, M_2), \ldots, \Pi_0(M_1, M_n))$ is a maximal chain of 2-divisible noncrossing partitions on $\langle 0, 2n-1 \rangle$. Each partition $\Pi_0(M_1, M_i)$ is noncrossing by Proposition 6.1(1), and 2-divisible since each block is a union of blocks of M_1. Since $\pi_0(M_1, M_1) = n$ and $\pi_0(M_1, M_n) = \pi_0(M_1, M_1(-1)) = 1$, and since we know that $\pi_0(M_1, M_{i+1})$ differs by at most 1 from $\pi_0(M_1, M_i)$, we deduce that $\pi_0(M_1, M_i) = n + 1 - i$. By Remark 6.4, $\Pi_0(M_1, M_i)$ is a refinement of $\Pi_0(M_1, M_{i+1})$ for each $i < n$. This completes the check.

Next we show that this map is injective. Suppose that a chain of 2-divisible noncrossing partitions $P_1, \ldots, P_n$ arises from a necklace $M_1, \ldots, M_n$; we must show that this necklace is uniquely determined. Certainly we must have $M_1 = \Pi_0(M_1, M_1) = P_1$. Suppose we know that $M_1, \ldots, M_i$ are determined uniquely, and suppose $M_1, \ldots, M_i, M'_{i+1}, \ldots, M'_n$ is a necklace yielding $P_1, \ldots, P_n$.

To fix notation, suppose that $P_{i+1} = \Pi_0(M_1, M'_{i+1})$ is obtained from $\Pi_0(M_1, M_i)$ by replacing the two blocks B_1, B_2 with their union. Returning to the analysis of Remark 5.3, observe that in the oriented cyclic ordering on the block $B_1 \cup B_2$, the elements of B_1 are traversed in the same order as they were in B_1 (with some elements of B_2 possibly intervening), and similarly for B_2. By repeated use of this observation, it follows that the oriented cyclic ordering on $B_1 \cup B_2$ in $\Pi_0(M_1, M'_{i+1})$ must be precisely the oriented cyclic ordering obtained by taking the oriented cyclic ordering on $\langle 0, 2n-1 \rangle$ in $\Pi_0(M_1, M'_n)$ and deleting all elements except those in $B_1 \cup B_2$. But $M'_n = M_n = M_1(-1)$, so this oriented cyclic ordering is uniquely determined by M_1! Given a cyclic ordering on B_1, B_2, and $B_1 \cup B_2$, there is at most one flip of edges in the loops of $G(M_1, M_i)$ corresponding to B_1 and B_2 that yields a loop with the desired ordering on $B_1 \cup B_2$. This must be the flip which transforms M_i to M_{i+1}, and so $M'_{i+1} = M_{i+1}$. Injectivity follows by induction.

We know that the set of strong necklaces of order n are a subset of the set of necklaces of order n, and we have now shown that there is an injective map from the set of necklaces of order n to the set of maximal chains of 2-divisible noncrossing partitions on $\langle 0, 2n-1 \rangle$. From Proposition 3.14, the number of strong necklaces of order n is $2(2n)^{n-2}$. By [Ede, Corollary 4.3], the number of maximal chains of 2-divisible noncrossing partitions on $\langle 0, 2n-1 \rangle$ is also $2(2n)^{n-2}$. Both parts of the theorem follow. □

7. A map the other way, and enumerative consequences

In this section we construct an injective map from the set of maximal chains of 2-divisible noncrossing partitions to the set of necklaces; combined with the first three paragraphs of the proof of Theorem 6.10, this furnishes another proof that these two sets are in bijection. The construction relies on recent work of Armstrong [Arm] that was not available when the first version of this article was written, and we thank the referee for bringing Armstrong's memoir to our attention, and suggesting the construction. Combined with results from Section 6, the construction yields some further enumerative properties of basketballs and necklaces.

