TRANSFER OPERATORS AND HANKEL TRANSFORMS: HOROSPHERICAL LIMITS AND QUANTIZATION

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ABSTRACT. Transfer operators are conjectural "operators of functoriality," which transfer test measures and (relative) characters from one homogeneous space to another. In previous work [Sak21, Sak], we computed transfer operators associated to spherical varieties of rank one, and gave an interpretation of them in terms of geometric quantization. In this paper, we study how these operators vary in the horospherical limits of these varieties, where they have a conceptual interpretation related to scattering theory. We also revisit Jacquet's Hankel transform for the Kuznetsov formula, which is related to the functional equation of the standard L-function of GL_n, and provide a quantization interpretation for it.

CONTENTS

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1. Introduction

1.1. **Outline.**

1.1.1. Let G be a connected reductive group over a local field F, and, for a G-space X, consider the quotient stack $\mathfrak{X} = (X \times X)/G$, where G acts diagonally. There is a notion of "relative characters" for G-representations associated to this quotient (generalizing characters on the adjoint quotient of the group), and many interesting questions that one can ask about them, such as

> Are there relations between representations and characters associated to different spaces $\mathfrak{X}, \mathfrak{Y}$ as above?

Such relations are often conjectured by the Langlands program and its generalization, the relative Langlands program. Relative characters are, roughly, G-invariant generalized functions on $X \times X$, or, equivalently, functionals on the Schwartz space $\mathcal{S}(\mathfrak{X})$ [Sak16], that are eigen- for the Bernstein center or the Harish-Chandra center. We would therefore like to answer such questions by describing "transfer operators" between Schwartz spaces

$$\mathcal{T} : \mathcal{S}(\mathfrak{Y}) \to \mathcal{S}(\mathfrak{X}),$$

which pull back characters for $\mathfrak{X}$ to characters for $\mathfrak{Y}$.

Such operators were described previously in [Sak21], when X is an affine homogeneous spherical variety of rank one, and $\mathfrak{Y}$ is the Kuznetsov quotient for the group $G^* = \mathrm{SL}_2$ or PGL_2 (hence, the analogous to the space X for $\mathfrak{Y}$ is the quotient of G^* by a nontrivial unipotent subgroup, "twisted" by a nontrivial character ψ of this subgroup). The surprising discovery was that, despite the non-abelian nature of the problem, the transfer operators $\mathcal{T}$ were given by explicit, abelian, Fourier transforms. A phenomenological "quantization" interpretation for this fact provided in [Sak]: namely, we can view $\mathcal{S}(\mathfrak{Y})$ and $\mathcal{S}(\mathfrak{X})$ as "quantizations" of (roughly) the *same* symplectic variety, and these Fourier transforms can be understood as direct analogs of the "operators of change of Schrödinger model" (albeit in a non-linear setting).

One goal of this paper is to provide another interpretation of these operators, in terms of the "asymptotic cones" (boundary degenerations) of $\mathfrak{X}$ and $\mathfrak{Y}$. In the case of $\mathfrak{X}$, this is the degeneration that one obtains when one lets the spherical variety X degenerate to its horospherical limit, and in the case of $\mathfrak{Y}$, when one lets the character ψ degenerate to the trivial character. In Section 3, we generalize to all rank-one (affine homogeneous) varieties a discovery of [Sak22a, §4.3, 5] for the special cases $X = \mathbb{G}_m \backslash \mathrm{PGL}_2$ and $X = \mathrm{SL}_2$, namely, that *the transfer operators for $\mathfrak{X}$ and its boundary degeneration have exactly the same form* (in appropriate coordinates); see Theorem 3.4.1. We will also compare the two "explanations" in § 3.5, studying how the cotangent stacks of $\mathfrak{X}$ and $\mathfrak{Y}$ vary in these degenerations.

1.1.2. The argument for the study of transfer operators in the horospherical limit is no different than the one used in [Sak22a], up to knowing the *scattering operators* in each case. Those are the operators that control the *asymptotics* of generalized matrix coefficients on X, and the *Plancherel formula* for the "most continuous" part of $L^2(X)$ [SV17, DHS21]. Thus, we start in Section 2 by computing the scattering operators for all spherical varieties of rank one. The calculation should be very familiar to anyone with experience in the calculation of "unramified/spherical (eigen)functions" (for the Hecke algebra): Indeed, the scattering operators are the "functional equations" of the Casselman–Shalika method and its generalizations. Thus, our methods do not differ significanty from the ones employed in [Sak13] to compute spherical functions.

To extend this calculation to the ramified (principal series) representations, we need to be very pedantic with the definitions of Radon transforms (intertwining operators), to eliminate scalar ambiguities. Unfortunately, this makes that section quite technical, and the reader might be wise to just skim through it at first reading. On the flip side, we obtain a very rigid formula, encoded in Theorem 2.4.2, that relates scattering operators to the *gamma factors* of the local functional equation of L-functions. These γ-factors will be our first encounter, in this paper, of the deep relations between local harmonic analysis and local L-functions.

1.1.3. Of similar nature to the transfer operators are the so-called *Hankel transforms*, which are the trace-formula-theoretic incarnations of the functional equations of local L-functions. For the purposes of the present paper, we do not need to recall general definitions or conjectures around those, as we will only be concerned with the standard L-function of GL_n. By the work of Godement and Jacquet [GJ72], the local functional equation for this L-function is afforded by the Fourier transform (depending on an additive character ψ)

$$\mathcal{F} : \mathcal{D}(\mathrm{Mat}_n) \xrightarrow{\sim} \mathcal{D}(\mathrm{Mat}_n^*),$$

between half-densities on the vector space of $n \times n$ matrices, and on its dual, both viewed as $G = \mathrm{GL}_n \times^{\mathbb{G}_m} \mathrm{GL}_n$-spaces. The natural embeddings of GL_n in both Mat_n and Mat_n^* allow us to view $\mathcal{F}$ as a G-equivariant map between certain spaces of half-densities (or, by fixing a Haar measure, of measures) on GL_n. It therefore acts by a scalar on characters of irreducible representations π (at least, a scalar varying meromorphically, as the representation is twisted by characters of the determinant, and therefore defined for almost every π), and this scalar is, by definition, the gamma factor

$$\gamma(\pi, \frac{1}{2}, \psi)$$

of the standard (local) L-function of π. Because of its equivariant nature, the Fourier transform descends to a *Hankel transform* between spaces $(N, \psi)^2$-coinvariants (where N is the upper triangular unipotent subgroup, and ψ

now also denotes a generic character of it)

$$\mathcal{H} : \mathcal{D}(\mathrm{Mat}_n)_{(N,\psi)^2} \xrightarrow{\sim} \mathcal{D}(\mathrm{Mat}_n^*)_{(N,\psi)^2},$$

and we can now think of both sides as spaces $\mathcal{D}^-$, $\mathcal{D}^+$ of half-densities for the Kuznetsov quotient stack $\mathfrak{Y}$ of GL_n. Jacquet [Jac03] has computed a formula for this transform – see Theorem 4.1.2.

In this paper, we will give a quantization interpretation for this formula, Theorem 4.2.11, similar to the one given in [Sak] to the rank-one transfer operators. Namely, we will view the spaces $\mathcal{D}^-$, $\mathcal{D}^+$ as two different "geometric quantizations" of the *same* cotangent stack $\mathfrak{M}$ (the latter being the two-sided Whittaker reduction of $T^* \mathrm{Mat}_n = T^* \mathrm{Mat}_n^*$), given by two different Lagrangian foliations on it. And we will show that Jacquet's Hankel transform is given by the integral along the leaves of these foliations, just as in the case of intertwining operators between Schrödinger models for the oscillator representation. This point of view also allows us to give a geometric reformulation of Jacquet's proof.

1.1.4. I view the results of this paper as further evidence for the microlocal nature of conjectural "operators of functoriality." This idea appeared already in my earlier works mentioned above, but the study of Jacquet's Hankel transform given here is the first time that it is being confirmed in higher rank. Similar ideas, but not in the context of trace formulas, have appeared in talks and unpublished notes of Vincent Lafforgue.

Moreover, concepts such as "geometric quantization of symplectic stacks" are essentially unexplored, and presently very vague. This paper provides some examples and hints as to what they might mean.

1.2. **Acknowledgments.** It is my pleasure to dedicate this paper to Toshiyuki Kobayashi, on the occasion of his 60th birthday. I met Professor Kobayashi in 2007, when I visited him with Joseph Bernstein in Kyoto, and I have since enjoyed the privilege of talking to him at various occasions in Japan, Israel, and elsewhere. I have always admired his originality and independence as a mathematical thinker, but also the breadth and depth of his knowledge beyond mathematics, which ranges from Japanese mountain vegetables to Greek mythology.

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1.3. **Notation.** We will be working over a local field F, which is non-Archimedean of characteristic zero in Sections 2, 3, and Archimedean in 4. When no confusion arises, we will simply write X for the F-points of a variety X over F.

Once we fix a Haar measure on F, every volume form ω on a smooth variety X gives rise to a density (measure) $|\omega|$ on its F-points, in the standard way. We mostly follow the standard habit of denoting differentials and the associated densities by the same symbols, dx etc., when the meaning is clear from the context; when not, we write $|\omega|$ for the density. One can also define associated half-densities, which will be denoted by $|\omega|^{\frac{1}{2}}$.

We write $X /\!\!/ G$ for the invariant-theoretic quotient $\mathrm{Spec}\, F[X]^G$ of a (typically affine) variety X by the right action of a group G. When quotients are denoted by a single slash, X/G, unless we say otherwise, we will mean the stack quotient. However, knowledge of stacks is not required to read this paper: when this is not a variety, this symbol will mostly be a placeholder for an explicitly-defined object associated to the G-action on X, such as a space of orbital integrals.

We will sometimes switch right actions to left actions, via the rule $g^{-1} \cdot x = x \cdot g$. The quotient of the product $X \times Y$ two right G-varieties by the diagonal action of G will be denoted by $X \times^G Y$.

The "universal" or "abstract" Cartan of a reductive group G is defined as the quotient $A = B/N$ of *any* Borel subgroup B by its unipotent radical N; different choices for the Borel give a canonically isomorphic torus, which also comes equipped with a based root datum. Many constructions in this paper are "universal" in this sense: they can be described using a Borel subgroup, but a different choice leads to a canonically isomorphic construction. In those cases, we will feel free to use a Borel subgroup B, without commenting on the choice.

Finally, if M is a Hamiltonian G-space (i.e., a symplectic G-variety equipped with a moment map to $\mathfrak{g}^*$, for some group G), and $f \in \mathfrak{g}^*$ is a G-invariant element, the Hamiltonian reduction of M at f, denoted $M /\!\!/\!\!/_f G$, is the symplectic "space" $(M \times_{\mathfrak{g}^*} \{f\})/G$. Often, this quotient does not make sense as a variety, i.e., the G-action is not free. In those cases, the proper way to think of $M /\!\!/\!\!/_f G$ is as a derived stack, i.e., we also need to understand the fiber product over $\mathfrak{g}^*$ as a derived fiber product. However, in this paper, we will usually restrict to an open subset which is a variety, or else explain some less sophisticated way of using this quotient.

2. SCATTERING OPERATORS IN RANK ONE

2.1. **Spherical varieties of rank one, and their asymptotic cones.**

2.1.1. Let $X = H\backslash G$ be an affine, homogeneous spherical variety of rank one over a local field F, with G and H split reductive groups. In this section and the next, we will assume that F is non-Archimedean in characteristic zero, because this is where a "cleaner" theory of asymptotics is available, by [SV17, § 5].

Remark 2.1.2. There are no serious obstacles to extending the theory of asymptotics to positive characteristic, at least for the spaces of Table (1) below, other than some care that needs to be taken when defining these spaces and their boundary degenerations in small characteristics. Moreover, the calculations of scattering operators in the present chapter extend to the scattering operators for *tempered* representations, developed in [Car97], [DKSBP22], in the Archimedean case.

We will assume that X is contained in the following table, which contains all such varieties, up to the action of the "center" $\mathcal{Z}(X) := \mathrm{Aut}^G(X)$.

	X	$P(X)$	$\check{G}_X$	γ	L_X
A_1	$\mathbb{G}_m\backslash \mathrm{PGL}_2$	B	SL_2	α	$L(\mathrm{Std}, \frac{1}{2})^2$
A_n	$\mathrm{GL}_n \backslash \mathrm{PGL}_{n+1}$	$P_{1,n-1,1}$	SL_2	$\alpha_1 + \cdots + \alpha_n$	$L(\mathrm{Std}, \frac{n}{2})^2$
B_n	$\mathrm{SO}_{2n}\backslash\mathrm{SO}_{2n+1}$	$P_{\mathrm{SO}_{2n-1}}$	SL_2	$\alpha_1 + \cdots + \alpha_n$	$L(\mathrm{Std}, n-\frac{1}{2})L(\mathrm{Std}, \frac{1}{2})$
C_n	$\mathrm{Sp}_{2n-2} \times \mathrm{Sp}_2 \backslash \mathrm{Sp}_{2n}$	$P_{\mathrm{SL}_2 \times \mathrm{Sp}_{2(n-2)}}$	SL_2	$\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-1} + \alpha_n$	$L(\mathrm{Std}, n-\frac{1}{2})L(\mathrm{Std}, n-\frac{3}{2})$
F_4	$\mathrm{Spin}_9 \backslash F_4$	P_{Spin_7}	SL_2	$\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4$	$L(\mathrm{Std}, \frac{11}{2})L(\mathrm{Std}, \frac{5}{2})$
G_2	$\mathrm{SL}_3 \backslash G_2$	P_{SL_2}	SL_2	$2\alpha_1 + \alpha_2$	$L(\mathrm{Std}, \frac{5}{2})L(\mathrm{Std}, \frac{1}{2})$
$\mathcal{D}_2$	$\mathrm{SL}_2 = \mathrm{SO}_3\backslash\mathrm{SO}_4$	B	PGL_2	$\alpha_1 + \alpha_2$	$L(\mathrm{Ad}, 1)$
$\mathcal{D}_n$	$\mathrm{SO}_{2n-1}\backslash\mathrm{SO}_{2n}$	$P_{\mathrm{SO}_{2n-2}}$	PGL_2	$2\alpha_1 + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n$	$L(\mathrm{Ad}, n-1)$
$\mathcal{D}_4''$	$\mathrm{Spin}_7 \backslash \mathrm{Spin}_8$	P_{Spin_6}	PGL_2	$2\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4$	$L(\mathrm{Ad}, 3)$
B_3''	$G_2\backslash \mathrm{Spin}_7$	P_{SL_3}	PGL_2	$\alpha_1 + 2\alpha_2 + 3\alpha_3$	$L(\mathrm{Ad}, 3)$

(1)

The table also shows the dual group $\check{G}_X$ of X, with its positive coroot γ (the "normalized spherical root" of X) and the "associated L-value." The dual group admits an embedding into the dual group $\check{G}$ of G with positive coroot γ, and its Weyl group $W_X \simeq \mathbb{Z}/2$ has a generator w which can also be considered as an element of the Weyl group of G.

Associating L-values (i.e., special values of L-functions – in this case, of local L-factors) to these varieties is motivated by number theory, but it turns out that these mysterious quantities control a great deal of harmonic analysis. In this paper, we will see how they control *scattering operators* on the asymptotic cone of X, and *transfer operators* for its relative trace formula. We postpone the discussion of these L-values until we encounter them in harmonic analysis.

