Introduction to Lie groups, Fall 2019: Exercises*

Yiannis Sakellaridis

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1. Classify all finite-dimensional representations of the group $G = \mathbb{Z}$ over $\mathbb{C}$, and over an arbitrary field k. Verify explicitly that those which factor through a finite quotient $\mathbb{Z}/n$ are semisimple, if k is of characteristic zero. Examine the case of positive characteristic.

2. If G is a finite group, use the Peter–Weyl theorem to deduce the following well-known formulas; here, if $\pi_1, \ldots, \pi_r$ are representatives for the isomorphism classes of irreducible representations (over $\mathbb{C}$), $d_i = \dim(\pi_i)$ and χ_i =the character of d_i. (Hint: identify the subspace spanned by characters on both sides of the Peter–Weyl theorem.)

   (a) $\sum_i d_i^2 = |G|$.

   (b) $r = \#$ of conjugacy classes in G.

   (c) "Row orthogonality"[1]:
   $$\frac{1}{|G|}\sum_{g\in G}\chi_i(g)\overline{\chi_j(g)} = \delta_{ij}.$$

   (d) "Column orthogonality":
   $$\frac{1}{|G_{g_1}|}\sum_{i=1}^{r}\chi_i(g_1)\overline{\chi_i(g_2)} = \begin{cases}1, & \text{if } g_1 \text{ and } g_2 \text{ are conjugate}\\ 0, & \text{otherwise.}\end{cases}$$

   Here, G_{g_1} denotes the centralizer of g_1.

3. Use the above character relations, and possibly some natural representations that you can think of, to compute the irreducible characters of the symmetric groups S_3 and S_4.

4. Let (V, q) be a two-dimensional real vector space, endowed with a non-degenerate quadratic form. Let G be the special orthogonal group of those linear transformations which preserve the quadratic form and act trivially on $\bigwedge^2 V$. Describe the group G, and classify its irreducible finite-dimensional complex representations.

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[1] By the way the character table is usually written: one row for each character, one column for each conjugacy class.

5. Let $G = \mathrm{SU}_2 \subset \mathrm{SL}_2(\mathbb{C})$ be the subgroup of those transformations which preserve an inner product on $\mathbb{C}^2$.

   (a) Show that G acts transitively on the set of complex lines in $\mathbb{C}^2$, and that the stabilizer of a line is isomorphic to the compact torus group $S^1 = \mathbb{R}/\mathbb{Z}$. Fix such a line, with stabilizer $T \subset G$, and deduce:

      - every element of G is conjugate to an element of T;
      - there is a fibration $G \to \mathbb{C}P^1$, with fiber isomorphic to S^1.

   (b) Identify $\mathbb{C}^2$ with the quaternion algebra $\mathbb{H}$, and G with the multiplicative group of quaternions of norm one. Deduce that SU_2 is homeomorphic to the 3-sphere S^3. Thus, we obtain the *Hopf fibration* $S^1 \hookrightarrow S^3 \to S^2$.

6. Classify all Lie algebras of dimensions ≤ 3 over a field k.

7. Consider the "fake Lie algebra" $\mathfrak{g}$ with generators x, y, z and bracket relations $[x, y] = x$, $[y, z] = z$, $[x, z] = z$. Confirm that the Jacobi identity fails, and then prove that the Poincaré–Birkhoff–Witt theorem also fails in this case: The natural surjection $S\mathfrak{g} \to \mathrm{gr}U$, where U is the quadratic algebra $T(\mathfrak{g})/(R)$, with $R = \mathrm{span}\{x \otimes y - y \otimes x - z, x \otimes z - z \otimes x - z, y \otimes z - z \otimes y - z\}$, is not an isomorphism. (You can prove, actually, that PBW fails every time that the Jacobi identity does.)

