BREUIL MODULES FOR RAYNAUD SCHEMES

DAVID SAVITT

Abstract. Let p be an odd prime, and let $\mathcal{O}_K$ be the ring of integers in a finite extension $K/\mathbb{Q}_p$. Breuil has classified finite flat group schemes of type $(p, \ldots, p)$ over $\mathcal{O}_K$ in terms of linear-algebraic objects that have come to be known as Breuil modules. This classification can be extended to the case of finite flat vector space schemes $\mathcal{G}$ over $\mathcal{O}_K$. When $\mathcal{G}$ has rank one, the generic fiber of $\mathcal{G}$ corresponds to a Galois character, and we explicitly determine this character in terms of the Breuil module of $\mathcal{G}$. Special attention is paid to Breuil modules *with descent data* corresponding to characters of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_{p^d})$ that become finite flat over a totally ramified extension of degree $p^d - 1$; these arise in Gee's work on the weight in Serre's conjecture over totally real fields.

Introduction

Let p be an odd prime, let $\mathcal{O}_K$ be the ring of integers in a finite extension $K/\mathbb{Q}_p$, and fix a uniformizer π of $\mathcal{O}_K$. Breuil [Bre00] has given a classification of finite flat group schemes of type $(p, \ldots, p)$ over $\mathcal{O}_K$, in terms of linear-algebraic objects that have come to be known as Breuil modules. The group schemes studied by Breuil are the integral models of group schemes over K arising from $\mathbb{F}_p$-representations of $G_K = \mathrm{Gal}(\overline{K}/K)$. We begin in Section 1 by giving an extension of Breuil's classification to the case of finite flat E-module group schemes, where E is an Artinian local $\mathbb{F}_p$-algebra. This follows formally from Breuil's work, and we make no claim of originality here; however, this extension does not seem to be in the literature, and will be useful here and elsewhere (e.g. in [Gee06]).

Suppose now that E is a finite field. If $\mathcal{G}$ is a finite flat E-vector space scheme of rank one over $\mathcal{O}_K$, then the generic fiber of $\mathcal{G}$ corresponds to a character $G_K \to E^\times$. In Section 2, we explicitly determine this character in terms of the Breuil module of $\mathcal{G}$, subject to the hypothesis (familiar from [Ray74]) that E contains the residue field of $\mathcal{O}_K$.

In the final section, we consider the same problem for Breuil modules *with descent data* corresponding to characters of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_{p^d})$ that become finite flat over a totally ramified extension of degree $p^d - 1$. This result is used in Gee's proof [Gee06] of many cases of the Buzzard-Diamond-Jarvis conjecture on weights of mod p Hilbert modular forms. In fact this note had originally been intended to be an appendix to Gee's paper; we are publishing it separately at the suggestion of the referee for that paper.

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1. Breuil modules with coefficients, and E-module schemes

We retain the notation of the introduction. Let $\mathbf{k}$ be the residue field of $\mathcal{O}_K$, let e be the absolute ramification index of K, and as above let E (the coefficients) be an Artinian local $\mathbb{F}_p$-algebra. Let ϕ denote the endomorphism of $(\mathbf{k}\otimes_{\mathbb{F}_p} E)[u]/u^{ep}$ which is the pth power map on $\mathbf{k}$ and u, and the identity on E. We define $\mathrm{BrMod}_{\mathcal{O}_K,E}$ to be the category of triples $(\mathcal{M}, \mathcal{M}_1, \phi_1)$ where:

- $\mathcal{M}$ is a finitely generated $(\mathbf{k}\otimes_{\mathbb{F}_p} E)[u]/u^{ep}$-module which is free when regarded as a $\mathbf{k}[u]/u^{ep}$-module,
- $\mathcal{M}_1$ is a $(\mathbf{k}\otimes_{\mathbb{F}_p} E)[u]/u^{ep}$-submodule of $\mathcal{M}$ containing $u^e\mathcal{M}$, and
- ϕ_1 is a ϕ-semilinear additive map $\mathcal{M}_1 \to \mathcal{M}$ such that $\phi_1(\mathcal{M}_1)$ generates $\mathcal{M}$ over $(\mathbf{k}\otimes_{\mathbb{F}_p} E)[u]/u^{ep}$.

The objects of $\mathrm{BrMod}_{\mathcal{O}_K,E}$ are called *Breuil modules with coefficients* (or simply Breuil modules). Morphisms of Breuil modules are $(\mathbf{k}\otimes_{\mathbb{F}_p} E)[u]/u^{ep}$-linear maps which preserve $\mathcal{M}_1$ and commute with ϕ_1. We will sometimes abuse notation and denote the Breuil module $(\mathcal{M}, \mathcal{M}_1, \phi_1)$ simply by $\mathcal{M}$.

Proposition 1.1. *For each choice of π, there is an anti-equivalence of categories between $\mathrm{BrMod}_{\mathcal{O}_K,E}$ and the category of finite flat E-module schemes over $\mathcal{O}_K$.*

Proof. When the coefficient ring E is $\mathbb{F}_p$, this result is Théorème 3.3.7 of [Bre00]. If $(\mathcal{M}, \mathcal{M}_1, \phi_1)$ is an object in $\mathrm{BrMod}_{\mathcal{O}_K,E}$, note that by forgetting the action of E we obtain an object in $\mathrm{BrMod}_{\mathcal{O}_K,\mathbb{F}_p}$. Indeed, the only thing to be checked is that $\phi_1(\mathcal{M}_1)$ generates $\mathcal{M}$ as a $\mathbf{k}[u]/u^{ep}$-module, which follows because ϕ_1 is E-linear. Note that morphisms in $\mathrm{BrMod}_{\mathcal{O}_K,E}$ are precisely the morphisms in $\mathrm{BrMod}_{\mathcal{O}_K,\mathbb{F}_p}$ which commute with the action of E.

By Théorème 3.3.7 and Proposition 2.1.2.2 of [Bre00] we have an anti-equivalence of categories $\mathcal{G}_\pi$ from $\mathrm{BrMod}_{\mathcal{O}_K,\mathbb{F}_p}$ to the category of finite flat group schemes of type $(p,\dots,p)$ over $\mathcal{O}_K$. Let $\mathcal{M}_\pi$ denote a quasi-inverse of $\mathcal{G}_\pi$. Let $\mathcal{M}^0$ denote $\mathcal{M}$ regarded as an object of $\mathrm{BrMod}_{\mathcal{O}_K,\mathbb{F}_p}$, and observe that we have a map $E \to \mathrm{End}_{\mathrm{BrMod}_{\mathcal{O}_K,\mathbb{F}_p}}(\mathcal{M}^0)$. It follows without difficulty that the group scheme $\mathcal{G}_\pi(\mathcal{M}^0)$ has the structure of an E-module scheme. Conversely, suppose that $\mathcal{G}$ is an E-module scheme. Then $\mathcal{M} = \mathcal{M}_\pi(\mathcal{G})$ is an object in $\mathrm{BrMod}_{\mathcal{O}_K,\mathbb{F}_p}$ with a map $E \to \mathrm{End}_{\mathrm{BrMod}_{\mathcal{O}_K,\mathbb{F}_p}}(\mathcal{M})$. Since endomorphisms of Breuil modules with $\mathbb{F}_p$-coefficients are $\mathbf{k}[u]/u^{ep}$-linear, we deduce that $\mathcal{M}$ is a $(\mathbf{k}\otimes_{\mathbb{F}_p} E)[u]/u^{ep}$-module. □