2.1.3. The purpose of this section is to recall and precisely calculate scattering operators, for all the varieties above. We must first recall the *asymptotic cone* (or *boundary degeneration*, in the language of [SV17]) of a (quasi-affine, homogeneous) spherical variety X. This is a *horospherical* variety $X_\varnothing$ (which, here, we will take to be homogeneous, by definition), which can be defined in several equivalent ways. One of them is by identifying $X_\varnothing$ with the open G-orbit in the normal bundle to a closed G-orbit in a "wonderful" (or rather, smooth toroidal) compactification of X [SV17, § 2.4]. In the case at hand, where our varieties are of rank 1, they all possess a canonical "wonderful" compactification $\bar{X}$, which is the union of X and a projective orbit isomorphic to the flag variety $\mathcal{B}_X = P(X)^-\backslash G$, for some parabolic $P(X)^-$. The opposite $P(X)$ of this parabolic (or rather, its conjugacy class) can be characterized as the stabilizer of the open Borel orbit $X^\circ \subset X$, and it admits a quotient to a torus A_X, which is the quotient by which it acts on $X^\circ/U_{P(X)}$. In the rank-one cases of the table above, $A_X \simeq \mathbb{G}_m$, either via the spherical root γ (when $\check{G} = \mathrm{SL}_2$) or via its square root (when $\check{G} = \mathrm{PGL}_2$). Let $q : P(X) \to A_X = \mathbb{G}_m$ denote the quotient map.

When X is symmetric (as are almost all[1] of the varieties in Table (1)), with $\theta =$ the involution associated to a point in the open orbit for a chosen Borel subgroup, $P(X)$ is known as the "minimal θ-split parabolic." The roots in the Levi of $P(X)$, for the varieties of Table (1), are those that are orthogonal to the spherical root γ, and the boundary degeneration is an A_X-torsor over $P(X)^-\backslash G$.

The quotient $q : P(X) \to A_X = \mathbb{G}_m$, also defines a character q^- for the opposite parabolic $P(X)^-$, by the natural identification of the abelianizations of $P(X)$ and $P(X)^-$. The boundary degeneration is an $A_X \times G$-variety which can be identified with

$$X_\varnothing = S\backslash G, \tag{2}$$

where $S = \ker q^-$, and our convention is that A_X acts on $X_\varnothing$ via $A_X \simeq P(X)^-/S$. For the cases of Table (1), $P(X)$ is self-dual, i.e., $P(X)^-$ is conjugate to $P(X)$, so we could have written $X_\varnothing$ as an A_X-torsor over $P(X)\backslash G$. However, our presentation helps remember our conventions for the A_X-action, and the action of the universal Cartan A of G: While A_X is identified as a quotient of the abstract Cartan via $B \to P(X) \to A_X$, it acts on $X_\varnothing$ via $P(X)^- \to A_X$.

Note that the identification (2) is by no means canonical – any translation by the action of A_X leads to another such identification. But the boundary degeneration $X_\varnothing$ is rigid, by construction, and this has consequences for our geometric and harmonic-analytic calculations (which will not always be invariant under the action of A_X). In particular, by [SV17, Theorem 5.1.1], there is a canonical "asymptotics" morphism

$$e_\varnothing^* : C^\infty(X) \to C^\infty(X_\varnothing), \tag{3}$$

with the property that, in a suitable sense (that we will not review here), Φ and $e_\varnothing^*\Phi$ "coincide in a neighborhood of the orbit at infinity."

As another manifestation of the rigidity of $X_\varnothing$, in a subsequent subsection we will show that there is a distinguished G-orbit $X_\varnothing^R$ in the "open Bruhat cell" of $X_\varnothing \times X_\varnothing$, that is, whose image in $\mathcal{B}_X \times \mathcal{B}_X$ under the map induced from $X_\varnothing \to \mathcal{B}_X$ (the map taking a point to the normalizer of its stabilizer) belongs to the open G-orbit.

[1] In fact, as abstract varieties, all are symmetric; but the varieties $\mathrm{SL}_3\backslash G_2$ and $G_2\backslash\mathrm{Spin}_7$ are symmetric under a *larger* group of automorphisms, namely, they isomorphic to $\mathrm{SO}_6\backslash\mathrm{SO}_7$ and $\mathrm{SO}_7\backslash\mathrm{SO}_8$, respectively.

2.1.4. Before we do that, let us recall an alternative, equivalent definition of $X_\varnothing$: It is the open G-orbit in the special fiber of the "affine degeneration" of X [SV17, 2.5]. This is a family $\mathcal{X} \to \overline{A_X}$ of affine varieties over a certain affine embedding of the torus A_X, which in our case can be identified with $\mathbb{A}^1$. The fibers over A_X are isomorphic to X, while the fiber over $0 \in \mathbb{A}^1$ is an affine horospherical variety, whose open G-orbit can be identified with $X_\varnothing$.

Example 2.1.5. Let $X = \mathrm{SO}_n \backslash \mathrm{SO}_{n+1}$, identified with the "unit sphere" in a maximally isotropic quadratic space V of the appropriate discriminant (if we want SO_n to be split, too). Then, in one version of the affine degeneration, we can consider V itself as a family of spaces containing X, with the map to $\mathbb{A}^1$ being the quadratic form. Note, however, that the fibers over $t \neq \mathbb{A}^1$ are only isomorphic over the algebraic closure, and depend on the square class of t over F. Therefore, it may be arithmetically preferable to define $\mathcal{X} = V \times_{\mathbb{A}^1} \mathbb{A}^1$, with $\mathbb{A}^1 \to \mathbb{A}^1$ the square map. This is the family over $\overline{A_X}$ mentioned above.

2.2. **Scattering operators.**

2.2.1. We generally normalize actions of various groups to be L^2-unitary. Specifically for the action of A_X on functions on $X_\varnothing$, if $\delta_{P(X)}$ denotes the modular character of $P(X)$ (the inverse of the modular character of $P(X)^-$), then this unitary action is given by

$$a \cdot \Phi(Sg) = \delta_{P(X)}^{\frac{1}{2}}(a)\Phi(Sag). \tag{4}$$

For χ a character of A_X in general position, consider the normalized (possibly degenerate) principal series representation

$$I_{P(X)^-}^G(\chi) = \mathrm{Ind}_{P(X)^-}^G(\chi\delta_{P(X)}^{-\frac{1}{2}}).$$

Up to a choice of scalar, we have an isomorphism

$$C^\infty(A_X\backslash X_\varnothing, \chi) \simeq I_{P(X)^-}^G(\chi), \tag{5}$$

where the space on the left is the space of smooth (A_X, χ)-eigenfunctions under the normalized action above.

The scattering maps that we would like to describe form a meromorphic (in χ) family of morphisms:

$$\mathscr{S}_{w,\chi} : C^\infty(A_X\backslash X_\varnothing, \chi^{-1}) \to C^\infty(A_X\backslash X_\varnothing, \chi),$$

such that $\mathscr{S}_{w,\chi^{-1}} \circ \mathscr{S}_{w,\chi} = I$ (note that ${}^w\chi = \chi^{-1}$), and characterized as follows:

For χ in general position, there is a unique up to scalar morphism $I_{P(X)^-}^G(\chi) \to C^\infty(X)$, and the scattering morphism $\mathscr{S}_{w,\chi}$ encodes its asymptotics, in the sense that the composition of this embedding with the asymptotics morphism (3) lives in the space of $\mathscr{S}_w$-invariant pairs of the direct sum

$$C^\infty(A_X\backslash X_\varnothing, \chi^{-1}) \oplus C^\infty(A_X\backslash X_\varnothing, \chi).$$

More precisely, the following commutative diagram characterizes the scattering morphisms, see [DHS21, § 10.17]:

$$\begin{tikzcd}
& C^\infty((A_X,\chi^{-1})\backslash X^h_\varnothing) \arrow[r, "\mathfrak{M}^{-1}_{\chi^{-1}}"] \arrow[dd, dashed] & C^\infty((A_X,\chi^{-1})\backslash X_\varnothing) \arrow[dd, "\mathscr{S}_{w,\chi}"] \\
C_c^\infty(X) \arrow[ur, "\mathfrak{N}_{\chi^{-1}}"] \arrow[dr, "\mathfrak{N}_\chi"'] & & \\
& C^\infty((A_X,\chi)\backslash X^h_\varnothing) \arrow[r, "\mathfrak{M}^{-1}_{\chi}"] & C^\infty((A_X,\chi)\backslash X_\varnothing),
\end{tikzcd} \tag{6}$$

where the notation is as follows:

(1) The space $X^h_\varnothing$ is the space of *generic horocycles* on X, or on $X_\varnothing$. It classifies pairs (P, Y), where $P \in \mathcal{B}_X$ (that is, in the class of parabolics $P(X)$), with unipotent radical U, and Y is a U-orbit in the open P-orbit of X, or of $X_\varnothing$; by [SV17, Lemma 2.8.1], X and $X_\varnothing$ give canonically isomorphic spaces by this construction. In the rank-one cases that we are considering, $X^h_\varnothing$ is G-isomorphic to $X_\varnothing$, but not canonically. Moreover, the A_X-action on $X_\varnothing$, defined above, gives rise to the *inverse* A_X-action on $X^h_\varnothing$, under such an isomorphism; it is therefore best to think of $X^h_\varnothing$ as $S^+\backslash G$, where $S^+ \subset P(X)$ is the kernel of the character q.

(2) The operator $\mathfrak{M}_\chi$, which can be thought of as the "standard intertwining operator", is the operator which, in a region of convergence, takes a function in $C^\infty(A_X, \chi\backslash X_\varnothing)$ and integrates it over generic horocycles. *Because there is no canonical measure on those horocycles, this operator depends on a choice of such measures*, and more canonically has image in the sections of a certain line bundle over $X^h_\varnothing$ (the line bundle dual to the line bundle whose fiber over a horocycle is the set of invariant measures on it — see [SV17, §15.2]). However, in the cases of Table (1) that we are interested in this paper (and, more generally, whenever X, hence also $X_\varnothing$, admits a G-invariant measure), such a choice can be made G-equivariantly, and it will not matter for the commutativity of the diagram — the important point here being that horocycles in $X_\varnothing$ and X are identified, and the choices of Haar measures must be made compatibly.

(3) The operator $\mathfrak{N}_\chi$ is, similarly, the integral over the horocycles on X, followed by an averaging over horocycles in the same B-orbit,

against the character χ^{-1} of A_X; that is, for a horocycle Y, considered both as a point in $X_\emptyset^h$ and as a subspace of X,

$$\mathfrak{R}_\chi\Phi(Y) = \int_{A_X}\left(\int_{aY}\Phi(y)dy\right)\chi^{-1}\delta_{P(X)}^{-\frac{1}{2}}(a)da.$$

In other words, this is the standard morphism to the principal series representation $C^\infty((A_X,\chi)\backslash X_\emptyset^h)$, given by an integral over the open Borel orbit. Again, the measure used on A_X does not matter for the commutativity of the diagram.

Note that, via the noncanonical identification (5), the morphism $\mathscr{S}_{w,\chi}$ has to be (for almost all χ) a multiple of the "standard intertwining operator"

$$\mathfrak{R}_\chi : I_{P(X)^-}^G(\chi^{-1}) \to I_{P(X)^-}^G(\chi)$$

given by the integral

$$\mathfrak{R}_\chi f(g) = \int_{U_{P(X)}^-} f(\tilde{w}ug)du,$$

for some lift $\tilde{w}$ of w to G. Hence, our goal is to describe this constant of proportionality, but we must first give a careful definition of these intertwining operators, since they depend on the isomorphism (5), and the lift $\tilde{w}$, as well as the measure u that appear in the definition of $\mathfrak{R}_\chi$. It turns out that there is a way to do define $\mathfrak{R}_\chi$ (which I will call "spectral Radon transform"), that is independent of the choices of these isomorphisms.

2.3. **The canonical Radon transform.**

2.3.1. The spectral Radon transforms $\mathfrak{R}_\chi$ will be obtained as Mellin transforms of a Radon transform $\mathfrak{R} : C_c^\infty(X_\emptyset) \to C^\infty(X_\emptyset)$ which, under isomorphisms $X_\emptyset \simeq S\backslash G$ as before, can be written

$$\mathfrak{R}_\chi\Phi(Sg) = \int_{U_{P(X)}^-}\Phi(S\tilde{w}ug)du.$$

Interpreting such an integral without fixing such an isomorphism or a lift $\tilde{w}$ for the Weyl group element, we must describe a distinguished G-orbit $X_\emptyset^R \subset X_\emptyset \times X_\emptyset$, living over the open Bruhat cell of $\mathcal{B}_X \times \mathcal{B}_X$, such that the Radon transform is given by:

$$\mathfrak{R}(\Phi)(y) = \int_{(x,y)\in X_\emptyset^R}\Phi(x)dx. \tag{7}$$

2.3.2. *Choice of measure for Radon transforms.* This integral (7) also depends on fixing (G-equivariantly) measures dy on the fibers of $X_\emptyset^R$ with respect to the first projection. The formulas for the scattering operators that we will present in this section are only true for one such choice of measure, because the spectral scattering operators $\mathscr{S}_{w,\chi}$ are completely canonical (and unitary on $L^2((A_X,\chi)\backslash X_\emptyset)$, for χ unitary). The measure will depend on a nontrivial, unitary additive character $\psi : F \to \mathbb{C}^\times$, which we fix, thereby fixing the corresponding self-dual Haar measure on F.

It is enough to describe the measure for the standard intertwining operators for *nondegenerate* principal series,

$$\mathfrak{R}_{\chi,\tilde{w}} : I_B^G({}^{w^{-1}}\chi) \to I_B(\chi),$$

defined for almost every character by meromorphic continuation of

$$\mathfrak{R}_{\chi,\tilde{w}} f(g) = \int_{(N\cap w^{-1}Nw)\backslash N} f(\tilde{w}ng)dn,$$

since the operator for degenerate principal series descends for those. This will be applied, in the course of our argument, to various spaces which are (noncanonically) quotients of $N\backslash G$ (i.e., horospherical), hence, being mindful of the noncanonical nature of such isomorphisms, we should think of f above as a function on a space Y which is isomorphic to $N\backslash G$, and of the representative $\tilde{w}$ of the Weyl element w as determining a distinguished G-orbit in $Y \times Y$, so that $\mathfrak{R}_{\chi,\tilde{w}}$ is given by an integral analogous to (7). Choosing a reduced decomposition $\tilde{w} = \tilde{w}_1 \cdots \tilde{w}_n$ into representatives for the *simple* reflections, it is well-known (and immediate to check) that $\mathfrak{R}_{\tilde{w}} = \mathfrak{R}_{\tilde{w}_1} \circ \cdots \circ \mathfrak{R}_{\tilde{w}_n}$, hence we only need to describe the measure for w a simple reflection in the Weyl group. In this setting, we are reduced to the case of SL_2, through the map from SL_2 to the Levi of the parabolic corresponding to the simple reflection.

Hence, we take $G = \mathrm{SL}_2$, with a chosen G-orbit on $(N\backslash \mathrm{SL}_2)^2$. We can then fix an isomorphism between SL_2 and the special linear group of a symplectic vector space (V,ω), with the chosen G-orbit equal to $V^R = \{(v_1,v_2)|\omega(v_1,v_2) = 1\}$. The canonical choice of measure, then, is the one described in [Sak22a, § 3.3], and it is the following: By our isomorphism, the intertwining operator descends from the following Radon transform on V:

$$\mathfrak{R}_{\tilde{w}}\Phi(v) = \int_{\omega(u,v)=1} \Phi(u)du,$$

with du the measure to be described. But the horocycle $\omega(u,v) = 1$, here, is canonically an F-torsor under $(x,u) \mapsto u + xv$, thus inheriting the Haar measure from F.

This completes our description of the measure used to define Radon transforms. We complement it with a well-known formula for its inverse. It uses the *gamma factors* of the local functional equation of L-functions:

$$\gamma(\chi,s,\psi)L(\chi,s) = \epsilon(\chi,s,\psi)L(\chi^{-1},1-s). \tag{8}$$

They are defined by the theory of local zeta integrals (Iwasawa–Tate), see [Sak22a, 2.1.4] for a recollection.

When χ is a character of a split torus T, and r is a representation of the dual torus, we will use Langlands' notation for L- and γ-factors:

$$\gamma(\chi, r, s, \psi) = \prod_i \gamma(\chi \circ \lambda_i, s, \psi), \tag{9}$$

where λ_i ranges over the weights of r (with multiplicities).