We begin by setting up some notation. Let S_n denote the symmetric group on $\{0, \ldots, n-1\}$, and let c denote the Coxeter element $(0\ 1\ \cdots\ n-1)$. If $\sigma \in S_n$, its length $\ell(\sigma)$ is defined to be the least number of transpositions required to express σ as a product of transpositions. If σ is a product of r disjoint cycles (including trivial cycles), then $\ell(\sigma) = n - r$. We write $\sigma \le \tau$ if $\ell(\sigma) + \ell(\sigma^{-1}\tau) = \ell(\tau)$, or equivalently if there exists an expression for τ as a product of $\ell(\tau)$ transpositions for which the first $\ell(\sigma)$ transpositions multiply to give σ. Since conjugation preserves cycle types, "first" can equivalently be replaced with "last" in the preceding sentence.

Let $NC(A_{n-1})$ denote the set of permutations σ such that $\sigma \le c$. (The notation $NC(A_{n-1})$ is chosen for consistency with [Arm].) The following property of $NC(A_{n-1})$ is standard: if $\mathcal{P} \in NC(n)$, then the map $\mathcal{P} \mapsto \sigma(\mathcal{P})$ defined in Section 4 is a lattice isomorphism $NC(n) \to NC(A_{n-1})$, where σ lies immediately below τ in $NC(A_{n-1})$ if there is a transposition t such that $\tau = \sigma t$ and $\ell(\tau) = \ell(\sigma) + 1$. If M is a noncrossing matching on $\langle 0, 2n-1\rangle$, we let $\sigma(M)$ denote the permutation $\sigma(\mathcal{P}(M))$. Let $\mathcal{S}(n)$ denote the set of n-tuples of permutations $(\sigma(M_1), \ldots, \sigma(M_n))$ associated to necklaces $(M_1, \ldots, M_n)$.

Let $NC^{(2)}(n)$ denote the set of 2-divisible noncrossing partitions of $\langle 0, 2n-1\rangle$, and let $NC^{(2)}(A_{n-1})'$ denote the set of all pairs $(\sigma, \tau) \in NC(A_{n-1}) \times NC(A_{n-1})$ such that $\ell(\sigma\tau) = \ell(\sigma) + \ell(\tau)$ and $\sigma\tau \le c$. More concretely, $(\sigma, \tau) \in NC^{(2)}(A_{n-1})'$ if and only if c can be written as a product of $n-1$ transpositions in which there is a block of $\ell(\sigma)$ transpositions that multiply to σ, to the right of which there is a block of $\ell(\tau)$ transpositions that multiply to τ. The set $NC^{(2)}(n)$ is a poset, where $\mathcal{P}$ lies below $\mathcal{P}'$ if the blocks of $\mathcal{P}'$ are unions of blocks of $\mathcal{P}$; the set $NC^{(2)}(A_{n-1})'$ is a poset, where $(\sigma, \tau) \le (\sigma', \tau')$ if and only if $\sigma \le \sigma'$ and $\tau \le \tau'$. The following is a theorem of Armstrong, adapted to our notation.

Theorem 7.1. [Arm, Theorem 4.3.8] *There is a poset anti-isomorphism from $NC^{(2)}(n)$ to $NC^{(2)}(A_{n-1})'$.*

Remark 7.2. To relate Armstrong's statement of this theorem to the statement above, note that Armstrong's $NC^{(2)}(A_{n-1})$ [Arm, Definition 3.3.1(2)] consists of pairs (π_1, π_2) such that $\pi_1 \le \pi_2 \le c$, with the ordering $(\pi_1, \pi_2) \le (\mu_1, \mu_2)$ if and only if $\mu_1^{-1}\mu_2 \le \pi_1^{-1}\pi_2$ and $\mu_2^{-1}c \le \pi_2^{-1}c$. Then set $\sigma = \pi_1^{-1}\pi_2$ and $\tau = \pi_2^{-1}c$.