We now examine more closely the structure of Breuil modules with coefficients. Let $\mathbf{k}_0$ be the largest subfield of $\mathbf{k}$ which embeds into E (equivalently, into the residue field of E), and let S denote the set of embeddings of $\mathbf{k}_0$ into E. We will allow φ to stand for the pth power map on any finite field. For each $\sigma \in S$ let $(\mathbf{k}E)_\sigma$ denote the Artinian local ring $\mathbf{k}\otimes_{\mathbf{k}_0,\sigma} E$, so that we have an algebra isomorphism

$$\mathbf{k}\otimes_{\mathbb{F}_p} E \cong \oplus_\sigma (\mathbf{k}E)_\sigma \,.$$

We can make this isomorphism explicit, as follows. For each $\sigma \in S$, define $e_\sigma = -\sum_{x\in\mathbf{k}_0^\times} x\otimes\sigma(x)^{-1}$. It is straightforward to check that:

- $e_\sigma^2 = e_\sigma$, and $e_\sigma e_\tau = 0$ if $\sigma \neq \tau$,
- $\sum_\sigma e_\sigma = 1$, and
- $(\varphi \otimes 1)(e_\sigma) = e_{\sigma\varphi^{-1}}$,

and we may then identify $(\mathbf{k}E)_\sigma$ with $e_\sigma(\mathbf{k} \otimes_{\mathbb{F}_p} E)$. The second of the above facts follows from the formula $\sum_{x \in \mathbf{k}_0^\times} x\mathrm{Tr}_{\mathbf{k}_0/\mathbb{F}_p}(x^{-1}) = -1$.

If M is any $(\mathbf{k} \otimes_{\mathbb{F}_p} E)$-module, set $M_\sigma = e_\sigma M$. Then $M = \oplus_\sigma M_\sigma$, and M_σ can be characterized as the subset of M consisting of elements m for which $(x \otimes 1)m = (1 \otimes \sigma(x))m$ for all $x \in \mathbf{k}_0$.

Proposition 1.2. *A Breuil module with coefficients $\mathcal{M}$ which is projective as a $(\mathbf{k} \otimes_{\mathbb{F}_p} E)[u]/u^{ep}$-module is free as a $(\mathbf{k} \otimes_{\mathbb{F}_p} E)[u]/u^{ep}$-module. In particular, this is always the case when E is a field.*

Proof. The proof is the same as the proof of Lemma (1.2.2)(4) in [Kis], but we repeat it since we will make use of some of the details. It suffices to check that the ranks of the free $(\mathbf{k}E)_\sigma[u]/u^{ep}$-modules $\mathcal{M}_\sigma$ are all equal, or equivalently that the rank rk_σ of $\mathcal{M}_\sigma$ as a $\mathbf{k}[u]/u^{ep}$-module does not depend on σ. Suppose that $m \in (\mathcal{M}_1)_\sigma$. If $x \in \mathbf{k}_0$, then

$$(x \otimes 1)\phi_1(m) = \phi_1((\varphi^{-1}x \otimes 1)m) = \phi_1((1 \otimes \sigma\varphi^{-1}x)m) = (1 \otimes \sigma\varphi^{-1}x)\phi_1(m)\,.$$

By the discussion preceding the Proposition we conclude that ϕ_1 maps $(\mathcal{M}_1)_\sigma$ to $\mathcal{M}_{\sigma\varphi^{-1}}$. The map $\overline{\phi}_1 : \mathcal{M}_1/u\mathcal{M}_1 \to \mathcal{M}/u\mathcal{M}$ therefore decomposes as a direct sum of maps

$$(\mathcal{M}_1)_\sigma/u(\mathcal{M}_1)_\sigma \to \mathcal{M}_{\sigma\varphi^{-1}}/u\mathcal{M}_{\sigma\varphi^{-1}}\,. \tag{1.3}$$

But the map $\overline{\phi}_1$ is bijective; see, for instance, the discussion before Lemma 5.1.1 of [BCDT01]. Therefore the map in (1.3) is bijective. Let $M[u]$ denote the kernel of multiplication by u on M. Since $\#M[u] = \#(M/uM)$ for any finite $\mathbf{k}[u]/u^{ep}$-module M, we see that $\#((\mathcal{M}_1)_\sigma/u(\mathcal{M}_1)_\sigma) \leq \#(\mathcal{M}_\sigma/u\mathcal{M}_\sigma)$. We deduce that $\mathrm{rk}_{\sigma\varphi^{-1}} \leq \mathrm{rk}_\sigma$, and since $\mathrm{Gal}(\mathbf{k}_0/\mathbb{F}_p)$ is cyclic, the first part of the result follows.

When E is a field, we have to check that $\mathcal{M}_\sigma$ is always a free $(\mathbf{k}E)_\sigma[u]/u^{ep}$-module. But by definition $\mathcal{M}$ is a free $\mathbf{k}[u]/u^{ep}$-module, so the direct summand $\mathcal{M}_\sigma$ is a projective $\mathbf{k}[u]/u^{ep}$-module, and thus also free. Since $(\mathbf{k}E)_\sigma$ is a field, it is easy to see that any $(\mathbf{k}E)_\sigma[u]/u^{ep}$-module that is free as a $\mathbf{k}[u]/u^{ep}$-module is free. □

Let $(\mathcal{M}, \mathcal{M}_1, \phi_1)$ be a Breuil module with $\mathcal{M}$ a projective $(\mathbf{k} \otimes_{\mathbb{F}_p} E)[u]/u^{ep}$-module; define the rank of this Breuil module to be the rank of $\mathcal{M}$ as a $(\mathbf{k} \otimes_{\mathbb{F}_p} E)[u]/u^{ep}$-module. The E-linear bijection $\mathcal{M}_1/u\mathcal{M}_1 \to \mathcal{M}/u\mathcal{M}$ yields a $(\mathbf{k} \otimes_{\mathbb{F}_p} E)$-isomorphism $\mathbf{k} \otimes_{\varphi,\mathbf{k}} (\mathcal{M}_1/u\mathcal{M}_1) \to \mathcal{M}/u\mathcal{M}$, whence $\mathcal{M}_1/u\mathcal{M}_1$ is a free $(\mathbf{k} \otimes_{\mathbb{F}_p} E)$-module of the same rank as the Breuil module $\mathcal{M}$. In particular, if $\mathcal{M}$ has rank n, then each $(\mathcal{M}_1)_\sigma$ can be generated by n elements over $(\mathbf{k}E)_\sigma[u]/u^{ep}$.

Suppose now that E is a field, so that each $(\mathbf{k}E)_\sigma$ is a field. Recall [Bre00, Proposition 2.1.2.5] that every object $(\mathcal{M}, \mathcal{M}_1, \phi_1)$ of $\mathrm{BrMod}_{\mathcal{O}_K,\mathbb{F}_p}$ possesses a *suitable basis* (*base adaptée*): a basis $m_1, \ldots, m_n$ of $\mathcal{M}$ (as a free $\mathbf{k}[u]/u^{ep}$ module) such that $\mathcal{M}_1$ is generated by $u^{r_1}m_1, \ldots, u^{r_n}m_n$ for integers $0 \leq r_1, \ldots, r_n \leq e$. Note, however, that the proof of [Bre00, Proposition 2.1.2.5] does not involve ϕ_1, only $\mathcal{M}$ and $\mathcal{M}_1$; hence the same argument proves the existence of a suitable basis of $\mathcal{M}_\sigma$ with respect to $(\mathcal{M}_1)_\sigma$. We thus obtain an analogous notion of suitable basis in $\mathrm{BrMod}_{\mathcal{O}_K,E}$.