Lemma 2.3.3. *The operator $\mathfrak{R}_{\chi,\tilde{w}}$ is defined and invertible almost everywhere, with inverse*

$$\mathfrak{R}_{\chi,\tilde{w}}^{-1} = \prod_{\alpha>0, w^{-1}\alpha<0} \gamma(\chi, \check{\alpha}, 1, \psi)\gamma(\chi, -\check{\alpha}, 1, \psi)\mathfrak{R}_{w^{-1}\chi,\tilde{w}^{-1}}. \tag{10}$$

Proof. This reduces again, by the same argument, to SL_2, where the formula can be proven by relating Radon transforms to Fourier transforms [Sak22a, (3.16)], and using the Fourier inversion formula. □

Remark 2.3.4. A technical detail: When $w^2 = 1$ (for example, in the case of SL_2) the operator $\mathfrak{R}_{w^{-1}\chi,\tilde{w}^{-1}}$ is not necessarily equal to the operator $\mathfrak{R}_{w^{-1}\chi,\tilde{w}}$ – which is why we stress the choice of representative $\tilde{w}$ in the notation. For example, for SL_2, we cannot find $\tilde{w}$ with $\tilde{w}^2 = 1$. More abstractly, in terms of the chosen G-orbit on $Y \times Y$ (where $Y \simeq N\backslash G$), this orbit does not necessarily get preserved by switching the two copies of Y. However, this will be the case for the cases of Table 1 – even when $X = \mathrm{SL}_2$, the group acting will be $(\mathrm{SL}_2 \times \mathrm{SL}_2)/\mu_2^{\mathrm{diag}} = \mathrm{SO}_4$, where the longest Weyl element is represented by an involution in the group.

2.3.5. *Review of the basic cases.* Such canonical G-orbits were described in [Sak22a, §3] for the basic cases of the two families of Table (1), denoted by A_1 and $\mathcal{D}_2$: $X = \mathrm{SO}_2\backslash\mathrm{SO}_3 = \mathbb{G}_m\backslash\mathrm{PGL}_2$ and $X = \mathrm{SO}_3\backslash\mathrm{SO}_4 \simeq \mathrm{SL}_2$. The word "canonical" is too strong, since there is nothing compelling about choosing these G-orbits over others. However, they are independent of choices of isomorphisms, and eventually their importance is that they are useful in obtaining exact formulas for the scattering operators.

Remark 2.3.6. The varieties of the second group of Table (1) admit various forms over the field F with G and H split, parametrized by $F^\times/(F^\times)^2$. Namely, since they are all isomorphic (as abstract varieties) to $\mathrm{SO}_{2n-1}\backslash\mathrm{SO}_{2n}$, the forms depend on the discriminant of the orthogonal complement of a $(2n-1)$-dimensional, maximally isotropic quadratic space V inside of a split $(2n)$-dimensional quadratic space U. We *choose* to compute only for the cases where this discriminant is (square equivalent to) 1, of which $X = \mathrm{SL}_2$ is the base case; minor modifications are needed to accommodate other discriminants. The same choice was made for the calculation of transfer operators in [Sak21], but this was not explicitly stated there.

2.3.7. In the case of $X \simeq \mathbb{G}_m\backslash\mathrm{PGL}_2$, we should fix a 2-dimensional symplectic vector space V, an isomorphism $G \simeq \mathrm{PGL}(V)$, and an isomorphism of $X \simeq \mathrm{SO}_2\backslash\mathrm{SL}_2$ with the space of quadratic forms of the form $q = xy$

for some choice of standard symplectic coordinates (i.e., such that the symplectic form is $dx \wedge dy$). Once the isomorphism $G \simeq \mathrm{PGL}(V)$ is fixed, such an identification for X is unique up to the G-automorphism group $\mathbb{Z}/2$ of X, which sends the quadratic form xy to the quadratic form $-xy$ (corresponding to the standard coordinates $(-y, x)$). The variety $X_\varnothing$ can, then, be identified with the variety of rank-one degenerate quadratic forms on V, which is the same as $V^{*\times}/\{\pm 1\}$ (where V^* is the dual, and the exponent $^\times$ denotes the complement of zero), since those quadratic forms, over the algebraic closure, are squares of linear forms. Via the identification $V \simeq V^*$ afforded by the symplectic form, we can also identify $X_\varnothing$ with $V^\times/\{\pm 1\}$. Evaluation of the quadratic form:

$$\mathrm{ev} : X_\varnothing \times V \to \mathbb{G}_a$$

on $\mathbb{G}_a$ gives rise to our "canonical" G-orbit $X_\varnothing^G$, as the image of $\mathrm{ev}^{-1}(1)$ under $X_\varnothing \times V^\times \to X_\varnothing \times X_\varnothing$. Applying the automorphism $xy \mapsto -xy$ to X also induces the analogous automorphism on $X_\varnothing$, fixing the distinguished orbit $X_\varnothing^R \subset X_\varnothing^2$.

With this distinguished orbit fixed, the scattering operator

$$\mathscr{S}_{w,\chi} : C^\infty((A_X, \chi^{-1})\backslash X_\varnothing) \to C^\infty((A_X, \chi^{-1})\backslash X_\varnothing)$$

was calculated in [Sak22a, Theorem 3.5.1]

$$\mathscr{S}_{w,\chi} = \gamma(\chi, \frac{\check\alpha}{2}, \frac{1}{2}, \psi^{-1})\gamma(\chi, \frac{\check\alpha}{2}, \frac{1}{2}, \psi)\gamma(\chi, -\check\alpha, 0, \psi) \cdot \mathfrak{R}_\chi. \tag{11}$$

Here, ψ is any nontrivial unitary character of the additive group F, and the Radon transforms are computed with respect to a measure that is directly proportional to the self-dual Haar measure on F with respect to ψ. Note that the change-of-character formula for γ-factors,

$$\gamma(\chi, s, \psi(a\bullet)) = \chi(a)|a|^{s-\frac{1}{2}}\gamma(\chi, s, \psi(\bullet)) \tag{12}$$

(see [Del73, 3.3.3], taking into account that the self-dual measure for $\psi(a\bullet)$ is $|a|^{\frac{1}{2}}$ the self-dual measure for ψ) implies that the factor

$$\gamma(\chi, \frac{\check\alpha}{2}, \frac{1}{2}, \psi^{-1})\gamma(\chi, \frac{\check\alpha}{2}, \frac{1}{2}, \psi)\gamma(\chi, -\check\alpha, 0, \psi)$$

will get multiplied by $|a|^{-\frac{1}{2}}$, which is the inverse of the factor by which the measure defining $\mathfrak{R}_\chi$ will be multiplied, making the above expression for $\mathscr{S}_{w,\chi}$ independent of ψ.

2.3.8. In the case of $X = \mathrm{SL}_2 = \mathrm{SL}(V)$, on the other hand, we can identify $X_\varnothing$ with the space of endomorphisms of V of rank 1, i.e., operators of the form $\tau : V/L \to L'$, where L and L' are lines in V. The distinguished G^{diag}-orbit, in that case, is the set of pairs $(\tau_1, \tau_2) \in X_\varnothing \times X_\varnothing$ with $\tau_1 + \tau_2 \in \mathrm{SL}(V)$. With this distinguished orbit at hand, we define the spectral Radon transforms $\mathfrak{R}_\chi$, and [Sak22a, Theorem 3.5.1] in this case gives the following formula for the scattering operator:

$$\mathscr{S}_{w,\chi} = \gamma(\chi, \check\alpha, 0, \psi^{-1})\gamma(\chi, -\check\alpha, 0, \psi) \cdot \mathfrak{R}_\chi. \tag{13}$$

Again, the product of gamma factors would be multiplied by $|a|^{-1}$, if we changed ψ to $\psi(a\bullet)$, which is exactly inverse the factor by which the measure defining $\mathfrak{R}_\chi$ would change. (Notice here that the intertwining operator $\mathfrak{R}_\chi$ is given by an integral over a 2-dimensional unipotent subgroup.)

2.3.9. *The other cases of Table (1).* The calculation of scattering operators for the general case of a variety of Table (1) will be reduced to the basic cases by means of the following lemma. For every subset I of the simple roots of G, we denote by P_I the corresponding class of parabolics (or a representative), by U_{P_I} its unipotent radical, and by L_I its Levi quotient.

Lemma 2.3.10. *Let $Z \subset X$ be a closed B-orbit.*

- *For the cases of Table (1) with $\check G_X = \mathrm{SL}_2$, Z is of rank zero, and for every simple root α such that $Y := ZP_\alpha \neq Z$, $Y_2 := Y/U_{P_\alpha}$ is L-isomorphic to $\mathrm{SO}_2\backslash\mathrm{SL}_2 \simeq \mathbb{G}_m\backslash\mathrm{PGL}_2$ under some morphism $L \to \mathrm{PGL}_2$.*
- *For the cases of Table (1) with $\check G_X = \mathrm{PGL}_2$, and for all simple roots α we have $ZP_\alpha = Z$, except for two orthogonal simple roots α, β for which, setting $Y := ZP_{\alpha\beta}$, $Y_2 := Y/U_{P_{\alpha\beta}}$ is L-isomorphic to $\mathrm{SO}_3\backslash\mathrm{SO}_4$ under some homomorphism $L \to \mathrm{SO}_4$.*

As a matter of notation, what we denote by Y/U_P here is the *geometric* quotient of Y by the U_P-action, not the stack quotient, which could be different because of nontrivial stabilizers.

Proof. This is a combination of Lemmas 2.2.4 and 2.3.5 of [Sak21]. □

In the remainder of this subsection, we will describe a "canonical" G-orbit in $X_\varnothing \times X_\varnothing$, living over the open Bruhat cell in $\mathcal{B}_X \times \mathcal{B}_X$. This will give rise to a "canonical" Radon transform, which we will use in the next section to describe the scattering operators.

A G-orbit on $X_\varnothing \times X_\varnothing$ over the open Bruhat cell in $\mathcal{B}_X \times \mathcal{B}_X$ is equivalent to an equivariant isomorphism $\iota : X_\varnothing \xrightarrow{\sim} X_\varnothing^h$, where $X_\varnothing^h$ is the space of generic horocycles on $X_\varnothing^h$, as above. Indeed, such an isomorphism defines the distinguished G-orbit

$$X_\varnothing^R = \{(x,y) \in X_\varnothing \times X_\varnothing | y \in \iota(x)\},$$

and, vice versa, can be recovered from it. Similarly, this is equivalent to describing a distinguished G-orbit

$$X_\varnothing^{R,h} \subset X_\varnothing^h \times X_\varnothing^h,$$

which by the canonical isomorphism $X_\varnothing^h \simeq X^h$ of [SV17, Lemma 2.8.1], can be understood as a distinguished G-orbit $X^{R,h} \subset X^h \times X^h$. This is the orbit that we will describe.

The orbit (and the Radon transform) will depend, a priori, on the choice of a closed B-orbit Z, as in Lemma 2.3.10; a posteriori, by the calculation of scattering operators, it doesn't, a fact that can also be proved directly (but we will not do). Hence, fix Z, let Y be as in Lemma 2.3.10, and let P be the parabolic P_α, resp. $P_{\alpha\beta}$, appearing in the two cases of the lemma. Let $Y^\circ \subset Y$ be the open B-orbit; without fixing a Borel subgroup, we can consider Y° as a G-orbit on $X \times \mathcal{B}$ — denoted by $\tilde{Y}^\circ$ to avoid confusion. Similarly to the definition of X^h, we can define

$$X^{h,Y} = \{(B,M)|B \in \mathcal{B},\ M \text{ is a } U_B\text{-orbit with } (z,B) \in \tilde{Y}^\circ \text{ for any } m \in M\}.$$

Hence, these are not generic horocycles, but horocycles corresponding to the B-orbit Y°.

Let $\tilde{X}^h$ be the base change $X^h \times_{\mathcal{B}_X} \mathcal{B}$ of X^h to the full flag variety $\mathcal{B}$. Fixing a Borel subgroup $B \in \mathcal{B}$, we have noncanonical isomorphisms:

$$\tilde{X}^h = A_X \times^B G, \tag{14}$$

$$X^{h,Y} = A_Y \times^B G, \tag{15}$$

where A_Y is the torus quotient by which B acts on the geometric quotient Y/U_B.

Let w' be the Weyl group element w_α or $w_\alpha w_\beta$, respectively (where w_α, w_β denote simple reflections), for each of the two cases of Lemma 2.3.10. A result of Knop [Kno95, § 6] implies that there is an element w_1 of the Weyl group of G, such that

$$Y^\circ \times^{N \cap w_1 N w_1^{-1}} \tilde{w}_1 N \xrightarrow{\sim} X^\circ \tag{16}$$

under the restriction of the action map to $w_1 N \subset G$; here, $\tilde{w}_1$ is any lift of w_1, thought of as a double coset of $B\backslash G/B$, to G. Note that this implies that $\mathrm{codim} Y^\circ = \mathrm{length}(w_1)$, and $A_Y^{w_1} = A_X$ Let $w_1 \in W$ be such an element. It is known [Bri01], [Sak13, § 6.2] that the nontrivial element $w \in W_X$ is equal to $w_1^{-1} w' w_1$.

By Lemma 2.3.10, the P-variety $Y_2 := Y/U_P$ is isomorphic either to $\mathbb{G}_m\backslash \mathrm{PGL}_2$ or to SL_2 under the action of the Levi L of P. In each of the two cases, a distinguished L-orbit on $Y_2^h \times Y_2^h$, living over the open Bruhat cell, was described in § 2.3.5 above. This corresponds to a G-orbit on $X^{h,Y}$, living over the Bruhat cell corresponding to w'. We denote by $X^{R,h,Y}$ this G-orbit.

Now, the product $X^{Y,h} \times \tilde{X}^h$ lives over the product $\mathcal{B} \times \mathcal{B}$. It follows from (16) that there is a *distinguished* G-orbit $X'^h \subset X^{Y,h} \times \tilde{X}^h$ that lives over the Bruhat cell corresponding to w_1, that is, if we fix a basepoint $B \in \mathcal{B}$, over the G-orbit of a pair $(B, B' = \tilde{w}_1 B \tilde{w}_1^{-1})$, where, as above, $\tilde{w}_1$ is any lift of $w_1 \in B\backslash G/B$ to G. The fiber of this orbit over (B,B') consists of all pairs of horocycles $((B,M),(B',xN'))$ with $N' \subset B'$ the unipotent radical and $x \in M$. Equation (16) implies that if the horocycle (B,M) belongs to $X^{Y,h}$, the horocycle (B', xN') is generic, i.e., belongs to X^h.

Consider the space of quadruples
$$(x_1, y_1, y_2, x_2)$$
with $x_1, x_2 \in X^h$, $y_1, y_2 \in X^{Y,h}$, $(y_i, x_i) \in X'^h$ (for $i = 1, 2$) and $(y_1, y_2) \in X^{R,h,Y}$. It lives over the G-orbit of the quadruple
$$(B, \tilde{w}_1 B \tilde{w}_1^{-1}, (\tilde{w}')^{-1}\tilde{w}_1 B \tilde{w}_1^{-1}\tilde{w}', \tilde{w}^{-1}B\tilde{w})$$
of Borel subgroups, where the tilde denotes, again, lifts to G, and $\tilde{w} = \tilde{w}_1^{-1}\tilde{w}'\tilde{w}_1$. *The distinguished G-orbit $X^{R,h} \subset X^h \times X^h$, now, is the image of this subset under the first and last projections, composed with the projection $\tilde{X}^h \to X^h$.* The reader can immediately check that this is indeed a G-orbit, using the noncanonical isomorphisms (14), (15).