We now explain how to associate a necklace (more precisely, an element of $\mathcal{S}(n)$) to a maximal chain of 2-divisible noncrossing partitions of $\langle 0, 2n-1\rangle$. It follows immediately from Theorem 7.1 that maximal chains of 2-divisible noncrossing partitions are in bijection with chains of pairs $(\sigma_1, \tau_1) < \cdots < (\sigma_n, \tau_n)$ in $NC^{(2)}(A_{n-1})'$ such that $\sigma_1 = \tau_1 = 1$, $\sigma_n\tau_n = c$, and for each $1 \le i < n$ there is a transposition t_i such that either $\sigma_{i+1} = t_i\sigma_i$ and $\tau_{i+1} = \tau_i$, or else $\tau_{i+1} = t_i\tau_i$ and $\sigma_{i+1} = \sigma_i$.

Let $\ell = \ell(\sigma_n)$ and $\ell' = \ell(\tau_n) = n - 1 - \ell$. Observe that there exist $a_1 < \cdots < a_\ell$ and $b_1 < \cdots < b_{\ell'}$ such that $\sigma_n = t_{a_\ell} \cdots t_{a_1}$ and $\tau_n = t_{b_{\ell'}} \cdots t_{b_1}$. If j, j' are the largest integers such that $a_j, b_{j'} < i$, then $\sigma_i = t_{a_j} \cdots t_{a_1}$ and $\tau_i = t_{b_{j'}} \cdots t_{b_1}$.

To our chain of pairs (σ_i, τ_i), we associate the sequence $\rho_i = \sigma_i\tau_n\tau_i^{-1}$ for $1 \le i \le n$. Note that $\rho_1 = \tau_n = \sigma_n^{-1}c$ and $\rho_n = \sigma_n$. Since the transpositions in the product

$$t_{a_j} \cdots t_{a_1} t_{b_{\ell'}} \cdots t_{b_{j'+1}} = \rho_i \tag{7.3}$$

are a subsequence of those in the product $\sigma_n\tau_n = c$, it follows that $\rho_i \le c$ and the product (7.3) is a shortest length product for ρ_i. Moreover, ρ_i and ρ_{i+1} are adjacent in $NC(A_{n-1})$, with $\ell(\rho_{i+1}) = \ell(\rho_i) + 1$ if $i = a_{j+1}$ and $\ell(\rho_{i+1}) = \ell(\rho_i) - 1$

if $i = b_{j'+1}$. It follows that $\rho_n, \ldots, \rho_1$ is the sequence of permutations associated to a necklace from M to $M(-1)$, where M is the matching for which $\sigma_n = \sigma(M)$ and $\tau_n = \sigma(M(-1))$.

On the other hand, from the sequence $\rho_n, \ldots, \rho_1$ we can recover the original permutations (σ_i, τ_i) as follows. Define $\sigma_1 = \tau_1 = 1$. If $\ell(\rho_{i+1}) = \ell(\rho_i) + 1$ with $\rho_{i+1} = t_i\rho_i$, set $\sigma_{i+1} = t_i\sigma_i$ and $\tau_{i+1} = \tau_i$. If $\ell(\rho_{i+1}) = \ell(\rho_i) - 1$ with $\rho_i = \rho_{i+1}t_i$, set $\sigma_{i+1} = \sigma_i$ and $\tau_{i+1} = t_i\tau_i$. We conclude that the map sending $(\sigma_1, \tau_1), \ldots, (\sigma_n, \tau_n)$ to $\rho_n, \ldots, \rho_1$ is one-to-one. By Theorem 6.10, it is a bijection onto the set $\mathcal{S}(n)$.

The following is an immediate consequence of the above argument.