8. Consider the affine variety $X = \mathrm{SO}_2 \backslash \mathrm{SO}_3$. We have seen (in the discussion of spherical harmonics) that the coordinate ring $\mathbb{R}[X]$ (or $\mathbb{C}[X]$) has a natural, G-stable (where $G = \mathrm{SO}_3$) grading as a vector space:

$$\mathbb{R}[X] = \bigoplus_{k \geq 0} \mathbb{R}[X]_k,$$

   corresponding to the degree of the spherical harmonics. Show that this fails to be an algebra grading, but it corresponds to an algebra filtration with $F^n\mathbb{R}[X] = \bigoplus_{0 \leq k \leq n} \mathbb{R}[X]_k$. Describe the G-variety corresponding to the associated graded $\mathrm{gr}\mathbb{R}[X]$, as well as the one corresponding to the Rees family $\bigoplus_{n \geq 0} t^n F^n \mathbb{R}[X]$.

9. In this exercise we will describe the irreducible representations of the symmetric group S_d.

   We consider $G = S_d$ as the permutation group on the set $\Sigma = \{1, \ldots, d\}$, and will say "partition of d" for every sequence $\lambda : \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0$, of positive integers with $\sum \lambda_i = d$. We will call "λ-partition of Σ" be a disjoint decomposition $\Sigma = \bigsqcup_i \Sigma_i$, with $|\Sigma_i| = \lambda_i$. (In particular, these subsets are indexed with decreasing size, but we distinguish between subsets of the same size.)

   The group G acts on the space Σ_λ of λ-partitions; we define this action as a right action. To invoke a vocabulary similar to that of Lie groups, although it is not standard in this case, let us say that a "parabolic subgroup" of G is the stabilizer of a λ-partition, for some λ. Hence, a parabolic subgroup is a subgroup of the form $S_{\Sigma_1} \times \cdots \times S_{\Sigma_n}$, where $\Sigma = \bigsqcup_i \Sigma_i$ is a λ-partition as before, and S_{Σ_i} denotes the group of permutations of the subset Σ_i.

Thus, the homogeneous space Σ_λ is isomorphic to $P\backslash G$, where P is the stabilizer of the "standard" λ-partition, where the integers are placed in the subsets Σ_λ in order.

We can think of λ as a *Young diagram*, that is, the diagram consisting of a row of λ_1 squares stacked over λ_2 squares (aligned on the left), etc, and we can also think of *Young tableaux*, which are ways to populate the squares of a given Young diagram with the elements of Σ (without repetitions):

5	6	3	7
1	2	4	
8			

Then, the parabolic P mentioned above is the stabilizer of the rows of the standard Young tableau, where the integers are placed in order.

The *dual partition* to λ is partition $\lambda^* : \lambda_1^* \ge \lambda_2^* \ge \dots \lambda_m^* > 0$ of d counting the sizes of the columns of the Young diagram of λ. The space Σ_{λ^*} of λ^*-partitions of Σ is the homogeneous space $Q\backslash G$, where $Q = G_{\lambda^*}$ is the stabilizer of the columns of the standard Young tableau.

We will construct the irreducible representations of G by inducing the trivial and sign representation from the groups P and Q. None of them is irreducible, but they share a unique irreducible component. Everything will be based on the following combinatorial lemma, which is the first thing that you are asked to prove:

Lemma 1. *If λ, μ are two partitions of d, and (σ, τ) is a pair consisting of a λ-partition σ of Σ and a μ^*-partition τ of Σ with no pair (k, l) of elements of Σ in the same subset of σ and of τ, then τ is the refinement of a λ^*-partition of Σ; that is, there is a Young tableau, whose rows are the subsets of the partition σ, and whose columns are* unions *of subsets of the partition τ.*

In particular, $\mu^* \le \lambda^*$, or equivalently, $\mu \ge \lambda$, in the lexicographic order, i.e., $\mu = \lambda$ or at the first index i where $\mu_i \ne \lambda_i$ we have $\mu_i > \lambda_i$.