By the discussion above, this G-orbit corresponds to a distinguished G-orbit $X_\varnothing^R \subset X_\varnothing \times X_\varnothing$, that we use to define the Radon transform $\mathfrak{R} : C_c^\infty(X_\varnothing) \to C^\infty(X_\varnothing)$ by (7). The spectral Radon transforms $\mathfrak{R}_\chi$ descend from it by Mellin transform. For the choice of measure used to define $\mathfrak{R}$, see § 2.3.2.

2.4. **Formula for the scattering operators.**

2.4.1. The main result of this section is the following, where we fix a unitary additive character ψ:

Theorem 2.4.2. *For the cases of Table (1), in terms of the spectral Radon transforms*
$$\mathfrak{R}_\chi : C^\infty(A_X\backslash X_\varnothing, \chi^{-1}) \to C^\infty(A_X\backslash X_\varnothing, \chi)$$
that descend from the canonical Radon $\mathfrak{R}$ described in § 2.3, the scattering operator $\mathscr{S}_{w,\chi}$ for the nontrivial element $w \in W_X$ is given by
$$\mathscr{S}_{w,\chi} = \mu_X(\chi) \cdot \mathfrak{R}_\chi, \tag{17}$$
where μ_X is given by the following formulas:

- *for the cases with $\check{G} = \mathrm{SL}_2$, with $L_X = L(\mathrm{Std}, s_1)L(\mathrm{Std}, s_2)$,*
$$\mu_X(\chi) = \gamma(\chi, \frac{\check{\gamma}}{2}, 1 - s_1, \psi^{-1})\gamma(\chi, \frac{\check{\gamma}}{2}, 1 - s_2, \psi)\gamma(\chi, -\check{\gamma}, 0, \psi), \tag{18}$$
- *for the cases with $\check{G} = \mathrm{PGL}_2$, with $L_X = L(\mathrm{Ad}, s_0)$,*
$$\mu_X(\chi) = \gamma(\chi, \check{\gamma}, 1 - s_0, \psi^{-1})\gamma(\chi, -\check{\gamma}, 0, \psi). \tag{19}$$

Here, $\gamma(\chi, \check{\lambda}, s, \psi)$ denotes the gamma factor (9) of the local functional equation for the abelian L-function associated to the composition of χ with the cocharacter $\check{\lambda} : \mathbb{G}_m \to A_X$. Notice that, by (12), if we replace ψ by $\psi(a\bullet)$, for some $a \in F^\times$, the factor $\mu_X(\chi)$ changes by a factor of $|a|^{-s}$, where $s = s_1 + s_2 - \frac{1}{2}$, in the first case, and $s = s_0$, in the second case. It so happens (see [Sak21, § 1.2]) that $2s = \dim X - 1 =$ the dimension of the unipotent radical of $P(X)$. Therefore, the measure used to define the Radon transform $\mathfrak{R}$, which is proportional to the self-dual measure with respect to

ψ to the power $\dim_{U_{P(X)}}$, changes by $|a|^s$, making the formula above for $\mathscr{S}_{w,\chi}$ independent of ψ.

2.4.3. The proof of the theorem will be given in a somewhat telegraphic fashion, because the arguments are essentially the same as the ones used to compute "functional equations" in [Sak13, Section 6]. The reader who wishes to read a detailed and explicit account of the arguments that follow is advised to look at that reference. The "added value" of the present work consists in the following:

- We adopt the formalism of scattering operators, introduced in [SV17]. This adds an extra layer of complication; for example, [Sak13] only considered the functional equations represented by the first vertical (dotted) arrow of (6), while the scattering operators are given by the second vertical arrow. The benefit, however, is that the scattering operators are directly related to the asymptotics (and other constructions such as the "unitary asymptotics" of the Plancherel formula), a fact that we will use in the next section.
- We allow for general principal series representations, not only unramified ones. This is quite straightforward and does not change any of the arguments of [Sak13], once the "basic cases" of § 2.3.5 have been computed.
- We "rigidify" certain constructions to eliminate ambiguities up to automorphism groups. In the unramified case of [Sak13], integral structures provided such rigidifications, but, to consider ramified characters, this is not enough. Therefore, we replace various explicit integrals and principal series representations by (noncanonically) isomorphic constructions that live over the various horocycle spaces already introduced. This, again, makes the discussion more pedantic (e.g., to fix the canonical Radon transforms introduced in § 2.3), but is a more conceptual description of these constructions.

Proof of Theorem 2.4.2. The proof will follow the argument of [Sak13, § 6.5]. Because of the isomorphism (16), the morphism $\mathfrak{N}_\chi$ of Diagram (6), given by an integral over the open $P(X)$-orbit X°, can be expressed in terms of a similar integral over the smaller B-orbit Y°. Namely, choosing a lift $\tilde{w}_1$ of w_1, we get a bijection $M \mapsto M\tilde{w}_1 N w_1^{-1}$ between horocycles in Y° for a given Borel subgroup B and generic horocycles for the Borel subgroup $\tilde{w}_1 B \tilde{w}_1^{-1}$. Then, for $\Phi \in C_c^\infty(X)$, we have

$$\mathfrak{N}_\chi \Phi(M\tilde{w}_1 N \tilde{w}_1^{-1}) = \int_{A_X} \int_{M\tilde{w}_1 N\tilde{w}_1^{-1}} \Phi(xa)dx\, \chi^{-1}\delta_{P(X)}^{-\frac{1}{2}}(a)da =$$
$$\int_{(N\cap\tilde{w}_1^{-1}N\tilde{w}_1)\backslash N} \int_{A_Y} \int_M \Phi(ya\tilde{w}_1 n\tilde{w}_1^{-1})dy \cdot$$
$$w_1^{-1}\left(\chi^{-1}\cdot \delta_{P(X)}^{-\frac{1}{2}} \cdot \delta_{(N\cap\tilde{w}_1^{-1}N\tilde{w}_1)\backslash N}\right)(a)da\, dn, \quad (20)$$

where we have factored the invariant measure dx on the horocycle $M\tilde{w}_1 N \tilde{w}_1^{-1}$ in terms of an invariant measure dy on the horocycle M and an invariant measure on $(N \cap \tilde{w}_1^{-1} N \tilde{w}_1)\backslash N$. The inner two integrals of (20) represent a morphism that we can denote

$$\mathfrak{N}_\chi^Y : C_c^\infty(X) \to C^\infty((A_Y, {}^{w_1^{-1}}\chi)\backslash X^{h,Y}).$$

There is some abuse of notation in the expressions above, due to the fact that the modular character $\delta_{(N\cap\tilde{w}_1^{-1}N\tilde{w}_1)\backslash N}$ is a character of the universal Cartan A of G which does not, in general, factor through A_X. Related to this is that the space $X^{h,Y}$ does not, in general, admit a G-invariant measure, that would give rise to a unitary A_Y-action. Therefore, the two inner integrals of (20) actually have image in a certain *equivariant sheaf* over $X^{h,Y}$, induced from a certain character of $\ker(A \to A_Y)$. These characters are explicated, e.g., in [Sak08, § 5.2], and there is no reason to dwell over them here, since they will not play a role in our calculation.

For clarity, we will explicate noncanonical isomorphisms with principal series representations, as we did for the asymptotic cone in (5). First of all, we can fix a "standard" split Cartan subgroup in a "standard" Borel subgroup B, which will determine the choice of lifts of the Weyl group elements, up to elements of that Cartan, so that it makes sense to write $w_1 B w_1^{-1}$.

- For the horospherical space $X^h = X^h_\varnothing$, we have $C^\infty((A_X,\chi)\backslash X^h) \simeq I^G_{P(X)}(\chi)$. Note that this is a subrepresentation of $I_B^G(\chi\delta_{L(X)}^{-\frac12})$, where $\delta_{L(X)}$ is the modular character of the Borel of $L(X)$.
- For the Y-horospherical variety $X^{h,Y}$, what we denoted above by $C^\infty((A_Y, {}^{w_1^{-1}}\chi)\backslash X^{h,Y})$ is isomorphic to $I^G_{w_1Bw_1^{-1}}(\chi\delta_{L(X)}^{-\frac12})$.
- The outer integral of (20) represents a standard intertwining operator
$$I^G_{w_1Bw_1^{-1}}(\chi{\delta_{L(X)}}^{-\frac12}) \to I^G_B(\chi\delta_{L(X)}^{-\frac12}),$$
that takes the image of $\mathfrak{N}_\chi^Y$ into the subspace $I^G_{P(X)}(\chi)$.

We can now expand the left-hand triangle of Diagram 6 to

$$\begin{tikzcd}
 & C^\infty((A_Y, {}^{w_1^{-1}}\chi^{-1})\backslash X^{h,Y}) \arrow[r, dashed, "\mathfrak{R}^Y_{\chi^{-1}}"] & C^\infty((A_X, \chi^{-1})\backslash X^h_\varnothing) \\
C_c^\infty(X) \arrow[ur, "\mathfrak{N}^Y_{\chi^{-1}}"] \arrow[urr, "\mathfrak{N}_{\chi^{-1}}"'] \arrow[dr, "\mathfrak{N}^Y_\chi"'] \arrow[drr, "\mathfrak{N}_\chi"] & & \\
 & C^\infty((A_Y, {}^{w_1^{-1}}w_1^{-1}\chi)\backslash X^{h,Y}) \arrow[r, dashed, "\mathfrak{R}^Y_\chi"] & C^\infty((A_X, \chi)\backslash X^h_\varnothing),
\end{tikzcd} \tag{21}$$

where $\mathfrak{R}^Y_\chi$, here, is the Radon transform represented by the outer integral of (20), and we used dotted arrows to signify that the map is only defined in the image of the maps $\mathfrak{N}^Y_{\chi^{-1}}$, resp. $\mathfrak{N}^Y_\chi$. Note that this Radon transform can be expressed in terms of the canonical G-orbit in $X^{h,Y} \times \tilde{X}^h$, described at the end of § 2.3, by a formula analogous to (7). Again, for clarity, we present the same diagram with the noncanonical isomorphisms discussed above,

$$\begin{tikzcd}
 & I^G_{w_1Bw_1^{-1}}(\chi^{-1}\delta_{L(X)}^{-\frac{1}{2}}) \arrow[r, dashed, "\mathfrak{R}^Y_{\chi^{-1}}"] & I^G_{P(X)}(\chi^{-1}) \\
C_c^\infty(X) \arrow[ur, "\mathfrak{N}^Y_{\chi^{-1}}"] \arrow[urr, "\mathfrak{N}_{\chi^{-1}}"'] \arrow[dr, "\mathfrak{N}^Y_\chi"'] \arrow[drr, "\mathfrak{N}_\chi"] & & \\
 & I^G_{w_1Bw_1^{-1}}(\chi\delta_{L(X)}^{-\frac{1}{2}}) \arrow[r, dashed, "\mathfrak{R}^Y_\chi"] & I^G_{P(X)}(\chi).
\end{tikzcd} \tag{22}$$

We can fix the isomorphisms between (21) and (22), so that the various Radon transforms in the latter are given by the standard intertwining operators

$$I_{wBw^{-1}}(\chi) \ni f \mapsto \int_{(N\cap wNw^{-1})\backslash N} f(n\bullet) \in I_B(\chi).$$

It is more convenient from now on to work with (22), to invoke a well-known fact about these intertwining operators. Namely, the top and bottom row of the diagram fit into commutative squares (we present only the one for the bottom):

$$\begin{tikzcd}
I^G_{w'w_1Bw_1^{-1}w'}(\chi\delta_{L(X)}^{-\frac{1}{2}}) \arrow[r, dashed, "\mathfrak{R}^Y_\chi"] \arrow[d] & I^G_{P(X)^-}(\chi) \arrow[d, "\mathfrak{M}_{\chi^{-1}}"] \\
I^G_{w_1Bw_1^{-1}}(\chi\delta_{L(X)}^{-\frac{1}{2}}) \arrow[r, dashed, "\mathfrak{R}^Y_\chi"] & I^G_{P(X)}(\chi),
\end{tikzcd} \tag{23}$$

where w', as before, is the involution w_α or $w_\alpha w_\beta$ corresponding to Lemma 2.3.10 (so that the nontrivial $w \in W_X$ is equal to $w_1^{-1}w'w_1$), and the vertical arrows are also standard intertwining operators.

In the upper right corner, we should recognize the space $C^\infty((A_X,\chi)\backslash X_\varnothing)$, with the Radon transform $\mathfrak{M}_{\chi^{-1}}$ that appears in Diagram 6. For the upper left corner, we have the following interpretation: Let P be the parabolic denoted by P_α, resp. $P_{\alpha\beta}$, in the two cases of Lemma 2.3.10, and let

$Y_2 = Y/U_P$. The operator $\mathfrak{N}^Y_\chi$ of (21) can be further factored in the form

$$\mathfrak{N}^Y_\chi \Phi(M) = \int_{A_Y} \int_{M_2} \int_{U_{P,y}\backslash U_P} \Phi(yua) du\, dy \cdot \\ {}^{w_1^{-1}}\left(\chi^{-1} \cdot \delta_{P(X)}^{-\frac{1}{2}} \cdot \delta_{(N\cap \tilde{w}_1^{-1} N \tilde{w}_1)\backslash N}\right)(a) da,$$

where we have written M_2 for the image of the horocycle M modulo U_P. The innermost integral clearly represents surjection $C_c^\infty(X) \twoheadrightarrow C_c^\infty(Y_2)$, which is equivariant under the derived group of the Levi $[P,P]$ in P. (It is also P-equivariant, if we twist by the appropriate modular character, but this does not matter for our calculation.) The upper left corner of (23) admits a restriction map to $w_1 P w_1^{-1}$:

$$I^G_{w' w_1 B w_1^{-1} w'}(\chi \delta_{L(X)}^{-\frac{1}{2}}) \to I^{w_1 P w_1^{-1}}_{w' w_1 B w_1^{-1} w'}(\chi \delta_{L(X)}^{-\frac{1}{2}}),$$

and by thinking of the right hand side as a representation of the derived group $[L,L]$ of the Levi of P, we can identify it with $C^\infty((A_Y, {}^{w_1^{-1}}\chi)\backslash Y_{2,\varnothing})$. Under the analogous restriction map of the lower left corner, the left vertical arrow is the analog of $\mathfrak{M}_{\chi^{-1}}$ for Y_2.

Putting together the above, *we deduce that the following diagram of scattering operators and Radon transforms commutes*:

$$\begin{tikzcd}
I^G_{w' w_1 B w_1^{-1} w'}(\chi^{-1} \delta_{L(X)}^{-\frac{1}{2}}) \arrow[r, dashed, "\mathfrak{R}^Y_\chi"] \arrow[d, "\mathscr{S}^{Y_2}_{w', {}^{w_1^{-1}}\chi}"] & I^G_{P(X)^-}(\chi^{-1}) \simeq C^\infty((A_X, \chi^{-1})\backslash X_\varnothing) \arrow[d, "\mathscr{S}_{w,\chi}"] \\
\mathscr{S}_{w,\chi} I^G_{w' w_1 B w_1^{-1} w'}(\chi \delta_{L(X)}^{-\frac{1}{2}}) \arrow[r, dashed, "\mathfrak{R}^Y_\chi"] & I^G_{P(X)^-}(\chi) \simeq C^\infty((A_X, \chi)\backslash X_\varnothing),
\end{tikzcd} \tag{24}$$

where we are using the symbol $\mathscr{S}^{Y_2}_{w', {}^{w_1^{-1}}\chi}$ for the scattering operator for Y_2, but also for its induction to

$$I^G_{w' w_1 B w_1^{-1} w'}(\chi^{-1} \delta_{L(X)}^{-\frac{1}{2}}) \simeq I^G_{w_1 P w_1^{-1}}(C^\infty((A_Y, {}^{w_1^{-1}}\chi^{-1})\backslash Y_{2,\varnothing})).$$

(Again, the action of P on the inducing data is twisted by a character that we do not need to specify.)

The theorem now follows from the formula for $\mathscr{S}^{Y_2}_{w', {}^{w_1^{-1}}\chi}$ recalled in § 2.3.5, and formula 10 for the inverse of Radon transform. The details of how the various gamma factors simplify to produce the final answer are essentially the same as in the unramified case, therefore I point the reader to [Sak13, § 6.5].

