Examples of reducible curves with 27 points

We give examples to show that each of the possibilities enumerated in Section 5.2 of [Sav] do indeed occur. Recall that $\eta \in F_8$ is a chosen root of $\eta^3 + \eta + 1 = 0$. Let $\beta$ be a generator of $F_{64}$ such that $\beta^9 = \eta$.

Each intersection described below has exactly 27 points over $F_8$, and each was found in the computer calculations whose output is available at http://www.math.mcgill.ca/~dsavitt/curves/.

- The intersection of $XY + ZW = 0$ with the cubic $(X + W)(X + \eta^3 Y + \eta^{-2} Z)(X + \eta^{-3} Y + \eta^{-3} W) = 0$ consists of three geometrically irreducible conics. Let the conics be $C_1$, $C_2$, and $C_3$ respectively (in the order of the factors of the cubic). Then $C_1$ and $C_2$ meet in the two $F_{64}$-points $[\beta^{24} : \beta^{-24} : 1 : 1]$ and $[\beta^5 : \beta^{-5} : 1 : 1]$; $C_1$ and $C_3$ meet in the two $F_{64}$-points $[\beta^2 : \beta^{-2} : 1 : 1]$ and $[\beta^4 : \beta^{-4} : 1 : 1]$; and $C_2$ and $C_3$ meet in the two $F_{64}$-points $[\beta^2 : \beta^{46} : 1]$ and $[\beta^{16} : \beta^{37} : \beta^{53} : 1]$. The quadric and the cubic intersect in 189 points over $F_{64}$.
- The intersection of $XY + ZW = 0$ with the cubic $(X + Y)(Y + Z + W)(X + Z + W) = 0$ consists of three geometrically irreducible conics. The three conics have no intersection over $F_8$, and over $F_{64}$ all three conics pass through the conjugate points $[\beta^{21} : \beta^{21} : \beta^{42} : 1]$ and $[\beta^{42} : \beta^{42} : \beta^{21} : 1]$. The quadric and the cubic intersect in 191 points over $F_{64}$.
- The intersection of $XY + ZW = 0$ with the cubic $X^3 Y + \eta X Y W + \eta^{-1} X Z W + \eta^{-3} X W^2 + \eta Y^2 Z + Y^2 W + \eta^{-5} Y Z^2 + \eta^{-1} Y Z W + Y W^2 = 0$ contains the line $[X : 0 : Z : 0]$ and a component of degree 5, and has 119 points over $F_{64}$.
- The intersection of $XY + ZW = 0$ with the cubic $X^3 Y + X Y Z + X Y W + X W^2 + \eta Y^2 Z + Y^2 W + \eta Y Z^2 + Y W^2 = 0$ contains the two non-intersecting lines $[X : 0 : Z : 0]$ and $[X : Y : Y : X]$ and a component of degree 4, and has 195 points over $F_{64}$.
- The intersection of $XY + ZW = 0$ with the cubic $W(X^2 + Z^2 + \eta X Y + \eta^{-3} X Z + \eta^{-1} X W + Y^2 + Y Z + Y W) = 0$ contains the two intersecting lines $[0 : Y : Z : 0]$ and $[X : 0 : Z : 0]$. The intersection of $XY + ZW = 0$ and $X^2 + Z^2 + \eta X Y + \eta^{-3} X Z + \eta^{-1} X W + Y^2 + Y Z + Y W = 0$ is a curve of arithmetic genus 1 with 10 $F_8$-points and 64 $F_{64}$-points, and is singular at $[\eta : \eta^{-3} : \eta^{-2} : 1]$. The curve of genus 1 meets the line $[0 : Y : Z : 0]$ at $[0 : 1 : \beta^{21} : 0]$ and $[0 : 1 : \beta^{42} : 0]$ and the line $[X : 0 : Z : 0]$ at $[1 : 0 : \beta^{28} : 0]$ and $[1 : 0 : \beta^{35} : 0]$. The intersection of the quadric and the cubic has 189 points over $F_{64}$.
- The intersection of $XY + ZW = 0$ with the cubic $W(X^2 + Z^2 + \eta X Y + X Z + \eta^2 X W + Y^2 + Y Z + Y W) = 0$ contains the two intersecting lines $[0 : Y : Z : 0]$ and $[X : 0 : Z : 0]$. The intersection of $XY + ZW = 0$ and $X^2 + Z^2 + \eta X Y + X Z + \eta^2 X W + Y^2 + Y Z + Y W = 0$ is an elliptic curve with 10 $F_8$-points and 80 $F_{64}$-points. The curve of genus 1 meets the line $[0 : Y : Z : 0]$ at $[0 : 1 : \beta^{21} : 0]$ and $[0 : 1 : \beta^{42} : 0]$ and the line $[X : 0 : Z : 0]$ at $[1 : 0 : \beta^{21} : 0]$ and $[1 : 0 : \beta^{12} : 0]$. The intersection of the quadric and the cubic has 205 points over $F_{64}$.
• The intersection of $XY + ZW = 0$ with the cubic $Y(X^2 + \eta YZ + YW + \eta^{-1} Z^2 + \eta^{-3} ZW + W^2) = 0$ contains the two intersecting lines $[X : 0 : Z : 0]$ and $[X : 0 : 0 : W]$. The intersection of $XY + ZW = 0$ and $X^2 + \eta YZ + YW + \eta^{-1} Z^2 + \eta^{-3} ZW + W^2 = 0$ is an elliptic curve with 12 $F_6$-points and 72 $F_{64}$-points. This elliptic curve intersects the line $[X : 0 : Z : 0]$ in a double-point $[\eta^3 : 0 : 1 : 0]$ and the line $[X : 0 : 0 : W]$ in a double-point $[1 : 0 : 0 : 0 : 0]$, and intersection of the quadric and the cubic has 199 points over $F_{64}$.