Proposition 7.4. *Suppose $M \in G_{ncm}$ and let $\ell = \ell(\sigma(M))$. Giving a necklace from M to $M(-1)$ is equivalent to giving the following data: a shortest length factorization of $\sigma(M)$; a shortest length factorization of $\sigma(M(-1)) = \sigma(M)^{-1}c$; and a subset of $\langle 1, n-1\rangle$ of size ℓ.*

Proof. Given a sequence $(\sigma_1, \tau_1), \ldots, (\sigma_n, \tau_n)$ corresponding to such a necklace, so that $\sigma_n = \sigma(M)$, the associated factorization of $\sigma(M)$ is $t_{a_\ell} \cdots t_{a_1}$, the associated factorization of $\sigma(M)^{-1}c$ is $t_{b_{\ell'}} \cdots t_{b_1}$, and the associated set of size ℓ is $\{a_1, \ldots, a_\ell\}$. In the reverse direction, the factorizations of $\sigma(M)$ and $\sigma(M)^{-1}c$ determine the sets $\{\sigma_1, \ldots, \sigma_n\}$ and $\{\tau_1, \ldots, \tau_n\}$ in the obvious manner, and the subset of size ℓ in $\langle 1, n-1\rangle$ is the set of integers i such that $\sigma_{i+1} \neq \sigma_i$; its complement is the set of integers i such that $\tau_{i+1} \neq \tau_i$. □

Definition 7.5. If σ is a permutation of length ℓ and cycle type $(m_1, \ldots, m_{n-\ell})$ including trivial cycles, set

$$\mathcal{B}(\sigma) = \binom{\ell}{m_1 - 1, \ldots, m_{n-\ell} - 1} m_1^{m_1-2} \cdots m_{n-\ell}^{m_{n-\ell}-2}$$

and

$$\mathcal{C}(\sigma) = C_{m_1} \cdots C_{m_{n-\ell}}$$

where C_m denotes the mth Catalan number.

Then $\mathcal{B}(\sigma)$ is the number of factorizations of σ into ℓ transpositions: indeed, it is well-known that the number of shortest factorizations of an m-cycle is m^{m-2} (see, e.g., [Kre]); each minimal factorization of σ consists of a minimal factorization of each cycle, and the binomial coefficient gives the number of ways of interleaving the factorizations of the cycles. Similarly, $\mathcal{C}(\sigma)$ is the number of permutations u such that $u \leq \sigma$.

Corollary 7.6. *Let $\sigma = \sigma(M)$. The number of necklaces from M to $M(-1)$ is*

$$\binom{n-1}{\ell(\sigma)} \mathcal{B}(\sigma)\mathcal{B}(\sigma^{-1}c)\,.$$

Proposition 7.7. *Let $\sigma = \sigma(M)$. The number of noncrossing matchings M' such that (M, M') is a basketball is*

$$\mathcal{C}(\sigma)\mathcal{C}(\sigma^{-1}c)\,.$$

Proof. By Proposition 6.5, the noncrossing matchings M' such that (M, M') is a basketball are precisely the noncrossing matchings which lie on necklaces from M to $M(-1)$; by the preceding discussion, these are the noncrossing matchings M' such that $\sigma(M') = uv$ with $u \leq \sigma$ and $v \leq \sigma^{-1}c$. To complete the proof, we have to check that the product uv uniquely determines u and v.

Let $\wedge$ denote meet (greatest lower bound) in the lattice $NC(A_{n-1})$. We will show that $u = \sigma \wedge uv$; certainly $u \le \sigma \wedge uv$. Set $K^{\nu}_{\mu}(w) = \mu w^{-1}\nu$. If $\mu \le \pi \le \nu$, then [Arm, Theorem 2.6.14] gives the formula

$$K^{\nu}_{\mu}(\pi) = \nu \wedge K^{c}_{\mu}(\pi)\,.$$

Putting $\mu = u$ and $\nu = \pi = \sigma$ gives $u = \sigma \wedge u(\sigma^{-1}c)$. Then

$$u \le \sigma \wedge uv \le \sigma \wedge u(\sigma^{-1}c) = u\,,$$

completing the proof. □

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