□

3. DEGENERATION OF TRANSFER OPERATORS

In the previous section, we saw how the L-value associated to the spherical varieties of Table (1) controls, through its gamma factors, the scattering operators associated to the theory of asymptotics.

On the other hand, in [Sak21] it was discovered that the L-value also has a different function: It controls certain "transfer operators," which translate stable orbital integrals for the quotient $\mathfrak{X} = (X \times X)/G$ to orbital integrals for the associated Kuznetsov quotient $\mathfrak{Y} = (N,\psi)\backslash G^*/(N,\psi)$, where $G^* = \mathrm{PGL}_2$ or SL_2, according as $\check{G}_X = \mathrm{SL}_2$ or PGL_2, respectively.

In this section, we will see how the two results are related via the *degeneration of the transfer operators to the asymptotic cone*.

3.1. **Transfer operators for rank-one spherical varieties.** We keep assuming that the field F is non-Archimedean. Here, we will need to work with measures, rather than functions, so we will use $\mathcal{S}(X)$ to denote the space of Schwartz measures (compactly supported, smooth) on the points of a variety X over F. Note that the varieties of Table (1) all carry a G-invariant measure, so the translation from functions to measures is quite innocuous (up to the noncanonical choice of such a measure). But measures have natural pushforwards, and, using $\mathfrak{X}$ as a symbol for the quotient of $X \times X$ by the diagonal action of G, we will denote by $\mathcal{S}(\mathfrak{X})$ the image of the pushforward map, from Schwartz measures on $X \times X$, to measures on the invariant-theoretic quotient $(X \times X) /\!\!/ G := \operatorname{Spec} F[X \times X]^G$, which in our cases is just an affine line.

The meaning of $\mathcal{S}(\mathfrak{Y})$ for the Kuznetsov quotient of G^* is similar, but because of the twist by the Whittaker character ψ, we need to fix some conventions, as in [Sak21, § 1.3]. The result is a space of measures on the quotient $N\backslash G /\!\!/ N$, which is again isomorphic to the affine line.

The main theorem of [Sak21] is the following.

Theorem 3.1.1 ([Sak21, Theorem 1.3.1])**.** *For the varieties of Table (1), and for appropriate identifications of the invariant-theoretic quotients $(X \times X) /\!\!/ G$ and $N\backslash G^* /\!\!/ N$ with the affine line (with coordinates that we denote by ξ or ζ, depending on whether $G^* = \mathrm{PGL}_2$ or SL_2, respectively), the operator $\mathcal{T}$ described below gives rise to an injection*

$$\mathcal{T} : \mathcal{S}(\mathfrak{Y}) \to \mathfrak{S}(\mathfrak{X}). \tag{25}$$

- *When $\check{G}_X = \mathrm{SL}_2$ with $L_X = L(\mathrm{Std}, s_1)L(\mathrm{Std}, s_2)$, $s_1 \geqslant s_2$,*

$$\mathcal{T}_\varnothing f(\xi) = |\xi|^{s_1-\frac{1}{2}} \left(|\bullet|^{\frac{1}{2}-s_1}\psi(\bullet)d\bullet\right) \star \left(|\bullet|^{\frac{1}{2}-s_2}\psi(\bullet)d\bullet\right) \star f(\xi).$$

- *When $\check{G}_X = \mathrm{PGL}_2$ with $L_X = L(\mathrm{Ad}, s_0)$,*

$$\mathcal{T}_\varnothing f(\zeta) = |\zeta|^{s_0-1} \left(|\bullet|^{1-s_0}\psi(\bullet)d\bullet\right) \star f(\zeta).$$

Here, $\star$ denotes multiplicative convolution on $F^\times$, $D\star f(x) = \int_{a\in F^\times} D(a)f(a^{-1}x)$.

The operator bijective for a certain explicit enlargement of the space $\mathcal{S}(\mathfrak{Y})$ of measures, that we will not recall here. Conjecturally, the transfer operator also translates *relative characters* of one quotient to relative characters of the other. This has been proven in several cases [GW21, Sak22a, Sak22b]. We will recall the notion of relative characters below, noting that in the case of $\mathrm{SL}_2 = \mathrm{SO}_3\backslash\mathrm{SO}_4$ they coincide with the usual *stable* characters, while in the case of the Kuznetsov quotient $\mathfrak{Y}$ they are often called "Bessel distributions."

3.2. **Asymptotics of test measures.** To relate transfer operators to the scattering maps computed in the previous section, we recall [DHS21, Theorem 1.8] that the asymptotics map (3) restricts to a morphism

$$e_\varnothing^* : \mathcal{S}(X) \to \mathcal{S}^+(X_\varnothing), \tag{26}$$

where $\mathcal{S}^+(X_\varnothing)$ denotes a certain enlargement of $\mathcal{S}(X_\varnothing)$, namely, a space of smooth measures on $X_\varnothing$, whose support has compact closure in an affine embedding (in this case, the "affine closure" $\operatorname{Spec} F[X_\varnothing]$). The "restriction" of the asymptotics map (3) from smooth functions to Schwartz measures makes sense, because by [SV17, § 4.2] an invariant measure on X canonically induces an invariant measure on $X_\varnothing$.

The spectral scattering maps $\mathscr{S}_{w,\chi}$ studied in the previous section are actually the Mellin transforms of a scattering operator $\mathfrak{S}_w$, an involution on $\mathcal{S}^+(X_\varnothing)$ which we think of as an action of the Weyl group $W_X \simeq \mathbb{Z}/2$. This involution is (A_X, w_γ)-equivariant, that is, it intertwines the action of $a \in A_X$ with the action of ${}^{w_\gamma}a = a^{-1}$, when this action is normalized to be unitary. Since we are working with measures, here, the unitary action analogous to (4) is

$$a \cdot f(Sg) = \delta_{P(X)}^{-\frac{1}{2}}(a) f(Sag). \tag{27}$$

Because of the equivariance property of $\mathfrak{S}_w$, it descends to an operator from the (A_X, χ^{-1})-coinvariants to the (A_X, χ)-coinvariants of $\mathcal{S}^+(X_\varnothing)$; for χ in general position, these can be identified with the corresponding coinvariants of the standard Schwartz space:

Lemma 3.2.1. *For an open dense set of $\chi \in \widehat{A_X}_{\mathbb{C}}$ (the complex Lie group of characters of A_X), the inclusion $\mathcal{S}(X_\varnothing) \hookrightarrow \mathcal{S}^+(X_\varnothing)$ induces an isomorphism on (A_X, χ)-coinvariants, hence identifying those with $\mathcal{S}(A_X\backslash X_\varnothing, \chi)$, the space of smooth measures on $A_X\backslash X_\varnothing$, valued in the sheaf whose sections are (A_X, χ)-equivariant functions (for the normalized action (4) on functions) on $X_\varnothing$. Moreover, the "twisted pushforward maps"*

$$\mathcal{S}^+(X_\varnothing) \to \mathcal{S}(A_X\backslash X_\varnothing, \chi), \tag{28}$$

are meromorphic in χ.

Proof. This follows directly from the description of $\mathcal{S}^+(X_\varnothing)$ as a "fractional ideal" in the space of rational sections of $\chi \mapsto \mathcal{S}(A_X\backslash X_\varnothing, \chi)$, in [DHS21, (1.18)]. □

We will denote the map (28) by $f \mapsto \check{f}(\chi)$, and think of it as a Mellin transform.

The spectral scattering morphisms of the previous section are the meromorphic family of operators descending from $\mathfrak{S}_w$ through this map:

$$\mathscr{S}_{w_\gamma,\chi} : \mathcal{S}((A_X, \chi^{-1})\backslash X_\varnothing) \to \mathcal{S}((A_X, \chi)\backslash X_\varnothing).$$

Note that, up to a choice of measure, $\mathcal{S}((A_X, \chi)\backslash X_\varnothing)$ is what was denoted before by $C^\infty((A_X, \chi)\backslash X_\varnothing)$.

Similar maps exist for the Whittaker model $\mathcal{S}(N, \psi\backslash G^*)$, with $e^*_\varnothing$ there mapping to a space $\mathcal{S}^+(N\backslash G^*)$ of measures on the space $N\backslash G^*$ (the "degenerate Whittaker model," with the trivial character on N). In what follows, we will also be denoting by Y the "quotient" $(N, \psi)\backslash G^*$ (i.e., the space $N\backslash G^*$ endowed with a sheaf defined by the nondegenerate character ψ), and by $Y_\varnothing$ its degeneration (the space $N\backslash G$ with the trivial sheaf). The image of the Schwartz space $\mathcal{S}(Y) := \mathcal{S}(N, \psi\backslash G^*)$ under the asymptotics map will be denoted by $\mathcal{S}^+(N\backslash G^*)$:

$$e^*_\varnothing : \mathcal{S}(N, \psi\backslash G^*) \to \mathcal{S}^+(N\backslash G^*).$$

Now, let us repeat the construction of test measures for the relative trace formula, for the asymptotic cones of X and Y. Denote by $\mathcal{S}^+(X_\varnothing \times X_\varnothing/G)$ the pushforward of $\mathcal{S}^+(X_\varnothing) \otimes \mathcal{S}^+(X_\varnothing)$ to the invariant-theoretic quotient $(X_\varnothing \times X_\varnothing) /\!\!/ G$. We adopt the same convention for $(N\backslash G^* \times N\backslash G^*)/G^* = N\backslash G^*/N$, letting $\mathcal{S}^+(N\backslash G^*/N)$ be the image of the pushforward map from $\mathcal{S}^+(N\backslash G^*) \otimes \mathcal{S}^+(N\backslash G^*)$ to $N\backslash G^* /\!\!/ N = (N\backslash G^*) \times (N\backslash G^*) /\!\!/ G^*$.

Composing the asymptotics maps with these pushforwards, we obtain maps

$$\begin{tikzcd} \mathcal{S}(N, \psi\backslash G^*) \otimes \mathcal{S}(N, \psi^{-1}\backslash G^*) \arrow[r, "(e^*_\varnothing \otimes e^*_\varnothing)_G"] & \mathcal{S}^+(N\backslash G^*/N) \arrow[d, dashed, "\mathcal{T}_\varnothing"] \\ \mathcal{S}(X \times X) \arrow[r, "(e^*_\varnothing \otimes e^*_\varnothing)_G"] & \mathcal{S}^+(X_\varnothing \times X_\varnothing/G) \end{tikzcd} , \tag{29}$$

with the map $\mathcal{T}_\varnothing$ to be introduced. The main result of this section will be a description of a canonical "transfer operator" $\mathcal{T}_\varnothing$, characterized by compatibility with relative characters; therefore, let us first discuss those.

3.3. **Relative characters.**

3.3.1. Let π be an admissible representation of G, with $\tilde{\pi}$ its contragredient. A *relative character* on the quotient $\mathfrak{X} = (X \times X)/G$ for π is a functional that factors

$$J_\pi : \mathcal{S}(X \times X) \to \pi \otimes \tilde{\pi} \to \mathbb{C},$$

with the first map $G \times G$-equivariant and the second the defining pairing between π and $\tilde{\pi}$.

A source of relative characters is the Plancherel formula for $L^2(X)$: Once we choose a Plancherel measure μ_X, as well as a G-invariant measure dx in order to embed $\mathcal{S}(X) \hookrightarrow L^2(X)$, we have a decomposition of the bilinear pairing $\langle f_1, f_2 \rangle = \int_X \frac{f_1 f_2}{dx}$ as

$$\langle f_1, f_2 \rangle = \int_{\hat{G}} J_\pi(f_1 \otimes f_2) \mu_X(\pi),$$

uniquely determined by this formula for μ_X-almost every π in the unitary dual $\hat{G}$.

On the other hand, we can build relative characters by pullback via the asymptotics maps. Of interest to us here will be the relative characters that we can pull back from the space $\mathcal{S}^+(X_\varnothing \times X_\varnothing/G)$. Namely, there is a *canonical* open embedding $A_X \hookrightarrow X_\varnothing \times X_\varnothing /\!\!/ G$, such that

- the identity maps to the image of the distinguished G-orbit $X_\varnothing^R \subset X_\varnothing \times X_\varnothing$, described in § 2.3.
- the map is equivariant with respect to the A_X-action descending from the action on the first, or equivalently the second, copy of $X_\varnothing$.

Consider the meromorphic family of functionals I_χ obtained as pullbacks of the composition of maps

$$I_\chi : \mathcal{S}(X \times X) \xrightarrow{(e_\varnothing^* \otimes e_\varnothing^*)_G} \mathcal{S}^+(X_\varnothing \times X_\varnothing/G) \xrightarrow{\int \chi^{-1} \delta_{P(X)}^{-\frac{1}{2}}} \mathbb{C},$$

where the last arrow denotes the integral against the pullback of the character $\chi^{-1}\delta_{P(X)}^{-\frac{1}{2}}$ from A_X to (a dense open subset of) $X_\varnothing \times X_\varnothing$. A priori, this arrow converges only on the subspace of such measures that are supported on $A_X \subset (X_\varnothing \times X_\varnothing) /\!\!/ G$, but it is not hard to make sense of it for almost every χ:

Lemma 3.3.2. *The functionals I_χ converge when $|\chi(\varpi^{\check{\gamma}})| \ll 1$, and extend meromorphically to all $\chi \in \widehat{A_{X\mathbb{C}}}$ (the complex Lie group of characters of A_X). For an open dense set of χ's, the functional I_χ is a relative character for the normalized principal series representation $\pi_\chi = I_{P(X)}(\chi)$, that is, it factors through a morphism*

$$\mathcal{S}(X \times X) \to \pi_\chi \otimes \widetilde{\pi_\chi} \xrightarrow{\langle \, , \, \rangle} \mathbb{C}.$$

Proof. The elements of $\mathcal{S}^+(X_\varnothing \times X_\varnothing)$ are of moderate growth, and their support has compact closure in the affine closure of $X_\varnothing \times X_\varnothing$. Therefore, the elements of $\mathcal{S}^+(X_\varnothing \times X_\varnothing/G)$ are of moderate growth, and their support has compact closure in $\mathbb{A}^1(F)$, when we identify $A_X \simeq \mathbb{G}_m$ via the character γ or $\frac{\gamma}{2}$. Convergence for $|\chi(\varpi^{\check{\gamma}})| \ll 1$ follows.

Before we sketch the proof of meromorphic continuation, let us explain why they are relative characters. The composition

$$I_\chi^\varnothing : \mathcal{S}^+(X_\varnothing) \otimes \mathcal{S}^+(X_\varnothing) \longrightarrow \mathcal{S}^+(X_\varnothing \times X_\varnothing/G) \xrightarrow{\int \chi^{-1} \delta_{P(X)}^{-\frac{1}{2}}} \mathbb{C},$$

is, by construction, (A_X, χ^{-1})-equivariant with respect to the normalized action of A_X on *either* $\mathcal{S}^+(X_\varnothing)$-factor, and therefore the map factors through the (A_X, χ)-coinvariants of each factor. By Lemma 3.2.1, for χ in general position, these coinvariants are equal to $\mathcal{S}((A_X, \chi)\backslash X_\varnothing)$, which is isomorphic to $I_{P(X)^-}(\chi) \simeq \pi_{\chi^{-1}}$. Hence, I_χ factors through $\pi_{\chi^{-1}} \otimes \pi_{\chi^{-1}}$. Since χ is w-conjugate to its inverse, for generic χ we have that $\pi_\chi \simeq \pi_{\chi^{-1}} = \widetilde{\pi_\chi}$.