If $\mu = \lambda$ and there is no pair (k, l) in the same subset of σ and of τ, then σ, τ are the rows, resp. columns, of a single Young tableau.

Now, we let M_λ, resp. A_λ, be the G-equivariant complex line bundles over the (discrete) space Σ_λ which are induced, respectively, from the trivial, resp. sign character, of the stabilizers. Explicitly, sections of M_λ are left-P-invariant functions on G, i.e., left-P-invariant elements of $\mathbb{C}[G]$, while sections of A_λ are functions on G which vary by the sign character under left translation by P. For notational simplicity, we will identify the bundles with their space of sections. The next exercise is just asking you to recall the concept of induction:

Lemma 2. *Show that (the spaces of sections of) M_λ, A_λ are the induced representations $Ind_P^G(1)$, $Ind_P^G(sgn)$, using the universal property of induction as the definition.*

Now we come to the beef:

Proposition 3. *We have* $dim Hom_G(M_\lambda, A_{\lambda^*}) = 1$.

If $\lambda > \mu$ *(in the lexicographic order), we have* $dim Hom_G(M_\lambda, A_{\mu^*}) = 0$.

To prove it, think of (not necessarily G-equivariant) maps from M_λ to A_{λ^*} as "kernel functions": Those are sections, over $\Sigma_\lambda \times \Sigma_{\mu^*}$, of the line bundle $L := M_\lambda \otimes A_{\mu^*}$, where a kernel function K corresponds to the operator

$$T_K(f)(y) = \sum_{x \in \Sigma_\lambda} f(x) K_f(x, y).$$

Then, the space of G-morphisms $\mathrm{Hom}_G(M_\lambda, A_{\mu^*})$ is identified with the space of G^{diag}-invariant sections of L. Use Lemma 1 to deduce that, when $\lambda > \mu$, there are no nonzero invariant kernels, and when $\lambda = \mu$, there is a one-dimensional space of invariant kernels, supported on the unique G-orbit corresponding to Young tableaux.

Finally, use the previous proposition to prove:

Theorem 4. *The image of a nonzero* G*-morphism* $M_\lambda \to A_{\lambda^*}$ *is an irreducible representation* V_λ*. For* λ, μ *different partitions,* V_λ, V_μ *are non-isomorphic, and these are all the irreducible representations of* S_d*.*

10. In this exercise, we establish Schur–Weyl duality, which refers to a correspondence between representations of symmetric groups and general linear groups (or their Lie algebras), realized inside the tensor powers $V^{\otimes^d}$ of a vector space.

    It is based on the following theorem from linear algebra, which you are asked to prove:

    Theorem 5 (Double centralizer theorem)**.** *If* V *is a finite-dimensional complex vector space,* $A \subset End_B(V)$ *is a semisimple subalgebra of operators, and* $B = End_A(V)$ *is its commutant, then*

    (a) B *is semisimple.*

    (b) $A = End_B(V)$.

    (c) There is a bijection $M_i \leftrightarrow N_i$ *between isomorphism classes of simple* A*-modules and isomorphism classes of simple* B*-modules, and an isomorphism of* $A \otimes B$*-modules*

    $$V = \bigoplus_i M_i \otimes N_i.$$

    To start the proof, let M_i range over all isomorphism classes of simple A-modules. Since V is A-semisimple,

    $$V = \bigoplus_i M_i \otimes \mathrm{Hom}_A(M_i, V), \tag{1}$$

    and $N_i = \mathrm{Hom}_A(M_i, V)$ is a module for the centralizer B of A. Now use the Artin–Wedderburn theorem, and Schur's lemma, to explicitly identify B, and its centralizer, inside of $\mathrm{End}(V)$.