We remark that in general this is no longer possible when E is not a field. Suppose, for instance, that $E = \mathbb{F}_p[t]/t^2$ and $e \geq 2$. Let $\mathcal{M}$ be free of rank two generated by m_1, m_2, and let $\mathcal{M}_1 = \langle um_1 + xm_2, um_2 \rangle$ with $x \in E$. Then the pair $\mathcal{M}, \mathcal{M}_1$ has a suitable basis over $(\mathbf{k} \otimes_{\mathbb{F}_p} E)[u]/u^{ep}$ if and only if $x \in E^\times$.

Definition 1.4. Let $\mathcal{M}$ be an object of $\mathrm{BrMod}_{\mathcal{O}_K,E}$. For our fixed choice of uniformizer π, we obtain a finite flat E-module scheme $\mathcal{G}_\pi(\mathcal{M})$, and we have an E-representation of $\mathrm{Gal}(\overline{K}/K)$ on the points $\mathcal{G}_\pi(\mathcal{M})(\overline{K})$, which we denote $V_{st}(\mathcal{M})$. Following [BM02] and [Sav05], we set

$$T_{st,2}(\mathcal{M}) = V_{st}(\mathcal{M})\widehat{\ }(1)$$

where $\widehat{\ }$ denotes the $\mathbb{F}_p$-dual, and (1) denotes a twist by the cyclotomic character. If E is a field, then the dimension of the E-representation $T_{st,2}(\mathcal{M})$ is equal to the rank of the Breuil module $\mathcal{M}$.

Warning 1.5. When E is not a field, then even if $\mathcal{M}$ is a projective $(\mathbf{k} \otimes_{\mathbb{F}_p} E)[u]/u^{ep}$-module, we do not have a general result which says that $T_{st,2}(\mathcal{M})$ is a free E-module: the proof of [BM02, Lemma 3.2.1.3] does not succeed when the ramification index e is large. However, [Sav05, Lemma 4.9(2)] tells us that $T_{st,2}(\mathcal{M})$ is a free E-module when $\mathcal{M} = \mathcal{M}_R/I\mathcal{M}_R$ for a strongly divisible R-module $\mathcal{M}_R$ and $R/I = E$, which is always the case in the applications in [Gee06].

2. Vector space schemes arising from characters

In the remainder of this paper, E will be a field. We remark that E can naturally be identified as a subfield of $(\mathbf{k}E)_\sigma$ via $x \mapsto (1 \otimes x)e_\sigma$. In particular if $\mathbf{k}_0 = \mathbf{k}$ we can identify E with $(\mathbf{k}E)_\sigma$. Suppose that $\mathcal{G}$ is a finite flat E-vector space scheme over $\mathcal{O}_K$, with $q = \#E$. If the dimension of the corresponding E-representation of G_K on $\mathcal{G}(\overline{\mathbb{Q}}_p)$ is n, then the rank of $\mathcal{G}$ as a finite flat group scheme is nq. We will refer to n as the rank of the E-vector space scheme $\mathcal{G}$, but we point out that some authors use this term to refer to nq.

Let $(\mathcal{M}, \mathcal{M}_1, \phi_1)$ be an object of $\mathrm{BrMod}_{\mathcal{O}_K,E}$ corresponding to a finite flat E-vector space scheme over $\mathcal{O}_K$ of rank one, so that $\mathcal{M}$ is a free $(\mathbf{k} \otimes_{\mathbb{F}_p} E)[u]/u^{ep}$-module of rank one, and each $\mathcal{M}_\sigma$ is a free $(\mathbf{k}E)_\sigma[u]/u^{ep}$-module of rank one. Let $d = [\mathbf{k}_0 : \mathbb{F}_p]$, let σ_0 be any element in S, and inductively define $\sigma_{i+1} = \sigma_i \circ \varphi^{-1}$, so that $\mathcal{M} = \oplus_{i=0}^{d-1} \mathcal{M}_{\sigma_i}$, and ϕ_1 maps $(\mathcal{M}_1)_{\sigma_i}$ to $\mathcal{M}_{\sigma_{i+1}}$. We will often abbreviate $(\mathbf{k}E)_{\sigma_i}$ by $(\mathbf{k}E)_i$. Note that ϕ maps $(\mathbf{k}E)_i$ to $(\mathbf{k}E)_{i+1}$, sending $(x \otimes y)e_{\sigma_i} \mapsto (\varphi x \otimes y)e_{\sigma_{i+1}}$.

Let m_0 be any generator of $\mathcal{M}_{\sigma_0}$. Then there is an integer $r_0 \in [0, e]$ such that $(\mathcal{M}_1)_{\sigma_0}$ is generated over $(\mathbf{k}E)_0[u]/u^{ep}$ by $u^{r_0}m_0$. Define $m_1 = \phi_1(u^{r_0}m_0) \in \mathcal{M}_{\sigma_1}$, which is necessarily a generator of $\mathcal{M}_{\sigma_1}$. Iterate this construction, so that we obtain $m_i \in \mathcal{M}_{\sigma_i}$ and $r_i \in [0, e]$ for each integer $0 \leq i \leq d-1$, satisfying $\phi_1(u^{r_i}m_i) = m_{i+1}$ for $i < d-1$. Moreover we have $\phi_1(u^{r_{d-1}}m_{d-1}) = \alpha m_0$ for some

$\alpha \in ((\mathbf{k}E)_0[u]/u^{ep})^\times$. It is easy to verify that each such collection of data defines a Breuil module.

Suppose we repeat this construction, using a different generator $m_0' = \beta m_0$ of $\mathcal{M}_{\sigma_0}$. One checks without difficulty that the integers $r_0, \ldots, r_{d-1}$ are unchanged, while α is replaced by $\alpha\phi^{(d)}(\beta)/\beta$, where $\phi^{(d)}$ is the map on $(\mathbf{k}E)_0[u]/u^{ep}$ which fixes E, is φ^d on $\mathbf{k}$, and sends u to u^{p^d}. In particular, choosing $\beta = \alpha$ replaces α by $\phi^{(d)}(\alpha)$. Note that every power of u appearing in $\phi^{(d)}(\alpha)$ is divisible by u^{p^d}. Recalling that $u^{ep} = 0$, we see by iterating this procedure that it is possible to choose m_0 so that α is an element in $(\mathbf{k}E)_0$. This element of $(\mathbf{k}E)_0$ is not uniquely defined, but it does define a unique coset αH where H is the subgroup of $(\mathbf{k}E)_0^\times$ consisting of elements of the form $\phi^{(d)}(\beta)/\beta$ for $\beta \in (\mathbf{k}E)_0^\times$. However, H is precisely the kernel of the norm map $N_{(\mathbf{k}E)_0/E} : (\mathbf{k}E)_0^\times \to E^\times$, where E is identified with a subfield of $(\mathbf{k}E)_0$ as above. So, finally, we see that to the Breuil module $\mathcal{M}$ we can associate a well-defined element $\gamma = N_{(\mathbf{k}E)_0/E}(\alpha) \in E^\times$, and γ is independent of the choice of σ_0 since $N_{(\mathbf{k}E)_0/E}(\alpha) = N_{(\mathbf{k}E)_i/E}(\phi^{(i)}(\alpha))$. We have therefore proved:

Theorem 2.1. *Let $d = [\mathbf{k}_0 : \mathbb{F}_p]$. The finite flat E-vector space schemes of rank one over $\mathcal{O}_K$ are in one-to-one correspondence with d-tuples $(r_0, \ldots, r_{d-1})$ satisfying $0 \le r_i \le e$, together with an element $\gamma \in E^\times$.*

Fix a uniformizer π of $\mathcal{O}_K$ and $\sigma_0 \in S$. The corresponding Breuil modules each have the form:

- $\mathcal{M}_{\sigma_i} = (\mathbf{k}E)_i \cdot m_i$,
- $(\mathcal{M}_1)_{\sigma_i} = u^{r_i}\mathcal{M}_{\sigma_i}$, *and*
- $\phi_1(u^{r_i}m_i) = m_{i+1}$ *for* $0 \le i < d-1$ *and* $\phi_1(u^{r_{d-1}}m_{d-1}) = \alpha m_0$, *where* $\alpha \in (\mathbf{k}E)_0^\times$ *is any element with* $N_{(\mathbf{k}E)_0/E}(\alpha) = \gamma$.

Remark 2.2. Theorem 2.1 is a generalization of [Ray74, Corollaire 1.5.2]. There, Raynaud enumerates the finite flat E-vector space schemes of rank one over $\mathcal{O}_K$, under the hypothesis that the coefficient field E embeds into the residue field $\mathbf{k}$; we remove this latter hypothesis. Alternatively, let K' be an unramified extension of K such that E embeds into its residue field. One could start from Raynaud's description of finite flat E-vector space schemes of rank one over $\mathcal{O}_{K'}$, and count how many ways these schemes can obtain descent data from $\mathcal{O}_{K'}$ to $\mathcal{O}_K$. We note that Ohta [Oht77, Proposition 1] uses this base extension trick to find the (inertial) characters which can arise from finite flat E-vector space schemes of rank one over $\mathcal{O}_K$, but not the vector space schemes themselves.

For the Breuil modules in Theorem 2.1, we would like to determine the corresponding character $G_K \to E^\times$. We consider first the situation of [Ray74], where E embeds into $\mathbf{k}$, so that $d = [E : \mathbb{F}_p]$, each $(\mathbf{k}E)_i = \mathbf{k}$, and each element $\sigma \in S$ is an isomorphism $\mathbf{k}_0 \cong E$. Let $(\mathcal{M}, \mathcal{M}_1, \phi_1)$ be a Breuil module as in Theorem 2.1, and let $\mathcal{G}$ be the corresponding finite flat E-vector space scheme of rank one. Let $F(x)$ be the polynomial such that $x^e - pF(x)$ is the Eisenstein polynomial for our chosen uniformizer π.

The affine algebra of $\mathcal{G}$ is described by [Bre00, Proposition 3.1.2]. Following the notation of [Bre00, Section 3.1], let π_1 be a fixed pth root of π and set $\mathcal{O}_{K_1} = \mathcal{O}_K[\pi_1]$. The ring $\mathbf{k}[u]/u^{ep}$ is identified with $\mathcal{O}_{K_1}/p$ via the map $\lambda u^i \mapsto \lambda^{p^{-1}}\pi_1^i$. (We thank Xavier Caruso for bringing this point to our attention.) Then, letting $\widetilde{\alpha}^{p^{-1}}, \widetilde{\alpha} \in W(\mathbf{k})$ denote the respective Teichmüller lifts of $\alpha^{p^{-1}}$ and α, the matrix $\mathcal{G}_\pi$ (again in the notation of [Bre00, Section 3.1]) can be taken to be the matrix

$$\begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ \widetilde{\alpha}^{p^{-1}} & 0 & \cdots & 0 \end{pmatrix}$$

whose entries immediately above the diagonal are all equal to 1, whose lower left-hand entry is $\widetilde{\alpha}^{p^{-1}}$, and whose other entries are zero. (We will continue to label our basis vectors for $\mathcal{M}$ from 0 to $d-1$, where Breuil uses the labels 1 to d.) Since this matrix lies in $\mathrm{GL}_d(\mathcal{O}_K)$, [Bre00, Proposition 3.1.2] applies, and we see that the affine algebra $R_{\mathcal{M}}$ of $\mathcal{G}$ is isomorphic to

$$\mathcal{O}_K[X_0, \ldots, X_{d-1}]/I$$

where I is the ideal generated by $X_i^p + \frac{\pi^{e-r_i}}{F(\pi)} X_{i+1}$ for $0 \le i < d-1$ together with $X_{d-1}^p + \widetilde{\alpha}^{p^{-1}} \frac{\pi^{e-r_{d-1}}}{F(\pi)} X_0$.

Next we must determine the action of $E^\times$ on $R_{\mathcal{M}}$. To do this, we examine the proof of [Bre00, Proposition 3.1.5]. There, Breuil constructs a canonical morphism

$$\mathrm{Hom}_{\mathcal{O}_K}(R_{\mathcal{M}}, \mathfrak{A}) \to \mathrm{Hom}_{'(Mod/S_1)}(\widetilde{\mathcal{M}}, \mathcal{O}_{1,\pi}^{cris}(\mathfrak{A}))\,. \tag{2.3}$$

Here $\widetilde{\mathcal{M}}$ is the S_1-module $S_1 \otimes_{\mathbf{k}[u]/u^{ep}} \mathcal{M}$ associated to $\mathcal{M}$ by [Bre00, Proposition 2.1.2.2], $\mathfrak{A}$ is a formal syntomic $\mathcal{O}_K$-algebra, and $\mathcal{O}_{1,\pi}^{cris}$ is the sheaf on the small p-adic formal syntomic site over $\mathcal{O}_K$ associated the presheaf

$$\mathfrak{A} \mapsto (\mathfrak{A}/p \otimes_{\mathbf{k},\varphi} \mathbf{k}[u])^{DP}$$

where the divided powers are as given in the proof of [Bre00, Proposition 3.1.5]. Let $\lambda \in E^\times$, and take $\mathfrak{A} = R_{\mathcal{M}}$; since the morphism (2.3) is canonical, we obtain a commutative square

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{O}_K}(R_{\mathcal{M}}, R_{\mathcal{M}}) & \longrightarrow & \mathrm{Hom}_{'(Mod/S_1)}(\widetilde{\mathcal{M}}, \mathcal{O}_{1,\pi}^{cris}(R_{\mathcal{M}})) \\ \big\downarrow [\lambda] & & \big\downarrow [\lambda] \\ \mathrm{Hom}_{\mathcal{O}_K}(R_{\mathcal{M}}, R_{\mathcal{M}}) & \longrightarrow & \mathrm{Hom}_{'(Mod/S_1)}(\widetilde{\mathcal{M}}, \mathcal{O}_{1,\pi}^{cris}(R_{\mathcal{M}})) \end{array} \tag{2.4}$$

in which the horizontal arrows are both isomorphisms by [Bre00, Corollaire 3.1.8]. Begin with the identity map in the upper left-hand corner; suppose this maps to g in the upper right-hand corner, and then to g' in the lower-right. In the notation of the proof of [Bre00, Proposition 3.1.5] we have: $\overline{\mathfrak{a}}_{i,0} = \overline{X}_i$ and $\overline{\mathfrak{a}}_{i,j} = 0$ for $j > 0$; and g is the map which sends m_i to $\overline{X}_i \otimes 1 + \gamma_p(\overline{X}_{i-1} \otimes u^{r_{i-1}})$ for $i > 0$, and which sends m_0 to $\overline{X}_0 \otimes 1 + (1 \otimes \alpha^{-1})\gamma_p(\overline{X}_{d-1} \otimes u^{r_{d-1}})$. Noting that the action of $[\lambda]$ on m_i is multiplication by $\sigma_i^{-1}(\lambda)$, we see that g' is the map which sends m_i to $(1 \otimes \sigma_i^{-1}(\lambda))(\overline{X}_i \otimes 1 + \gamma_p(\overline{X}_{i-1} \otimes u^{r_{i-1}}))$ for $i > 0$, and similarly for m_0.