Finally, for the claim of meromorphicity, it is not hard to express the map $I_\chi^\varnothing$, on the tensor product $\mathcal{S}((A_X, \chi)\backslash X_\varnothing) \otimes \mathcal{S}((A_X, \chi)\backslash X_\varnothing)$ of these coinvariant spaces, in terms of standard intertwining operators and the standard pairing between π_χ and $\pi_{\chi^{-1}}$. More precisely, if we use the canonical Radon transform introduced in § 2.3 (translated to measures, by multiplying both sides by a G-invariant measure):

$$\mathfrak{R}_\chi : \mathcal{S}((A_X, \chi^{-1})\backslash X_\varnothing) \to \mathcal{S}((A_X, \chi)\backslash X_\varnothing),$$

then the map $I_\chi^\varnothing$ factors through

$$\mathcal{S}((A_X, \chi)\backslash X_\varnothing) \otimes \mathcal{S}((A_X, \chi)\backslash X_\varnothing) \xrightarrow{I \times \mathfrak{R}_{\chi^{-1}}} \mathcal{S}((A_X, \chi)\backslash X_\varnothing) \otimes \mathcal{S}((A_X, \chi^{-1})\backslash X_\varnothing) \xrightarrow{\langle \bullet \rangle} \mathbb{C}, \quad (30)$$

where the arrow labeled $\langle \bullet \rangle$ is the integral over the diagonal of $A_X \backslash X_\varnothing$, against an invariant measure that is prescribed by the measure used for the Radon transform. (I leave the details of this measure to the reader.) $\square$

The factorization (30) will be very useful in what follows, so let us record it as

$$I_\chi^\varnothing (f_1 \otimes f_2) = \left\langle \check{f}_1(\chi), \mathfrak{R}_{\chi^{-1}} \check{f}_2(\chi) \right\rangle . \quad (31)$$

Of course, we could have applied Radon transform to the first, instead of the second factor.

Similarly, consider the Kuznetsov quotient for G^*. Identifying the Cartan A^* of G^* with the torus of diagonal elements through the upper triangular Borel, the embedding

$$A^* \to \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} A^* \subset G^*$$

descends to an embedding of A^* in $N\backslash G^* /\!\!/ N$. We similarly have a relative character J_χ on $\mathcal{S}(N, \psi\backslash G^*) \otimes \mathcal{S}(N, \psi^{-1}\backslash G^*)$, obtained as the composition

$$\mathcal{S}(N, \psi\backslash G^*) \otimes \mathcal{S}(N, \psi^{-1}\backslash G^*) \xrightarrow{e_\varnothing^* \otimes e_\varnothing^*} \mathcal{S}^+(N\backslash G^*/N) \xrightarrow{\int \chi^{-1} \delta_{B^*}^{-\frac{1}{2}}} \mathbb{C} .$$

Notice that here we are integrating here against the character $\chi^{-1}\delta_{B^*}^{-\frac{1}{2}}$ of $A^* \subset N\backslash G^* /\!\!/ N$, where $B^* \subset G^*$ is the Borel subgroup of G^*; this makes J_χ a relative character for the normalized principal series $I_{B^*}^{G^*}(\chi)$. The analogous statements of Lemma 3.3.2 all hold in this setting.

We identify A^* with the Cartan A_X of X via the identification of the dual groups with SL_2 or PGL_2, i.e., so that the positive root of G^* corresponds to the spherical root γ.

3.3.3. We will now discuss the relation of the relative characters I_χ, J_χ to the Plancherel formulas for $L^2(X)$, resp. $L^2((N,\psi)\backslash G^*)$. Recall [SV17, Theorem 7.3.1] that the spaces $L^2(X)$, $L^2(Y)$ have discrete and continuous spectra, with the continuous spectra naturally parametrized by unitary characters of A_X, modulo inversion (i.e., modulo the action of W_X). More precisely, using the index $\varnothing$ for the orthogonal complement of the subspace spanned by relative discrete series (the images of irreducible subrepresentations $\pi \hookrightarrow L^2(X)$ or $L^2(Y)$), there are Plancherel decompositions

$$L^2(X)_\varnothing \text{ or } L^2(Y)_\varnothing = \int_{\widehat{A_X}/W_X} \mathcal{H}_\chi d\chi, \tag{32}$$

where $\widehat{A_X}$ denotes the unitary dual of A_X, and the unitary representation $\mathcal{H}_\chi$ can be identified with "the" (A_X,χ)-equivariant (for the normalized action) Hilbert space completion of $\mathcal{S}(X_\varnothing)$ (resp. of $\mathcal{S}(Y_\varnothing)$.

To be precise, $L^2(X)$ is not quite a completion of $\mathcal{S}(X)$ (it is, rather, a completion of a space of half-densities), but, fixing a Haar measure on X (and hence a Haar half-density), we can consider it to be so. We can similarly fix a Haar measure on $N\backslash G^*$, and take $d\chi$, in the decompositions above, to be such that it pulls back to a fixed Haar measure on $\widehat{A_X}$ under the finite map $\widehat{A_X} \to \widehat{A_X}/W_X$. The Plancherel decompositions (32), then, give rise to *relative characters* J_χ^Z *(where* $Z = X$ *or* Y*)*, which are the pullbacks of the Hermitian forms of $\mathcal{H}_\chi$ to $\mathcal{S}(Z)$ – but our convention will be to consider them as bilinear forms, i.e., as functionals on $\mathcal{S}(X)\otimes\mathcal{S}(X)$, resp. on $\mathcal{S}(N,\psi\backslash G^*)\otimes\mathcal{S}(N,\psi^{-1}\backslash G^*)$.

Proposition 3.3.4. *Let* $\mu(\chi)$ *be the function introduced in Theorem 2.4.2.*

The product $I_\chi\mu_X(\chi)$ *is a family of relative characters on* X *that is invariant under the* W_X*-action* $\chi\mapsto\chi^{-1}$.

For suitable choices of invariant measures, the most continuous part of the Plancherel decomposition of $L^2(X)$ *(corresponding to the canonical subspace* $L^2(X)_\varnothing \subset L^2(X)$*) reads:*

$$\langle f_1, \overline{f_2}\rangle_\varnothing = \int_{\widehat{A_X}/W_X} I_\chi(f_1\otimes f_2)\mu_X(\chi)d\chi, \tag{33}$$

where $d\chi$ *is such that it pulls back to a Haar measure on* $\widehat{A_X}$.

Similarly, for the Whittaker space $Y = (N,\psi)\backslash G^*$*, the family of relative characters* $\gamma(\chi,-\check\gamma,0,\psi)J_\chi$ *is invariant under* $\chi\mapsto\chi^{-1}$*, and the Plancherel formula for the most continuous part of* $L^2(N,\psi\backslash G^*)$ *reads:*

$$\langle f_1, \overline{f_2}\rangle_\varnothing = \int_{\widehat{A_X}/W_X} J_\chi(f_1\otimes f_2)\gamma(\chi,-\check\gamma,0,\psi)d\chi. \tag{34}$$

Proof. The proof, which is based on Theorem 2.4.2 and the Plancherel formula of [SV17, Theorem 7.3.1], is identical to that of [Sak22a, Theorem 3.6.3]. The Whittaker case is already included in [Sak22a, Theorem 3.6.3] (in the case of $G^* = \mathrm{SL}_2$, but the case of $G^* = \mathrm{PGL}_2$ is identical, since the scattering operators only depend on the pullback to SL_2). □

The relative characters $I_\chi(f_1 \otimes f_2)\mu_X(\chi)$ and $J_\chi(f_1 \otimes f_2)\gamma(\chi, -\check{\gamma}, 0, \psi)$ of the proposition, which appear in the Plancherel formulas of two different spaces with the *same* Plancherel measure, or any multiple of this pair by the same scalar, will be called "matching" relative characters.

3.4. **Asymptotic transfer operators.** Putting everything together, we can now prove the main result of this section, which is a generalization of the results of [Sak22a, §4.3, 5] to all varieties of Table (1). It relies on the notion of "matching relative characters," which will be explained in the next subsection, but we first formulate the theorem for motivation. Moreover, having identified an open subset of $(X \times X) /\!\!/ G$ and of $N\backslash G^* /\!\!/ N$ with $A_X = A^*$, we will use the following coordinate for this space:

- when $\check{G}_X = \mathrm{SL}_2$, an element of A^* will be denoted $\begin{pmatrix} \xi & \\ & 1 \end{pmatrix}$;
- when $\check{G}_X = \mathrm{SL}_2$, an element of A^* will be denoted $\begin{pmatrix} \zeta & \\ & \zeta^{-1} \end{pmatrix}$.

Theorem 3.4.1. *There is a unique A_X-equivariant operator*

$$\mathcal{T}_\varnothing : \mathcal{S}^+(N\backslash G^*/N) \to \mathcal{S}^+(X_\varnothing \times X_\varnothing/G),$$

such that, for all almost all $\chi \in \widehat{A_X}$, the pullbacks of

$$\begin{array}{ccccc} \mathcal{S}(N,\psi\backslash G^*) \otimes \mathcal{S}(N,\psi^{-1}\backslash G^*) & \xrightarrow{e^*_\varnothing \otimes e^*_\varnothing} & \mathcal{S}^+(N\backslash G^*/N) & & \\ & & \big\downarrow \mathcal{T}_\varnothing & & \\ \mathcal{S}(X \times X) & \xrightarrow{e^*_\varnothing \otimes e^*_\varnothing} & \mathcal{S}^+(X_\varnothing \times X_\varnothing/G) & \xrightarrow{\int \chi^{-1}\delta_{P(X)}^{-\frac{1}{2}}} & \mathbb{C}, \end{array} \tag{35}$$

are matching relative characters for X and $(N,\psi)\backslash G^*$.

Moreover, in the coordinates fixed above, the operator is given by the following formula:

- *When $\check{G}_X = \mathrm{SL}_2$ with $L_X = L(\mathrm{Std}, s_1)L(\mathrm{Std}, s_2)$, $s_1 \geqslant s_2$,*
$$\mathcal{T}_\varnothing f(\xi) = |\xi|^{s_1-\frac{1}{2}} \left(|\bullet|^{\frac{1}{2}-s_1}\psi(\bullet)d\bullet\right) \star \left(|\bullet|^{\frac{1}{2}-s_2}\psi(\bullet)d\bullet\right) \star f(\xi).$$
- *When $\check{G}_X = \mathrm{PGL}_2$ with $L_X = L(\mathrm{Ad}, s_0)$,*
$$\mathcal{T}_\varnothing f(\zeta) = |\zeta|^{s_0-1} \left(|\bullet|^{1-s_0}\psi(\bullet)d\bullet\right) \star f(\zeta).$$

The term "A_X-equivariant", here, refers to the normalized action of A_X on $\mathcal{S}^+(X_\varnothing \times X_\varnothing/G)$ that descends from (27), and, similarly, its analogously normalized action (but using the modular character δ_{B^*} instead of $\delta_{P(X)}$)

on $\mathcal{S}^+(N\backslash G^*/N)$. The factor $|\xi|^{s_1-\frac{1}{2}}$, resp. $|\zeta|^{s_0-1}$, in the formula for $\mathcal{T}_\varnothing$ is due to the difference between the characters $\delta_{B^*}^{-\frac{1}{2}}$ and $\delta_{P(X)}^{-\frac{1}{2}}$ in the definition of the relative characters I_χ and J_χ; in terms of the torus A_X, this factor can be written $|e^{\rho_{P(X)}-\frac{\gamma}{2}}| = |e^{\rho_{P(X)}-\rho_{B^*}}| = \delta_{P(X)}^{\frac{1}{2}}\delta_{B^*}^{-\frac{1}{2}}$.

It would have been more natural to work with half-densities instead of measures, in order to avoid these factors; the downside would be that the pushforward maps from half-densities on $X \times X$ to half-densities on $X \times X /\!\!/ G$ are not completely canonical. However, there seems to be a distinguished way to define such pushforwards, that was used in [Sak] to give a quantization interepretation of the transfer operators. Although we will discuss that picture in the next subsection, we will not dwell any further, in this article, on the reformulation of these results using half-densities.

Proof. Knowing the scattering operators by Theorem 2.4.2, the proof is then identical to that of [Sak22a, Theorem 4.3.1]. ☐

The remarkable feature of the formulas of Theorem 3.4.1 is that the transfer operators for the boundary are given by *exactly* the same formulas as for the original spaces, Theorem 3.1.1.

I conjecture that (35) descends to a commutative diagram

$$\begin{tikzcd} \mathcal{S}(N,\psi\backslash G^*/N,\psi) \arrow[r, "e^*_\varnothing\otimes e^*_\varnothing"] \arrow[d, "\mathcal{T}"] & \mathcal{S}^+(N\backslash G^*/N) \arrow[d, "\mathcal{T}_\varnothing"] \\ \mathcal{S}(X\times X/G) \arrow[r, "e^*_\varnothing\otimes e^*_\varnothing"] & \mathcal{S}^+(X_\varnothing\times X_\varnothing/G) \end{tikzcd}, \tag{36}$$

where $\mathcal{T}$ is the transfer operator of Theorem 3.1.1. This would imply that the relative characters under $\mathcal{T}$ satisfy:

$$\mathcal{T}^*(\mu_X(\chi)I_\chi) = \gamma(\chi, -\check{\gamma}, 0, \psi)J_\chi. \tag{37}$$

This was proven for the basic cases A_1 and $\mathcal{D}_2$ in [Sak22a].

3.5. **Degeneration of cotangent bundles.** In [Sak], I offered an interpretation of the transfer operators of Theorem 3.1.1 in terms of "geometric quantization" for the "cotangent spaces" of the quotients $\mathfrak{X} = (X \times X)/G$ and $\mathfrak{Y} = (N,\psi)\backslash G/(N,\psi)$. Roughly speaking the "cotangent spaces" of these stacks are birational to each other, and the spaces $\mathcal{S}(\mathfrak{X})$, $\mathcal{S}(\mathcal{Y})$ are "geometric quantizations" of these stacks, corresponding to two different Lagrangian foliations $\mathscr{F}_{\mathrm{ver}}$, $\mathscr{F}_{\mathrm{hor}}$ (termed "vertical" and "horizontal"), respectively. This interpretation, which literally makes sense when $F = \mathbb{R}$ but can provide formulas that can be used uniformly for every local field, requires describing a line bundle L_X on this cotangent space, equipped with a connection whose curvature is proportional to the symplectic form. The transfer operators $\mathcal{T}$, then, are simply given by integrals along the leaves of the foliation, at least when G^* is simply connected ($\check{G}_X = \mathrm{PGL}_2$); when $\check{G}^* = \mathrm{PGL}_2$, they descend from an analogous construction on GL_2.

Without getting into the details of such an interpretation, here, it will be interesting to discuss how the cotangent spaces of the aforementioned stacks degenerate to the corresponding spaces for the asymptotic cones, along the affine family $\mathcal{X} \to \mathbb{A}^1$ recalled in § 2.1. A particularly interesting outcome of this discussion will be that, although the asymptotic cones $X_\varnothing$ and $Y_\varnothing$ are very "similar" (they are both horospherical varieties of rank one, and in the basic case A_1, i.e., $X = \mathbb{G}_m\backslash\mathrm{PGL}_2$, they are even isomorphic), the coordinates of their cotangent spaces along this family are completely different; the picture of two *different* foliations "realizing" the two quotient stacks survives in the limit, providing a similar interpretation for the asymptotic transfer operators $\mathcal{T}_\varnothing$, as well (which we will not get into).

This subsection to be completed. It is not needed elsewhere, but it provides some insights, and a nice bridge to the next section.

4. Hankel transforms for the standard L-function of GL_n

In this section, we change our setting, to discuss a close analog of transfer operators, the *Hankel transforms* which realize the functional equation of L-functions at the level of trace formulas. Our goal is to give an interpretation of a theorem of Jacquet, and of its proof, from the point of view of quantization.