• The intersection of $XY + ZW = 0$ with the cubic $Y(\eta^2 XW + \eta^3 YZ + \eta YW + Z^2 + \eta^{-3} ZW + W^2) = 0$ contains the two intersecting lines $[X : 0 : Z : 0]$ and $[X : 0 : 0 : W]$. The intersection of $XY + ZW = 0$ and $\eta^2 XW + \eta^3 YZ + \eta YW + Z^2 + \eta^{-3} ZW + W^2$ is an elliptic curve with 12 $F_6$-points and 72 $F_{64}$-points. This elliptic curve intersects the line $[X : 0 : Z : 0]$ in a double-point $[1 : 0 : 0 : 0]$ and meets the line $[X : 0 : 0 : W]$ singly at $[1 : 0 : 0 : 0]$ and $[1 : 0 : 0 : \eta^3]$. The intersection of the quadric and the cubic has 199 points over $F_{64}$.

• The intersection of $XY + ZW = 0$ with the cubic $XYW + XZ^2 + \eta XZW + \eta YZ^2 + YZ + YZW + YW^2 = 0$ contains the lines $[\eta W : Y : Z : W]$ and $[X : 0 : 0 : W]$. The cubic is $(XY + ZW)(\eta^{-1} W + \eta Z) + (\eta Y + Z)(YZ + XZ + \eta XW + \eta^{-1} W^2 + \eta ZW + \eta^{-1} YW)$. The intersection of $XY + ZW = 0$ with $YZ + XZ + \eta XW + \eta^{-1} W^2 + \eta ZW + \eta^{-1} YW = 0$ is an elliptic curve with 12 $F_6$-points and 72 $F_{64}$-points. It meets the line $[X : 0 : 0 : 0]$ at the two points $[1 : 0 : 0 : 0]$ and $[1 : 0 : 0 : \eta^2]$, and meets the line $[\eta W : Y : Z : W]$ at the two Galois-conjugate points $[\beta^{\overline{3}} : 1 : \beta^3 : \beta^{60}]$ and $[\beta^{\overline{3}} : 1 : \beta^9 : \beta^{22}]$. The intersection of the quadric and the cubic has 197 points over $F_{64}$.

• The intersection of $XY + ZW = 0$ with the cubic $W(XZ + \eta XW + Y^2 + \eta YZ + YW) = 0$ contains the two intersecting lines $[X : 0 : Z : 0]$ and $[0 : Y : Z : 0]$. The intersection of $XY + ZW = 0$ and $XZ + \eta XW + Y^2 + \eta YZ + YW = 0$ is an elliptic curve with 13 $F_6$-points and 65 $F_{64}$-points. The elliptic curve meets both lines at $[0 : 0 : 0 : 1]$ and also meets $[X : 0 : Z : 0]$ and $[0 : Y : Z : 0]$ at $[1 : 0 : 0 : 0]$ and $[0 : \eta^3 : 1 : 0]$ respectively. The intersection of the quadric and the cubic has 191 points over $F_{64}$.

• The intersection of $XY + ZW = 0$ with the cubic $Y(XZ + \eta XW + \eta^{-2} X^2 + \eta YZ + YW + \eta^{-3} Z^2 + ZW + W^2) = 0$ contains the two lines $[X : 0 : Z : 0]$ and $[X : 0 : 0 : W]$. The intersection of $XY + ZW = 0$ and $XZ + \eta XW + \eta^{-2} X^2 + \eta YZ + YW + \eta^{-3} Z^2 + ZW + W^2$ is an elliptic curve with 14 $F_6$-points and 56 $F_{64}$-points. The elliptic curve meets the line $[X : 0 : Z : 0]$ at the points $[1 : 0 : \eta^2 : 0]$ and $[1 : 0 : \eta^{-2} : 0]$, and the line $[X : 0 : 0 : W]$ at the points $[1 : 0 : 0 : n]$ and $[1 : 0 : 0 : \eta^3]$. The intersection of the quadric and the cubic has 181 points over $F_{64}$.

• The intersection of $XY + ZW = 0$ with the cubic $\eta^2 X^2 Z + \eta X Y Z + \eta^3 X Y W + \eta^{-2} X Z^2 + \eta^3 X Z W + Y^2 W + \eta^3 Y Z W + Y W^2 = 0$ contains the three non-intersecting lines $[0 : Y : Z : 0]$, $[X : 0 : 0 : W]$, and $[X : Y : X : Y]$ and three lines defined over $F_{612}$. The intersection has 195 points over $F_{64}$.

References