1. Classify all finite-dimensional representations of the group $G = \mathbb{Z}$ over $\mathbb{C}$, and over an arbitrary field $k$. Verify explicitly that those which factor through a finite quotient $\mathbb{Z}/n$ are semisimple, if $k$ is of characteristic zero. Examine the case of positive characteristic.

2. If $G$ is a finite group, use the Peter–Weyl theorem to deduce the following well-known formulas: here, if $\pi_1, \ldots, \pi_r$ are representatives for the isomorphism classes of irreducible representations (over $\mathbb{C}$), $d_i = \dim(\pi_i)$ and $\chi_i =$ the character of $\pi_i$. (Hint: identify the subspace spanned by characters on both sides of the Peter–Weyl theorem.)

\begin{enumerate}
    \item \(\sum_i d_i^2 = |G|\).
    \item \(r = \# \text{ of conjugacy classes in } G\).
    \item \text{“Row orthogonality”}:
      \[
      \frac{1}{|G|} \sum_{g \in G} \chi_i(g) \overline{\chi_j(g)} = \delta_{ij}.
      \]
    \item \text{“Column orthogonality”}:
      \[
      \frac{1}{|G|} \sum_{i=1}^{r} \chi_i(g_1) \overline{\chi_i(g_2)} = \begin{cases} 
      1, & \text{if } g_1 \text{ and } g_2 \text{ are conjugate} \\
      0, & \text{otherwise}.
    \end{cases}
      \]
\end{enumerate}

3. Use the above character relations, and possibly some natural representations that you can think of, to compute the irreducible characters of the symmetric groups $S_3$ and $S_4$.

4. Let $(V, q)$ be a two-dimensional real vector space, endowed with a non-degenerate quadratic form. Let $G$ be the special orthogonal group of those linear transformations which preserve the quadratic form and act trivially on $\wedge^2 V$. Describe the group $G$, and classify its irreducible finite-dimensional complex representations.

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*Updated periodically. Last update: September 19, 2019

1By the way the character table is usually written: one row for each character, one column for each conjugacy class.
5. Let $G = \text{SU}_2 \subset \text{SL}_2(\mathbb{C})$ be the subgroup of those transformations which preserve an inner product on $\mathbb{C}^2$.

(a) Show that $G$ acts transitively on the set of complex lines in $\mathbb{C}^2$, and that the stabilizer of a line is isomorphic to the compact torus group $S^1 = \mathbb{R}/\mathbb{Z}$. Fix such a line, with stabilizer $T \subset G$, and deduce:

- every element of $G$ is conjugate to an element of $T$;
- there is a fibration $G \to \mathbb{C}P^1$, with fiber isomorphic to $S^1$.

(b) Identify $\mathbb{C}^2$ with the quaternion algebra $\mathbb{H}$, and $G$ with the multiplicative group of quaternions of norm one. Deduce that $\text{SU}_2$ is homeomorphic to the 3-sphere $S^3$. Thus, we obtain the Hopf fibration $S^1 \hookrightarrow S^3 \to S^2$.

6. Classify all Lie algebras of dimensions $\leq 3$ over a field $k$.

7. Consider the “fake Lie algebra” $\mathfrak{g}$ with generators $x, y, z$ and bracket relations $[x, y] = [x, z] = [y, z] = z$. Confirm that the Jacobi identity fails, and then prove that the Poincaré–Birkhoff–Witt theorem also fails in this case: The natural surjection $S\mathfrak{g} \to \text{gr}U$, where $U$ is the quadratic algebra $T(\mathfrak{g})/(R)$, with $R = \text{span}\{x \otimes y - y \otimes x - z, x \otimes z - z \otimes x - z, y \otimes z - z \otimes y - z\}$, is not an isomorphism.

8. Consider the affine variety $X = \text{SO}_2 \setminus \text{SO}_3$. We have seen (in the discussion of spherical harmonics) that the coordinate ring $\mathbb{R}[X]$ (or $\mathbb{C}[X]$) has a natural, $G$-stable (where $G = \text{SO}_3$) grading as a vector space:

$$\mathbb{R}[X] = \bigoplus_{k \geq 0} \mathbb{R}[X]_k,$$

corresponding to the degree of the spherical harmonics. Show that this fails to be an algebra grading, but it corresponds to an algebra filtration with $F^n \mathbb{R}[X] = \bigoplus_{0 \leq k \leq n} \mathbb{R}[X]_k$. Describe the $G$-variety corresponding to the associated graded $\text{gr}\mathbb{R}[X]$, as well as the one corresponding to the Rees family $\bigoplus_{n \geq 0} \mathcal{R} F^n \mathbb{R}[X]$. 