Here, we will take $F = \mathbb{R}$, in order to use the language of geometric quantization (which involves line bundles and connections). The results can then be transferred formally to any local field, and indeed the theorem of Jacquet holds over any local field.[2] For example, the flat sections of the connection $\nabla = \nabla^0 - i\hbar dx$ on the trivial (complex) line bundle over $\mathbb{R}$, where ∇^0 is the usual connection, are the multiples of the exponential $e^{i\hbar x}$, and this can be replaced by an additive character when F is any other local field. Similarly, Jacquet's formula (see Theorem 4.1.2 below) has a "natural" meaning (and is correct) over any local field. The reformulation of Jacquet's proof that we provide also makes sense over any such field, due to the theory of the Weil representation. However, since we do not develop a general theory for how to understand geometric quantization over general local fields, there is little benefit to complicating the presentation by including other local fields F; it should be straightforward for the interested reader to fill in the translations and verify that every step makes sense over any F.

Remark 4.0.1. An important note on proofs and notation: The arguments that we present are mere reformulations of Jacquet's arguments. There is therefore no point in commenting again on convergence issues and other technical details that have been dealt with in [Jac03]. For the same reason, we will take the freedom to be a bit vague with some of our notation; in particular, we will often denote by $\mathcal{D}(X)$ an unspecified space of half-densities

[2] In fact, Jacquet wrote [Jac03] for non-Archimedean fields, but the proof of the Theorem 4.1.2 that we are discussing here holds over Archimedean fields, as well.

on a variety X. The rigorous mathematical interpretation of these spaces is that we *start* with well-defined spaces, such as the space $\mathcal{D}(V)$ of *Schwartz* half-densities on a vector space, and then the other spaces $\mathcal{D}(X)$ are images of this space under various integral constructions that make sense, as proven by Jacquet. Finally, we will sometimes write $\mathcal{D}(X)$ for a space of half-densities defined on an open dense subset $X^\bullet$ of X; it will be clear (and we will usually say) what this subset is.

4.1. **The theorem of Jacquet.** In this section, we denote by G the group $G = \mathrm{GL}_n$, and we will work with the Kuznetsov formula for G, with respect to the standard character $\psi : (x_{ij})_{ij} \mapsto \psi(\sum_{i=1}^{n-1} x_{i,i+1})$ of the upper triangular unipotent subgroup $N \subset G$, where, again, we use ψ both for a fixed nontrivial unitary character of F, and for this character of N. Since $F = \mathbb{R}$, ψ has the form $\psi(x) = e^{i\hbar x}$, for some nonzero constant $\hbar$; we will be writing dx for the self-dual Haar measure with respect to ψ (or $|dx|$, when we want to distinguish the measure from the differential form).

As in the case of G^* in the previous section, we will identify the dense subset $N\backslash G_B/N \subset N\backslash G /\!\!/ N$, where G_B denotes the open Bruhat cell, with the Cartan A of diagonal matrices, but this time (for compatibility with the literature), we will use the identification that descends from $A \to wA$, where w is the antidiagonal permutation matrix (i.e., its entries are all 1, rather than the antidiagonal of $(1,-1)$ that we used previously in this paper). Fortunately, these noncanonical choices will not play a role, once we reformulate our theorem in terms of geometric quantization – our reformulation of § 4.2 will not depend on choices of representatives for the orbits.

It is essential here to work with half-densities, so we let $\mathcal{D}(X)$ be the space of Schwartz half-densities on the F-points of a smooth variety X. Those are products of Schwartz functions by half-densities of the form $|\omega|^{\frac12}$, where ω is a nowhere-vanishing polynomial volume form on X. (In the examples at hand, one can always find such a form; in the general case, one would have to be more careful with the definitions.)

We are particularly interested in the case where $X = \mathrm{Mat}_n$, the space of $n \times n$-matrices, or Mat_n^*, its linear dual. In calculations that follow, we will be identifying Mat_n^*, as a space, with Mat_n, via the (symmetric) trace pairing $(A, B) = \mathrm{tr}(AB)$; note, however, that, defining the right $G \times G$-action on Mat_n as

$$A \cdot (g_1, g_2) := g_1^{-1} A g_2, \tag{38}$$

the dual action on Mat_n^* is

$$B \cdot (g_1, g_2) = g_2^{-1} B g_1. \tag{39}$$

The two natural embeddings $G \hookrightarrow \mathrm{Mat}_n$ and $G \hookrightarrow \mathrm{Mat}_n^*$ differ by $g \mapsto g^{-1}$, once we apply the identification $\mathrm{Mat}_n^* = \mathrm{Mat}_n$. These embeddings allow us to restrict half-densities to G, and then we define twisted pushforwards to the double quotient space $N\backslash G /\!\!/ N$, or rather to its open subset identified, as above, with the Cartan A: If $\varphi = \Phi(dg)^{\frac12}$ is a half-density on G, where dg is a Haar measure on G and Φ is a function, its twisted push-forward to $A \hookrightarrow N\backslash G /\!\!/ N$ is defined as the product

$$\delta^{\frac{1}{2}}(a)O_a(\Phi)(da)^{\frac{1}{2}}, \tag{40}$$

where da is a Haar measure on A, and O_a is the Kuznetsov orbital integral

$$O_a(\Phi) = \int_{N\times N} \Phi(n_1 wan_2)\psi^{-1}(n_1n_2)dn_1dn_2, \tag{41}$$

where, again, w is the antidiagonal permutation matrix. The choice of Haar measures used here will not affect any of the results of this section.

Remark 4.1.1. We pause to emphasize the importance of the definition (40) for the pushforward of a half-density. Canonically, only measures admit pushforwards. The pushforward of a measure of the form Φdg is easily computed to be equal to $\delta(a)O_a^0(\Phi)(da)$ (for compatible choices of Haar measures), where now O_a^0 is the orbital integral (41), but with the trivial character instead of ψ. (Or, better, use O_a and think of it as a twisted pushforward.) Less canonically, the function $a \mapsto O_\bullet(\Phi)$ can be thought of as (twisted) "pushforward of functions" – it is the natural definition that we obtain from fixing the same Haar measure $|d\underline{n}|$ on $N\times N$ for all its orbits (in the open Bruhat cell), and integrating; it is ambiguous up to a constant, that depends on the choice of Haar measure. Similarly, the definition (40) corresponds to fixing a Haar measure on $N \times N$, taking its square root $|d\underline{n}|^{\frac{1}{2}}$ (a Haar half-density), and multiplying φ. The product $\varphi \cdot |d\underline{n}|^{\frac{1}{2}}$ is a "measure along the fibers of the map to A, multiplied by a half-density in the transverse direction," and it makes sense to push it forward to a half-density on A. We should note, however, that, although natural, it is not completely clear why we should take the same Haar measure for every $N \times N$-orbit; the full justification of this choice will come with the proof of the main theorem, see Lemma 4.3.6.

For reasons that have to do with the Godement–Jacquet method of representing L-functions, we will adopt the notations of [Sak19], and denote the images of $\mathcal{D}(\mathrm{Mat}_n)$, $\mathcal{D}(\mathrm{Mat}_n^*)$ under these pushforward maps by $\mathcal{D}^-_{L(\mathrm{Std},\frac{1}{2})}(\mathfrak{Y})$, $\mathcal{D}^-_{L(\mathrm{Std}^\vee,\frac{1}{2})}(\mathfrak{Y})$, respectively, where $\mathfrak{Y}$ stands as a symbol for the twisted Kuznetsov quotient $(N,\psi)\backslash G/(N,\psi)$. These are understood as spaces of half-densities on A, and they are genuinely different; for example, when $n=1$, the embedding $A \hookrightarrow \mathrm{Mat}_1$ attaches the point 0 to $A \simeq F^\times$, while the embedding $A \hookrightarrow \mathrm{Mat}_1^*$ attaches the point ∞.

The theorem that follows is due to Jacquet [Jac03, Theorem 1]; I present it in the reformulation of [Sak19, Theorem 9.1]. Thinking of half-densities for $\mathfrak{Y}$ as half-densities on the torus A of diagonal element, we also identify the latter with the "universal" Cartan of G, via the quotient $B \to A$ of the upper-triangular Borel subgroup. Thus, it makes sense to write $e^\alpha : A \to \mathbb{G}_m$ for a root (where we use exponential notation for the character of A, reserving the additive notation α for its differential), and we also denote by $e^{\check{\epsilon}_i} : \mathbb{G}_m \to A$ the cocharacter into the i-th entry of the diagonal. The Hankel transform of the theorem that follows will be expressed in terms of operators of the form "multiplication by a function $\psi(e^{\alpha})$ on A" as well as *multiplicative Fourier convolutions* along the cocharacters $e^{-\check{\epsilon}_i}$. Those will be denoted by $\mathcal{F}_{-\check{\epsilon}_i,\frac{1}{2}}$, and are given, explicitly, by

$$\mathcal{F}_{-\check{\epsilon}_i,\frac{1}{2}}\varphi(\mathrm{diag}(a_1,\dots,a_n)) = \int_{F^\times} \varphi(\mathrm{diag}(a_1,\dots,a_i x,\dots,a_n))|x|^{\frac{1}{2}}\psi^{-1}(x)d^\times x.$$

(Caution: Compared to the notation of [Sak19]*, we have changed ψ to ψ^{-1} here, and we will make some corresponding changes below.)*

Theorem 4.1.2. *Let $G = \mathrm{GL}_n$. Consider the diagram*

$$\begin{tikzcd} \mathcal{D}(\mathrm{Mat}_n) \arrow[r, "\mathcal{F}"] \arrow[d] & \mathcal{D}(\mathrm{Mat}_n^*) \arrow[d] \\ \mathcal{D}^-_{L(\mathrm{Std},\frac{1}{2})}(\mathfrak{Y}) \arrow[r, dashed, "\mathcal{H}_{\mathrm{Std}}"] & \mathcal{D}^-_{L(\mathrm{Std}^\vee,\frac{1}{2})}(\mathfrak{Y}) \end{tikzcd}$$

where $\mathcal{F}$ denotes the equivariant Fourier transform:

$$\mathcal{F}(\varphi)(y) = \left(\int_{\mathrm{Mat}_n} \varphi(x)\psi(-\langle x,y\rangle)dx^{\frac{1}{2}}\right)dy^{\frac{1}{2}}$$

(for dual Haar measures dx, dy on Mat_n and Mat_n^* with respect to the character ψ).

There is a linear isomorphism $\mathcal{H}_{\mathrm{Std}}$ as above, making the diagram commute. Moreover, $\mathcal{H}_{\mathrm{Std}}$ is given by the following formula:

$$\mathcal{H}_{\mathrm{Std}} = \mathcal{F}_{-\check{\epsilon}_1,\frac{1}{2}} \circ \psi(e^{-\alpha_1}) \circ \mathcal{F}_{-\check{\epsilon}_2,\frac{1}{2}} \circ \cdots \circ \psi(e^{-\alpha_{n-1}}) \circ \mathcal{F}_{-\check{\epsilon}_n,\frac{1}{2}}. \tag{42}$$

Explicitly, and denoting diagonal matrices simply as n-tuples, (42) reads:

$$\mathcal{H}_{\mathrm{Std}} f(b_1,\dots,b_n) = \int f(b_1p_1,\dots,b_np_n)\psi(-\sum_{i=1}^n p_i + \sum_{i=1}^{n-1}\frac{b_{i+1}}{p_ib_i})|p_1\cdots p_n|^{\frac{1}{2}}d^\times p_1\cdots d^\times p_n. \tag{43}$$

Remark 4.1.3. Opposite to the convention in [Sak19] and [Jac03] (where ψ is denoted by θ^{-1}), we have changed the convention of Fourier transform, using the character ψ^{-1} instead of ψ. This resulted in some sign changes, but will simplify some expressions in the rest of the section.

4.2. **Cotangent reformulation of Jacquet's theorem.**

4.2.1. The reformulation of Jacquet's theorem that we will present is directly analogous to the interpretation of transfer operators given in [Sak]: Namely, we will interpret both of the spaces $\mathcal{D}^-_{L(\mathrm{Std},\frac{1}{2})}(N,\psi\backslash G/N,\psi)$ and $\mathcal{D}^-_{L(\mathrm{Std}^\vee,\frac{1}{2})}(N,\psi\backslash G/N,\psi)$ as geometric quantizations corresponding to two different Lagrangian foliations on the *same* symplectic space, and then the operator of Formula (42) represents integrals over the leaves of such a foliation.

Let us start by introducing the notions of foliations and integrals over the leaves; our definitions are more restrictive than usual, but good enough for our purposes.

Definition.
(1) We will call *foliation* a smooth morphism of smooth varieties $\mathscr{F}: X \to Y$, and *leaves* its fibers; we will also be denoting Y by $X/\mathscr{F}$. If X is symplectic, a foliation is *Lagrangian* if its leaves are Lagrangian subvarieties.
(2) Given a foliation as above, let (L,∇) be a smooth vector bundle on (the real points of) X, with a connection that is flat with trivial monodromies along the leaves of $\mathscr{F}$. The *space of parallel half-densities along* $\mathscr{F}$, $\mathcal{D}_{\mathscr{F}}(X,L)$, is defined as follows: Its elements are half-densities on Y, valued in a line bundle $L_{\mathscr{F}}$, whose sections are those sections of L that are ∇-flat along the leaves of Y.
(3) Suppose, now, that $\mathscr{F}_1$, $\mathscr{F}_2$ are two Lagrangian foliations on a symplectic space X, and (L,∇) is a smooth vector bundle on X, with a connection whose curvature is equal to $i\hbar\omega$ – in particular, flat along the leaves of Lagrangian foliations. Assume, also, that the restriction of each foliation to the leaves of the other is also a foliation; in particular, the intersections of two leaves are smooth, and their tangent spaces coincide with the intersections of the tangent spaces of the leaves. If Z is a leaf, say of $\mathscr{F}_1$, we will be writing $Z/\mathscr{F}_2 \subset Y_2$ for its space of leaves for the restriction of the $\mathscr{F}_2$-foliation.

In this setting, the *integral* of an element $\varphi \in \mathcal{D}_{\mathscr{F}_1}(X,L)$ along the leaves of $\mathscr{F}_2$ is the element of $\mathcal{D}_{\mathscr{F}_2}(X,L)$ whose "value" at $x \in X$ is given by

$$\int_{\mathscr{F}_{2,x}/\mathscr{F}_1} T_{x,z}\varphi(z)|\omega|^{\frac{1}{2}\dim(\mathscr{F}_{2,x}/\mathscr{F}_1)}, \tag{44}$$

provided that the integral converges. Here, $\mathscr{F}_{2,x}$ denotes the $\mathscr{F}_2$-leaf containing x, and $T_{x,z}$ denotes parallel transport from the point z to the point x. The integral makes sense as a parallel half-density along $\mathscr{F}_2$, exactly as in the linear case [Li08, § 3.4] (see Remark 4.2.2 below for an explanation). In particular, our convention is that, for a volume form Ω on a smooth variety, $|\Omega|$ denotes the corresponding positive measure on the $\mathbb{R}$-points of the variety, that is induced from the self-dual measure on $F = \mathbb{R}$ with respect to the character $x \mapsto e^{i\hbar x}$.

Remark 4.2.2. We explain why the outcome of integral (44) represents a half-density on $X/\mathscr{F}_2$ (valued in the line bundle $L_{\mathscr{F}_2}$):

Let $x \in X$, set $M = T_x X$, and let $\ell_1, \ell_2 \subset M$ be the tangent spaces of the leaves of $\mathscr{F}_1$, $\mathscr{F}_2$ passing through x. The symplectic form ω restricts to a symplectic form the quotient $(\ell_1 + \ell_2)/(\ell_1 \cap \ell_2)$, giving rise to a volume form $\omega \wedge \cdots \wedge \omega$ ($\frac{1}{2}\dim(\ell_1 + \ell_2)/(\ell_1 \cap \ell_2)$ times). We have a canonical isomorphism of lines

$$\det(M/\ell_1) \otimes \det((\ell_1 + \ell_2)/(\ell_1 \cap \ell_2)) = \det(M/\ell_2) \otimes \det(\ell_2/(\ell_1 \cap \ell_2))^{\otimes 2}. \tag{45}$$

Taking the "square root of the absolute value" of the dual spaces, we deduce that the product of a half-density along $X/\mathscr{F}_1$ by $|\omega|^{\frac{1}{2}\dim(\mathscr{F}_{2,x}/\mathscr{F}_1)} := |\omega \wedge \cdots \wedge \omega|^{\frac{1}{2}}$ can be canonically understood as a density (measure) on $\mathscr{F}_{2,x}/\mathscr{F}_1$ times a half-density on $X/\mathscr{F}_2$. This is how the integral (44) gives rise to a half-density on $X/\mathscr{F}_2$, valued in the appropriate line bundle. For further discussion of these integrals along Lagrangians, see § 4.3.4 below.