    Now we apply this to S_d and $\mathfrak{gl}(V)$:

Theorem 6 (Schur–Weyl duality)**.** *Consider the space $V^{\otimes^d}$ under the commuting actions of S_d and $\mathfrak{gl}(V)$, i.e., as a representation of the algebra $A \otimes B$, where $A = \mathbb{C}[S_d]$ and $B = U(\mathfrak{gl}(V))$. Then, the images $\bar{A}$, $\bar{B}$ of A and B in $End(V^{\otimes^d})$ are each others' commutants, that is,*

$$\bar{A} = End_B(V^{\otimes^d}), \text{ and}$$

$$\bar{B} = End_A(V^{\otimes^d}).$$

We have a decomposition

$$V^{\otimes^d} = \bigoplus_{\tau} \tau \otimes \theta(\tau), \tag{2}$$

where τ ranges all isomorphism classes of irreducible representations of S_d, and the $\theta(\tau) := Hom_{S_d}(\tau, V^{\otimes^d})$ are either zero, or distinct irreducible representations of $\mathfrak{gl}(V)$.

Notice that the action of $\mathfrak{gl}(V)$ on $V^{\otimes^d}$ is defined as we define tensor products of representations of Lie algebras, i.e., the image of $e \in \mathfrak{gl}(V)$ in $\mathrm{End}(V^{\otimes^d})$ is the element $\mathcal{S}_d e := \sum_{i=1}^{d} 1 \otimes \cdots \otimes e$ (i-th factor) $\otimes \cdots \otimes 1$.

Since both subalgebras are semisimple (complete reducibility), by Theorem 5 it is enough to prove the second claim.

Do some linear algebra to identify $\mathrm{End}_A(V^{\otimes^d})$ with the d-th symmetric power of the endomorphism ring (viewed as a vector space), $S^d\mathrm{End}(V)$. The d-th symmetric power of any vector space E is spanned by the symmetric tensors $e \otimes \cdots \otimes e$, for $e \in E$. Use the theory of symmetric polynomials to deduce that, for all $e \in \mathrm{End}(V)$, the element $e \otimes \cdots \otimes e$ is in the subalgebra generated by the images $\mathcal{S}_d(x)$ of elements $x \in \mathfrak{gl}(V)$.

Finally, prove:

Lemma 7. *Every irreducible representation of $\mathfrak{gl}_n$ restricts irreducibly to $\mathfrak{sl}_n$.*

Proof. We have $\mathfrak{gl}_n = \mathfrak{z} \oplus \mathfrak{sl}_n$, where $\mathfrak{z}$ is the center, but the center acts by a scalar, by Schur's lemma, so any $\mathfrak{sl}_n$-invariant subspace is also $\mathfrak{gl}_n$-invariant. □

Therefore, the irreducible representations of $\mathfrak{gl}_n$ constructed in Theorem 6 are also irreducible for $\mathfrak{sl}_n$.

11. This exercise combines the previous two.

    For the Lie algebra $\mathfrak{gl}_n$ with the standard Cartan of diagonal elements and the standard Borel of upper triangular elements, the dominant, integral weights are of the form

    $$\mathrm{diag}(z_1, z_2, \ldots, z_n) \mapsto \lambda_1 z_1 + \cdots + \lambda_n z_n,$$

    with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ some integers.

For the Lie algebra $\mathfrak{sl}_n$, the positive, integral weights are described similarly, except that the λ_i's are determined modulo the operation of adding the same constant to all of them. To reduce ambiguity, we can always take $\lambda_n = 0$ (but won't be doing that).

Prove that, if $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$ are integers, $d = \sum_i \lambda_i$, $\tau = V_\lambda$ is the irreducible representation of S_d constructed in Exercise 9 (ignoring the λ_i's which are zero), and V is an n-dimensional vector space, then the irreducible representation $\theta(\tau)$ of Schur–Weyl duality (2) has heighest weight λ.

On the other hand, if $\lambda_n > 0$ and V is a vector space of dimension $< n$, then $\theta(\tau) = 0$, i.e., τ does not appear in the decomposition of $V^{\otimes^d}$.

[Some hints for this exercise will be added later.]