Let $\widetilde{\lambda}_i$ denote the Teichmüller lift of $\sigma_i^{-1}(\lambda)$, so that $\widetilde{\lambda}_i = \widetilde{\lambda}_0^{p^i}$. We can now check that the map g' is exactly the one which comes, via the bottom horizontal arrow in the diagram (2.4), from the map sending $X_i \mapsto \widetilde{\lambda}_0^{p^{i-1}} X_i$. Indeed, again tracing through the proof of [Bre00, Proposition 3.1.5] we find that the map obtained from $X_i \mapsto \widetilde{\lambda}_0^{p^{i-1}} X_i$ sends $m_i \mapsto \sigma_{i-1}^{-1}(\lambda)\overline{X_i} \otimes 1 + \gamma_p(\sigma_{i-2}^{-1}(\lambda)\overline{X}_{i-1} \otimes u^{r_{i-1}})$ for $i > 0$, and similarly for $i = 0$. Since $\gamma_p(\sigma_{i-2}^{-1}(\lambda)\overline{X}_{i-1} \otimes u^{r_{i-1}}) = (\sigma_{i-1}^{-1}(\lambda) \otimes 1)\gamma_p(\overline{X}_{i-1} \otimes u^{r_{i-1}})$ and since $\sigma_{i-1}^{-1}(\lambda) \otimes 1 = 1 \otimes \sigma_i^{-1}(\lambda)$ on $\mathcal{O}_{1,\pi}^{cris}$, the claim follows. We have therefore proved the following.

Proposition 2.5. *Suppose in Theorem 2.1 that E embeds into $\mathbf{k}$. The affine algebra of the finite flat E-vector space scheme of rank one over $\mathcal{O}_K$ corresponding to $\mathcal{M}$ is*

$$\mathcal{O}_K[X_0, \dots, X_{d-1}]/I$$

where I is the ideal generated by $X_i^p + \frac{\pi^{e-r_i}}{F(\pi)} X_{i+1}$ for $0 \le i < d-1$ together with $X_{d-1}^p + \widetilde{\alpha}^{p^{-1}} \frac{\pi^{e-r_{d-1}}}{F(\pi)} X_0$. Moreover, $\lambda \in E^\times$ acts as $[\lambda] X_i = \widetilde{\lambda}_0^{p^{i-1}} X_i$.

Let $q = p^d = \#E$, and let j_q denote the tame character $j_q : I_K \to \mu_{q-1}(K)$, as defined in [Ray74, Section 3.1]. Let ψ_i denote the composition of the reduction map $\mu_{q-1}(K) \to \mathbf{k}_0^\times$ with the isomorphism $\sigma_i : \mathbf{k}_0 \to E$.

Corollary 2.6. *With notation as in Proposition 2.5, set*

$$\eta = (-p)^{1/(p-1)} (\widetilde{\alpha} \cdot \pi^{-(r_1 p^{d-1} + \cdots + r_{d-1} p + r_0)})^{1/(q-1)}.$$

Then $V_{st}(\mathcal{M})$ is the character $\psi(g) = \psi_0(g(\eta)/\eta)$. In particular, $\psi|_{I_K} = \Psi \circ j_q$, where $\Psi = \psi_0^{e-r_0} \psi_1^{e-r_1} \cdots \psi_{d-1}^{e-r_{d-1}}$, and so $T_{st,2}(\mathcal{M})|_{I_K} = (\psi_0^{r_0} \psi_1^{r_1} \cdots \psi_{d-1}^{r_{d-1}}) \circ j_q$.

Proof. The first statement follows easily from the fact that X_1 satisfies the equation $X_1^q = \eta^{q-1} X_1$ (recall that $\pi^e/F(\pi) = p$), together with the fact that $[\lambda] X_1 = \widetilde{\lambda}_0 X_1$. The second statement follows in the manner of [Ray74, Théorème 3.4.1]. Note that $\omega_K|_{I_K} = \psi_0^e \cdots \psi_{d-1}^e$, where ω_K is the mod p cyclotomic character of G_K. □

Now let us return to the general situation, and suppose $[E : \mathbb{F}_p] = nd$. In this case we will only determine the inertial character. Let $(\mathcal{M}, \mathcal{M}_1, \phi_1)$ be a Breuil module as in 2.1, and define the integers $r_0, \dots, r_{d-1}$ as before. As in [Oht77], let K' be the unramified extension of K of degree n, so that E embeds onto a subfield $\mathbf{k}_0'$ of its residue field $\mathbf{k}'$. Let $\mathcal{G}$ be the finite flat E-vector space of rank one over $\mathcal{O}_K$ corresponding to $\mathcal{M}$, and let $\mathcal{G}' = \mathcal{G} \times_{\mathcal{O}_K} \mathcal{O}_{K'}$. Let ψ and ψ' be the characters associated to $\mathcal{G}$ and $\mathcal{G}'$ respectively; since K'/K is unramified, we have $\psi'|_{I_{K'}} = \psi|_{I_K}$, and so to find $\psi|_{I_K}$ we can reduce to the Raynaud situation.

By [BCDT01, Corollary 5.4.2], the Breuil module associated to $\mathcal{G}'$ is $\mathcal{M}' = \mathbf{k}' \otimes_{\mathbf{k}} \mathcal{M}$, with the action of E coming from the E-vector space scheme structure acting on the second factor. Let $(r_0', \dots, r_{nd-1}')$ be the nd-tuple arising from $\mathcal{M}_1'$, as in Theorem 2.1. Let σ be any embedding $\mathbf{k}_0 \to E$; since $\mathcal{M}_\sigma$ is the set of elements $m \in M$ such that $(x \otimes 1)m = (1 \otimes \sigma(x))m$ for all $x \in \mathbf{k}_0$, it follows that $\mathbf{k}' \otimes_{\mathbf{k}} \mathcal{M}_\sigma$ decomposes as the sum $\oplus_\tau (\mathcal{M}')_\tau$, the sum taken over embeddings $\mathbf{k}_0' \to E$ such that $\tau|_{\mathbf{k}_0} = \sigma$. We deduce immediately that $r_j' = r_i$ where i is the residue of j (mod d) in the interval $[0, d-1]$. We conclude the following.

Corollary 2.7. *Let $q = p^d = \#\mathbf{k}_0$, and let j_q denote the tame character $j_q : I_K \to \mu_{q-1}(K)$, as defined in* [Ray74, Section 3.1]*. Let $\psi_i : \mu_{q-1}(K) \to E^\times$ denote the composition of the reduction map $\mu_{q-1}(K) \to \mathbf{k}_0$ with the embedding σ_i.*

Let $\mathcal{M}$ be a Breuil module as given in Theorem 2.1. Then $V_{st}(\mathcal{M})\,|_{I_K} = \Psi \circ j_q$, where $\Psi = \psi_0^{e-r_0}\psi_1^{e-r_1}\cdots\psi_{d-1}^{e-r_{d-1}}$, and $T_{st,2}(\mathcal{M})\,|_{I_K} = (\psi_0^{r_0}\psi_1^{r_1}\cdots\psi_{d-1}^{r_{d-1}}) \circ j_q$.

Proof. Number the embeddings $\tau : \mathbf{k}_0' \hookrightarrow E$ so that $\tau_0\,|_{\mathbf{k}_0} = \sigma_0$ and $\tau_{i+1} = \tau \circ \varphi^{-1}$. Let ψ_i' denote the composition of $\mu_{p^{nd}-1}(K') \to \mathbf{k}_0'$ with τ_i, and let $j_{p^{nd}}$ denote the tame character $j_{p^{nd}} : I_{K'} \to \mu_{p^{nd}-1}(K')$. We see easily from Corollary 2.6 and our calculation of r_j' that $\psi\,|_{I_K} = N_{E/E_0} \circ \Psi' \circ j_{p^{nd}}$, where E_0 is the subfield of E that is isomorphic to $\mathbf{k}_0$ and $\Psi' = (\psi_0')^{e-r_0}(\psi_1')^{e-r_1}\cdots(\psi_{d-1}')^{e-r_{d-1}}$. But $N_{E/E_0} \circ \psi_i' \circ j_{p^{nd}}$ is precisely $\psi_i \circ j_q$: this follows directly from the definition of the tame character j (see the very end of [Ray74, Section 3.1], and note that since K'/K is unramified, j_q is the same map for K and K'). $\square$

3. Descent data

Let $\mathcal{G}$ be a finite flat E-vector space scheme over $\mathcal{O}_K$. If $\lambda \in E$, let $[\lambda]$ denote the corresponding endomorphism both of $\mathcal{G}$ and of the Breuil module $\mathcal{M}(\mathcal{G})$.

Suppose now that the underlying finite flat group scheme is endowed with descent data relative to L in the sense of the discussion in [BCDT01, Section 4.1], so that the Breuil module corresponding to the underlying finite flat group scheme obtains descent data from K to L, again in the sense of [BCDT01]. For any $g \in \mathrm{Gal}(K/L)$, let the superscript ^{g} denote base change by g. Let $\langle g \rangle$ denote the g-semilinear descent data map $\mathcal{G} \to \mathcal{G}$, and also the corresponding descent data map $\mathcal{M}(\mathcal{G}) \to \mathcal{M}(\mathcal{G})$. Finally, let $[g]$ be the corresponding morphism $\mathcal{G} \to^g \mathcal{G}$ of finite flat group schemes (see e.g. the diagram on [Sav05, p.155]).

Proposition 3.1. *The action of E on $\mathcal{G}$ commutes with the descent data — i.e., the descent data is actually descent data on the finite flat E-vector space scheme, and not just the underlying finite flat group scheme — if and only if the action of E on $\mathcal{M}(\mathcal{G})$ commutes with the descent data on $\mathcal{M}(\mathcal{G})$.*

Proof. Choose $\lambda \in E$, and note that $\langle g \rangle$ commutes with $[\lambda]$ on $\mathcal{G}$ if and only if ${}^g[\lambda] \circ [g] = [g] \circ [\lambda]$, if and only if the morphisms f_1, f_2 of Breuil modules $\mathcal{M}({}^g\mathcal{G}) \to \mathcal{M}(\mathcal{G})$ corresponding to ${}^g[\lambda] \circ [g]$ and $[g] \circ [\lambda]$ are equal. However, one checks without difficulty that the maps $[\lambda] \circ \langle g \rangle, \langle g \rangle \circ [\lambda] : \mathcal{M}(\mathcal{G}) \to \mathcal{M}(\mathcal{G})$ are obtained by composing f_1, f_2 respectively with the isomorphism of Corollary 5.4.5(1) of [BCDT01]. $\square$

Suppose henceforth that K/L is a tamely ramified Galois extension with relative ramification degree $e(K/L)$, and suppose $\pi \in K$ is a uniformizer such that $\pi^{e(K/L)} \in L$. Let $\mathbf{l}$ be the residue field of L. The group $\mathrm{Gal}(K/L)$ acts on $\mathbf{k} \otimes_{\mathbb{F}_p} E$ via $\mathrm{Gal}(\mathbf{k}/\mathbf{l})$ on the first factor and trivially on the second. Let $\eta : \mathrm{Gal}(K/L) \to K^\times$ be the function sending $g \mapsto g(\pi)/\pi$, and let $\overline{\eta}$ be the reduction of η modulo π. We sometimes write G for $\mathrm{Gal}(K/L)$ and I for the inertia subgroup of G.

Let $\mathcal{G}$ be a finite flat E-vector space scheme over $\mathcal{O}_K$, with $\mathcal{M}$ the corresponding object in $\mathrm{BrMod}_{\mathcal{O}_K,E}$. Combining Proposition 3.1 with [Sav04, Theorem 3.5], we immediately obtain the following.

Proposition 3.2. *Giving descent data relative to L on $\mathcal{G}$ is equivalent to giving, for each $g \in \mathrm{Gal}(K/L)$, an additive bijection $[g] : \mathcal{M} \to \mathcal{M}$ satisfying:*

- *each $[g]$ preserves $\mathcal{M}_1$ and commutes with ϕ_1,*
- *$[1]$ is the identity and $[g][h] = [gh]$, and*
- *$[g](au^i m) = g(a)(\overline{\eta}(g)^i \otimes 1)u^i g(m)$ for $m \in \mathcal{M}$ and $a \in \mathbf{k} \otimes_{\mathbb{F}_p} E$.*

Suppose now that $\mathcal{G}$ is a rank one E-vector space scheme with descent data, so that $\mathcal{M}$ is a free $(\mathbf{k} \otimes_{\mathbb{F}_p} E)[u]/u^{ep}$-module of rank one. If $g \in \mathrm{Gal}(K/L)$, define the integer $\alpha(g)$ so that the image of g in $\mathrm{Gal}(\mathbf{k}_0/\mathbb{F}_p)$ is $\varphi^{\alpha(g)}$; one checks that $g(e_i) = e_{i+\alpha(g)}$. Let D denote the index of the image of $\mathrm{Gal}(K/L)$ in $\mathrm{Gal}(\mathbf{k}_0/\mathbb{F}_p)$, i.e., D is the greatest common divisor of d and all the $\alpha(g)$. For any integer i, let $[i]$ denote the residue of $i \pmod D$ in the interval $[0, D-1]$. We have the following.

Proposition 3.3. *There exists a generator $m \in \mathcal{M}$ and integers $0 \le k_i < e(K/L)$ for $i = 0, \dots, D-1$ such that $[g]m = (\sum_{i=0}^{d-1} (\overline{\eta}(g)^{k_{[i]}} \otimes 1) e_{\sigma_i})m$ for all $g \in \mathrm{Gal}(K/L)$.*

Proof. This Proposition follows as in [Sav04, Proposition 5.3], provided that we can prove the analogue of [Sav04, Lemma 4.1] with $\mathbf{k}$ replaced everywhere by $\mathbf{k} \otimes_{\mathbb{F}_p} E$. The proof of the latter goes through *mutatis mutandis*, except for the justification that $H^1(\mathrm{Gal}(\mathbf{k}/\mathbf{l}), (\mathbf{k} \otimes_{\mathbb{F}_p} E)^\times) = H^2(\mathrm{Gal}(\mathbf{k}/\mathbf{l}), (\mathbf{k} \otimes_{\mathbb{F}_p} E)^\times) = 0$, and the calculation of $\mathrm{Hom}(I, (\mathbf{k} \otimes_{\mathbb{F}_p} E)^\times))^{G/I}$.

For the latter, every element of $\mathrm{Hom}(I, (\mathbf{k} \otimes_{\mathbb{F}_p} E)^\times)$ has the form $\sum_{i=0}^{d-1} (\overline{\eta}\,|_I^{k_i} \otimes 1)e_{\sigma_i}$ with $0 \le k_i < e(K/L)$, and one verifies that this is invariant by $g \in \mathrm{Gal}(K/L)$ if and only if $k_i = k_{i+\alpha(g)}$; it follows that $k_i = k_{[i]}$ for all i.

For the former, note that the vanishing of these two groups is equivalent (see e.g. [Ser79, Chapter 8, Proposition 8]); for H^2 this amounts to the surjectivity of the norm map $N_{\mathbf{k}/\mathbf{l},E} : (\mathbf{k} \otimes E)^\times \to (\mathbf{l} \otimes E)^\times$, which we will now prove. Let $\mathbf{l}_0 = \mathbf{k}_0 \cap \mathbf{l}$ denote the largest subfield of $\mathbf{l}$ that embeds into E. If $\sigma' : \mathbf{l}_0 \hookrightarrow E$, note that $(\mathbf{l}E)_{\sigma'}$ is naturally a subspace of $(\mathbf{k}E)_\sigma$ for any $\sigma \in S_{\sigma'}$, where $S_{\sigma'} = \{\sigma \in S \ : \ \sigma\,|_{\mathbf{l}_0} = \sigma'\}$. Then the inclusion $\mathbf{l} \otimes E \subset \mathbf{k} \otimes E$ embeds $(\mathbf{l}E)_{\sigma'}$ diagonally into $\oplus_{\sigma \in S_{\sigma'}} (\mathbf{k}E)_\sigma$. Write $\sigma'_i = \sigma_i\,|_{\mathbf{l}_0}$, and let $(\mathbf{l}E)^i$ denote the image of $(\mathbf{l}E)_i$ in $(\mathbf{k}E)_i$.

Let $\delta = [\mathbf{l} : \mathbb{F}_p]$ and $\Delta = [\mathbf{k}_0\mathbf{l} : \mathbb{F}_p]$, so that $\Delta = \mathrm{LCM}(d, \delta)$. Recall that $\varphi^\delta \in \mathrm{Gal}(\mathbf{k}/\mathbf{l})$ induces a map $(\mathbf{k}E)_i \to (\mathbf{k}E)_{i+\delta}$, and observe that $\varphi^\Delta : (\mathbf{k}E)_i \to (\mathbf{k}E)_i$ is a generator of $\mathrm{Gal}((\mathbf{k}E)_i/(\mathbf{l}E)^i)$. If $s = \sum_i s_i$ with $s_i \in (\mathbf{k}E)_i^\times$ for all i, it follows without difficulty that the σ_i-component of $N_{\mathbf{k}/\mathbf{l},E}(s)$ is equal to

$$t_i = N_{(\mathbf{k}E)_i/(\mathbf{l}E)^i}(s_i \varphi^\delta(s_{i-\delta}) \cdots \varphi^{(f-1)\delta}(s_{i-(f-1)\delta}))$$

where $f = \Delta/\delta$. Then $t_{i+\delta} = \varphi^\delta t_i$; but since φ^δ is trivial on $\mathbf{l}$ and E, it follows that the element $\sum_{j=0}^{f-1} t_{i+\delta j} \in \oplus_{\sigma \in S_{\sigma'_i}} (\mathbf{k}E)_\sigma$ actually lies in $(\mathbf{l}E)_i$. Moreover, since the $s_{i+\delta j}$ are arbitrary and the usual norm $N_{(\mathbf{k}E)_i/(\mathbf{l}E)^i}$ is surjective, this sum can be made to be any nonzero element in $(\mathbf{l}E)_i$. This can be done independently for each value of i mod $[\mathbf{l}_0 : \mathbb{F}_p] = \mathrm{GCD}(d, \delta)$, from which the result follows. $\square$

For additive bijections $[g]$ as in Proposition 3.3 (extended to all of $\mathcal{M}$ in the necessary manner) to form descent data, one must impose the conditions that each $[g]$ preserves $\mathcal{M}_1$ and commutes with ϕ_1. For the former, it is necessary and sufficient that $r_i \geq r_{i+\alpha(g)}$ for all i and g; this is equivalent to the equality $r_i = r_{[i]}$ for all i. For the latter, write $\underline{u^r} = \sum_i u^{r_i} e_{\sigma_i}$, so that $\mathcal{M}_1$ is generated by $\underline{u^r}m$, and suppose $\phi_1(\underline{u^r}m) = cm$ with $c \in ((\mathbf{k} \otimes_{\mathbb{F}_p} E)[u]/u^{ep})^\times$. Then the relation $\phi_1 \circ [g](\underline{u^r}m) = [g] \circ \phi_1(\underline{u^r}m)$ becomes:

$$\left(\sum_{i=0}^{d-1} \overline{\eta}(g)^{p(k_{[i-1]}+r_{[i-1]})} e_{\sigma_i}\right) cm = \left(\sum_{i=0}^{d-1} \overline{\eta}(g)^{k_{[i]}}\right) g(c)m \,,$$

or equivalently $g(c)/c = \sum_{i=0}^{d} \overline{\eta}(g)^{p(k_{[i-1]}+r_{[i-1]})-k_{[i]}} e_{\sigma_i}$. But this equation shows that the right-hand side is a coboundary in $H^1(G, (\mathbf{k} \otimes E)^\times)$; since the restriction of this cocycle to $H^1(I, (\mathbf{k} \otimes E)^\times) = \mathrm{Hom}(I, (\mathbf{k} \otimes E)^\times)$ must also be trivial, it is necessary and sufficient that

$$k_{[i]} \equiv p(k_{[i-1]} + r_{[i-1]}) \pmod{e(K/L)} \tag{3.4}$$

for all i, in which case $g(c) = c$.

Now we can apply the argument preceding Theorem 2.1: setting $m' = cm$, we see that $[g]$ still acts on m' as in Proposition 3.3, while $\phi_1(\underline{u^r}m') = \phi(c)m'$. Repeating this process, we see that we can suppose $c \in (\mathbf{k} \otimes_{\mathbb{F}_p} E)^\times$, and in fact since $g(c) = c$ we have $c \in (\mathbf{l} \otimes_{\mathbb{F}_p} E)^\times$. In summary, we have proved the following.

Theorem 3.5. *With π chosen as above, every rank one object of $\mathrm{BrMod}_{\mathcal{O}_K,E}$ with (tame) descent data relative to L has the form:*

- $\mathcal{M} = ((\mathbf{k} \otimes_{\mathbb{F}_p} E)[u]/u^{ep}) \cdot m$,
- $(\mathcal{M}_1)_{\sigma_i} = u^{r_{[i]}} \mathcal{M}_{\sigma_i}$,
- $\phi_1(\sum_{i=0}^{d-1} u^{r_{[i]}} e_{\sigma_i} m) = cm$ *for some* $c \in (\mathbf{l} \otimes_{\mathbb{F}_p} E)^\times$, *and*
- $[g]m = (\sum_{i=0}^{d-1} (\overline{\eta}(g)^{k_{[i]}} \otimes 1) e_{\sigma_i})m$ *for all* $g \in \mathrm{Gal}(K/L)$,

where $0 \leq r_{[i]} \leq e$ *and* $0 \leq k_{[i]} < e(K/L)$ *are sequences of integers satisfying* $k_{[i]} \equiv p(k_{[i-1]} + r_{[i-1]}) \pmod{e(K/L)}$ *for* $[i] = 0, \ldots, D-1$.

Remark 3.6. Given $r_0, \ldots, r_{D-1}$, a necessary and sufficient condition for such a sequence $\{k_{[i]}\}$ to exist is that $p^{D-1}r_0 + \ldots + r_{D-1}$ be divisible by $(e(K/L), p^D - 1)$, and then k_0 can be any solution of $p(p^{D-1}r_0 + \ldots + r_{D-1}) \equiv (1 - p^D)k_0 \pmod{e(K/L)}$.

Example 3.7. Suppose we are in the situation of [Gee06]: suppose $\mathbf{k}$ embeds into E, set $L = W(\mathbf{k})[1/p]$, and fix $\pi = (-p)^{1/(p^d-1)}$ with $d = [\mathbf{k} : \mathbb{F}_p] = [\mathbf{k}_0 : \mathbb{F}_p]$. Set $K = L(\pi)$, so that $e(K/L) = p^d - 1$, K/L is totally ramified, and $\mathrm{Gal}(K/L)$ acts trivially on $\mathbf{k} \otimes_{\mathbb{F}_p} E$. Then $D = d$, and the condition in Remark 3.6 is simply $p^{d-1}r_0 + \ldots + r_{d-1} \equiv 0 \pmod{p^d - 1}$; if this is satisfied, k_0 may be arbitrary. Let $\mathcal{M}$, then, be a Breuil module with descent data as in the statement of Theorem 3.5. Since $\mathbf{k} = \mathbf{l}$ we can use the argument of the paragraph preceding Theorem 2.1 to assume that c has the form $(1 \otimes a^{-1})e_{\sigma_0} + \sum_{i=1}^{d-1} e_{\sigma_i}$ for some $a \in E^\times$, and we do so. We will determine $T_{st,2}(\mathcal{M})$ using the method of Section 5 of [Sav05].

Let $s_i = p(r_i p^{d-1} + r_{i+1} p^{d-2} + \cdots + r_{i+d-1})/(p^d - 1)$ with subscripts taken modulo d, and define $\kappa_i = k_i + s_i$. Observe from (3.4) that $\kappa_i \equiv p^i \kappa_0 \pmod{p^d - 1}$. Define another rank one Breuil module with descent data $\mathcal{M}'$ with generator m', satisfying $\mathcal{M}'_1 = \mathcal{M}'$, $\phi_1(m') = cm'$, and $[g]m' = (\sum_{i=0}^{d-1} (\overline{\eta}(g)^{p^i \kappa_0} \otimes 1) e_{\sigma_i}) m' = (1 \otimes \sigma_0(\overline{\eta}(g))^{\kappa_0}) m'$. We can define a morphism $\mathcal{M}' \to \mathcal{M}$ by mapping $e_{\sigma_i} m' \mapsto u^{s_i} e_{\sigma_i} m$. One checks that this is a morphism of Breuil modules with descent data: for instance, the filtration is preserved since $s_i \geq r_i$, and the morphism commutes with ϕ_1 because $s_{i+1} = p(s_i - r_i)$. By an application of [Sav04, Proposition 8.3], we see that $T_{st,2}(\mathcal{M}) = T_{st,2}(\mathcal{M}')$.

Let $F = W(E)[1/p]$, let $\widetilde{\sigma}_i$ be a lift of σ_i to an embedding $L \hookrightarrow F$, and let $\widetilde{e}_i$ be the idempotent in $L \otimes_{\mathbb{Q}_p} F$ corresponding to $\widetilde{\sigma}_i$, so that $\widetilde{e}_i$ is a lift of e_{σ_i}. Note that the image of η lies in $L^\times$, and that since K/L is totally ramified, η is actually a character of $\mathrm{Gal}(K/L)$ and (abusing notation) of $\mathrm{Gal}(\overline{L}/L)$. Let $\widetilde{a}$ be the Teichmüller lift of a, and let $\lambda_{\widetilde{a}}, \lambda_a$ denote the characters of $\mathrm{Gal}(\overline{L}/L)$ sending arithmetic Frobenius Frob_L to $\widetilde{a}, a$ respectively. Set $\widetilde{c} = (1 \otimes \widetilde{a}^{-1}) \widetilde{e}_0 + \sum_{i=1}^{d-1} \widetilde{e}_i$.

By the method of Examples 2.13 and 2.14 of [Sav05], and using the notation and conventions of Section 2.2 of *loc. cit.*, the admissible filtered $(\varphi, N, K/L, F)$-module $D = D_{st,2}^K((\widetilde{\sigma}_0 \circ \eta^{\kappa_0}) \lambda_{\widetilde{a}})$ is a module $(L \otimes_{\mathbb{Q}_p} F)\mathbf{e}$ satisfying

$$N = 0, \qquad \varphi(\mathbf{e}) = p\widetilde{c}\mathbf{e}, \qquad g(\mathbf{e}) = (1 \otimes (\widetilde{\sigma}_0 \circ \eta(g)^{\kappa_0}))\mathbf{e} \text{ for } g \in \mathrm{Gal}(K/L),$$

and $\mathrm{Fil}^i(K \otimes_L D)$ is 0 for $i \geq 2$ and $(K \otimes_L D)$ for $i \leq 1$. For instance, one checks easily that D is admissible (indeed $t_H(D') = t_N(D') = m$ for any (φ, L)-submodule D' of dimension m), and the fact that $\varphi^d(\mathbf{e}) = p^d(1 \otimes \widetilde{a}^{-1})\mathbf{e}$ implies that the unramified part of $V_{st,2}^L(D)$ sends Frob_L to $\widetilde{a}$.

Let $S_{K,W(E)}$ be the period ring of [Sav05, Section 4]. One checks without difficulty that $S_{K,W(E)}[1/p] \otimes_{L \otimes_{\mathbb{Q}_p} F} D$ contains a strongly divisible module with $W(E)$-coefficients $\mathcal{M}$ (in the sense of [Sav05, Section 4]), namely $\mathcal{M} = S_{K,W(E)}\mathbf{e}$, and that $(\mathcal{M}/p\mathcal{M}) \otimes_{S_K} \mathbf{k}[u]/u^{ep} = \mathcal{M}'$. Combining Theorem 3.14 and Corollary 4.12(1) of [Sav05] and the discussion in Section 4.1 of *loc. cit.*, we deduce that $(\widetilde{\sigma}_0 \circ \eta^{\kappa_0}) \lambda_{\widetilde{a}}$ is a lift of $T_{st,2}(\mathcal{M}')$, so that

$$T_{st,2}(\mathcal{M}) = T_{st,2}(\mathcal{M}') = (\sigma_0 \circ \overline{\eta}^{\kappa_0}) \lambda_a .$$

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Department of Mathematics, The University of Arizona, 617 North Santa Rita Avenue, Tucson AZ, 85721, USA; savitt@math.arizona.edu