4.2.3. For the case at hand, let us adopt a basis-independent point of view, and write V for an n-dimensional vector space, V^* for its linear dual. The role of Mat_n will be played by $\mathrm{End}(V) \simeq V^* \otimes V$, and we identify its linear dual with $\mathrm{End}(V^*) \simeq V \otimes V^*$. The cotangent spaces of both are identified with the space $M := \mathrm{End}(V) \oplus \mathrm{End}(V^*)$, and the foliations $M \to \mathrm{End}(V)$ and $M \to \mathrm{End}(V^*)$ will be called the "vertical" and "horizontal" foliation, respectively. Breaking the symmetry,[3] we need to fix a sign for the symplectic form on this vector space, and we choose it to be

$$\omega_M = \sum_j dT_j \wedge dT_j^*, \tag{46}$$

where $(T_j)_j, (T_j^*)_j$ are dual bases for $\mathrm{End}(V), \mathrm{End}(V^*)$.

The group $G = \mathrm{GL}(V) \simeq \mathrm{GL}(V^*)$ admits an open embedding into both $\mathrm{End}(V)$ and $\mathrm{End}(V^*)$, giving rise to two *distinct* embeddings of T^*G into M.

Now we will define a smooth line bundle L on (the real points of) M, equipped with a connection ∇, whose curvature is $i\hbar\omega_M$, for some nonzero constant $\hbar$. Let ψ be the additive character $x \mapsto e^{i\hbar x}$. In order not to privilege one Lagrangian over the other, we will define L in a symmetric fashion: Smooth sections of L are pairs (Φ, Φ^*) consisting of $\Phi, \Phi^* \in C^\infty(M)$, with the property that

$$\Phi^*(A, B) = \psi(-\langle A, B\rangle)\Phi(A, B). \tag{47}$$

[3]Only the product of the symplectic form by an chosen nonzero constant $\hbar$ matters for what follows; we are not really breaking the symmetry – just choosing opposite constants for the two quotients.

If ∇^0 denotes the standard flat connection on the trivial line bundle on M, then the connection ∇ is defined as

$$\nabla = (\nabla^0 - i\hbar \sum_j T_j^* dT_j, \nabla^0 + i\hbar \sum_j T_j dT_j^*), \tag{48}$$

where again we have used a dual basis. In particular, a function f on $\mathrm{End}(V)$ pulls back to a section $\tilde{f} = (\Phi, \Phi^*)$ of L on M, given by

$$\Phi(A, B) = f(A),\ \Phi^*(A, B) = f(A)\psi(-\langle A, B\rangle),$$

with the property that $\nabla_Z \tilde{f} = 0$ for any vector field Z which is parallel to the leaves of the "vertical" Lagrangian foliation $M \to \mathrm{End}(V)$. Similarly, a function f on $\mathrm{End}(V^*)$ pulls back to

$$(\Phi(A, B) = f(B)\psi(\langle A, B\rangle),\ \Phi^*(A, B) = f(B)),$$

which is ∇-flat along the leaves of the "horizontal" foliation $M \to \mathrm{End}(V^*)$.

4.2.4. We now introduce the *Kuznetsov cotangent space* $T^*\mathfrak{Y}$. We first give a rather clumsy presentation of the Whittaker model, by fixing a maximal unipotent subgroup $N \subset G$, and a homomorphism $f : N \to \mathbb{G}_a$ which is a generic (i.e., nonzero on every simple root space). The Whittaker "space" Y is then the homogeneous space $N\backslash G$, but we will think of every point Ng of it as the set of triples (Ng, N', f'), where N' is the maximal unipotent subgroup $g^{-1}Ng$, and f' is the homomorphism $N' \to \mathbb{G}_a$ obtained from f via conjugation by g. There are nicer presentations of the Whittaker model, which do not depend on the choice of a base point; for example, if $\dim V = 2$, we can let $Y =$ the set of pairs (v, ω), where $v \in V^\times$ and ω is a non-zero alternating form on V, and we can endow the stabilizer $N \subset G$ of such a point with the homomorphism f such that $v^* - n \cdot v^* = f(n)v$, for every $v^* \in V$ with $\omega(v, v^*) = 1$. (Also, if the group were adjoint, the set of pairs (N, f) consisting of a maximal unipotent subgroup and a generic morphism to $\mathbb{G}_a$ would constitute the Whittaker model.) However, for reasons of conciseness, we will fix a base point.

We may, and will, identify f with its differential, as well, which is an element of $\mathfrak{n}^*$. We will then symbolically write $f + n^\perp$ for the preimage of f under the restriction map $\mathfrak{g}^* \to \mathfrak{n}^*$, and define the Whittaker cotangent space T^*Y as the space $T^*Y = (f + n^\perp) \times^N G$. We define the opposite Whittaker cotangent space as $T^*Y^- = (-f + n^\perp) \times^N G$.

The inclusion $(f + n^\perp) \hookrightarrow \mathfrak{g}^*$ gives rise to a G-equivariant map $T^*Y \to \mathfrak{g}^*$, and T^*Y is naturally a Hamiltonian space with this moment map; same for T^*Y^-.

Finally, the Kuznetsov cotangent space $T^*\mathfrak{Y}$ is defined as the Hamiltonian reduction of $T^*Y^- \times T^*Y$ with respect to the diagonal G-action:

$$T^*\mathfrak{Y} = T^*Y^- \times^G_{\mathfrak{g}^*, \pm} T^*Y, \tag{49}$$

where $\pm$ denotes that the images under the moment map should be *opposite*. Of course, the map $-f \mapsto f$ induces an equivariant map $T^*Y^- \to T^*Y$

which inverts the moment map, so we could also present $T^*\mathfrak{Y}$ as $T^*Y \times^G_{\mathfrak{g}^*} T^*Y$, but in order to avoid sign confusions in the calculations that follow, it is better to think of the presentation (49).

For completeness, we use the last presentation to provide a standard description of this space, although we will not use it further in this paper. Recall the *group scheme of regular centralizers* $J_G \to \mathfrak{c}^*$, where $\mathfrak{c}^* := \mathfrak{g}^* /\!\!/ G$; it comes equipped with an isomorphism of its pullback to the regular locus $\mathfrak{g}^{*,\mathrm{reg}}$ with the inertia group scheme for the adjoint action of G. A well-known result of Kostant says that T^*Y is a J_G-torsor over $\mathfrak{g}^{*,\mathrm{reg}}$. As a corollary, $T^*\mathfrak{Y}$ is *canonically* isomorphic to J_G, with the isomorphism induced by the (action, projection) map $J_G \times_{\mathfrak{c}^*} T^*Y \to T^*Y \times_{\mathfrak{g}^*} T^*Y$.

4.2.5. Next, we study the double Hamiltonian reduction $\mathfrak{M} = N\backslash\!\!\!\backslash_{\!f}\, M /\!\!\!/_{f}\, N$. Our notation stands for Hamiltonian reduction at (f, f) with respect to a *left* and a *right* action of N; when we convert this to a right $(N \times N)$-action, it will correspond to Hamiltonian reduction at $(-f, f)$. The two embeddings of T^*G into M induce two distinct embeddings of $J_G \simeq T^*\mathfrak{Y}$ into it.

The coordinate-dependent embedding $A \hookrightarrow N\backslash G /\!\!/ N$ that we defined when we identified G with GL_n is more canonically an A-torsor $A_1 \hookrightarrow N\backslash \mathrm{GL}(V) /\!\!/ N$, which further maps into the quotient $N \backslash\!\!\backslash \mathrm{End}(V) /\!\!/ N$. The coordinate ring of $N \backslash\!\!\backslash \mathrm{End}(V) /\!\!/ N$ is spanned by the semiinvariants (highest weight vectors) for B on $F[\mathrm{End}(V)]$, and similarly for the other spaces, which shows that A_1, the locus where these semiinvariants are nonvanishing, is open in each of those spaces, and its preimage is the open $B\times B$-orbit (Bruhat cell), which we will denote by $G_B \subset G$. We have the analogous torsor $A_2 \subset N \backslash\!\!\backslash \mathrm{End}(V^*) /\!\!/ N$. In particular, the restriction of the maps

$$N\backslash\!\!\!\backslash_{\!f}\, M /\!\!\!/_{f}\, M \to N \backslash\!\!\backslash \mathrm{End}(V) /\!\!/ N,$$
$$N\backslash\!\!\!\backslash_{\!f}\, M /\!\!\!/_{f}\, M \to N \backslash\!\!\backslash \mathrm{End}(V^*) /\!\!/ N,$$

to A_1, resp. A_2, coincides with the maps

$$N\backslash\!\!\!\backslash_{\!f}\, T^*G_B /\!\!\!/_{f}\, M \to A_1 \subset N \backslash\!\!\backslash \mathrm{End}(V) /\!\!/ N,$$
$$N\backslash\!\!\!\backslash_{\!f}\, T^*G_B /\!\!\!/_{f}\, M \to A_2 \subset N \backslash\!\!\backslash \mathrm{End}(V^*) /\!\!/ N,$$

respectively, but note again that these two refer to two *different* embeddings of T^*G_B into M. We will denote by M° the intersection of the two embeddings, which is an open subset of M. Explicitly, in coordinates identifying M with pairs (A, B) of $n\times n$ matrices, $M^\circ = NwAN\times NAwN$, with this presentation corresponding to how we think of the GIT quotients as A-torsors. Clearly, M° is an $N \times N$-torsor over $A_1 \times A_2$.

The following lemma will be useful – we formulate it for $\mathrm{End}(V)$, but of course we will also apply it to $\mathrm{End}(V^*)$:

Lemma 4.2.6. *Let $\mathrm{End}(V)_B$ denote the open subset which is the union of G_B and the open $B \times B$-orbit in the set of endomorphisms of rank $n-1$. The group $N \times N$ acts freely on $\mathrm{End}(V)_B$, and the quotient $\mathrm{End}(V)_B/(N \times N)$, under the action of A descending from right multiplication, is isomorphic to the an open* *subset $\bar{A}^\bullet \subset \bar{A} := \mathbb{A}^n$ whose complement has codimension $\geqslant 2$. Here, $\bar{A} = \mathbb{A}^n$ is considered as an A-space, with action given by the simple roots and the lowest weight of the standard representation, $(a_1, \dots, a_n) \mapsto a_n$.*

Proof. This is elementary, and left to the reader. □

To summarize, we have an open, $G \times G$-equivariant subset $M^\circ \subset M$, and the double Hamiltonian reduction $J^\circ := N \backslash\!\!\backslash_f M^\circ /\!\!/_f M$ admits two Lagrangian fibrations, $J^\circ \to A_1$ and $J^\circ \to A_2$, identifying J° with the product $\simeq A_1 \times A_2$. The symplectic space J° can be identified as an open subset of the Kuznetsov cotangent space $J_G = T^*\mathfrak{Y}$, but this identification depends on which of the two embeddings $T^*G \hookrightarrow M$ we choose.

Let us, again, be explicit about the symplectic form on J°. Fixing again coordinates as above, it is given on $A_1 \times A_2 \simeq A^2 \ni (a_1, \dots, a_n, b_1, \dots, b_n)$ by

$$\omega_J = \sum_j db_j \wedge da_j. \tag{50}$$

This follows immediately from the expression (46) for the symplectic form for M.

Lemma 4.2.7. *Setting $M^\circ_{-f,f} = M^\circ \times_{\mathfrak{n}^* \times \mathfrak{n}^*} (\{-f\} \times \{f\})$ (with respect to the right action of $N \times N$), the fibers of the composition $M^\circ_{-f,f} \to J^\circ \to A$ ($A = A_1$ or A_2) are Lagrangian subspaces. In particular, the connection ∇ on the line bundle L restricts to a flat connection. The monodromy of this flat connection is trivial.*

Proof. The first statement is a general statement about Hamiltonian spaces for a group H, and their Hamiltonian reduction at an H-fixed point $f \in \mathfrak{h}^*$.

To prove that the monodromy is trivial, we embed M° into the larger space $M^\bullet := T^* \operatorname{End}(V)_B \cap T^* \operatorname{End}(V^*)_B$. Then, $M^\bullet \times_{\mathfrak{n}^* \times \mathfrak{n}^*} (\{-f\} \times \{f\})$ admits $N \times N$-invariant maps to $\bar{A}_1^\bullet$ and $\bar{A}_2^\bullet$ (notation as in Lemma 4.2.6, with indices to distinguish the two quotients). The preimage of any point of $\bar{A}_1^\bullet$ is an $N \times N$-torsor over $\bar{A}_2^\bullet$ under these maps, and hence simply connected. Since the flat connection extends over this preimage, its monodromy is trivial. □

Note, also that the fibers of $M^\circ_{-f,f} \to J^\circ$ are $N \times N$-torsors. The line bundle, with its connection, descend to J°, and will also be denoted by (L, ∇). We will call the foliations $\mathscr{G}_1 : J^\circ \to A_1$ and $\mathscr{G}_2 : J^\circ \to A_2$ the “vertical” and “horizontal” foliations, respectively, but I emphasize that these are *different* foliations than the “vertical” and “horizontal” foliations on M: their preimages in $M^\circ_{-f,f} \to J^\circ$ *do not* coincide with leaves of the foliations on M.

4.2.8. We are now ready to reformulate Jacquet's Theorem 4.1.2. We will think of $\mathcal{D}(\operatorname{End}(V))$ and $\mathcal{D}(\operatorname{End}(V^*))$ as geometric quantizations of M, given by the data (L, ∇). This means that we consider functions on $\operatorname{End}(V)$ (resp. on $\operatorname{End}(V^*)$) as sections of L which are flat along the leaves of the “vertical” (resp. "horizontal") foliation, as explained before, and we will be writing $\mathcal{D}(\mathrm{End}(V)) = \mathcal{D}_{\mathrm{hor}}(M, L)$ (resp. $\mathcal{D}(\mathrm{End}(V^*)) = \mathcal{D}_{\mathrm{ver}}(M, L)$), thinking of the elements as (Schwartz) half densities on the space of leaves for the corresponding foliation, valued in the line bundle of flat sections along the leaves.

Fourier transform, then, becomes the isomorphism

$$\mathcal{F} : \mathcal{D}_{\mathrm{hor}}(M, L) \to \mathcal{D}_{\mathrm{ver}}(M, L)$$

given by the standard intertwiners of "integration along the leaves of $\mathscr{F}_{\mathrm{hor}}$," as in Definition 4.2.1. Note that the inverse Fourier transform is also given by integration along the leaves, this time of $\mathscr{F}_{\mathrm{ver}}$.

4.2.9. The pushforwards (40) of half-densities (restricted to M°) can now be seen as maps

$$\begin{aligned} \mathcal{D}_{\mathrm{hor}}(M, L) &\to \mathcal{D}_{\mathrm{hor}}(J^\circ, L), \\ \mathcal{D}_{\mathrm{ver}}(M, L) &\to \mathcal{D}_{\mathrm{ver}}(J^\circ, L). \end{aligned} \tag{51}$$

We can understand these maps as integrals along the Lagrangian leaves of the map $M^\circ_{-f,f} \to A$ (where $A = A_1$, resp. A_2), but some care is required in understanding those integrals, since the expression (44) does not make sense in the absence of a Lagrangian foliation on the entire space M.

Rather, what we should do is pick a Haar measure on $N \times N$, and use it (or rather, its square root half-density) to integrate the given half-densities along the fibers of the map $M^\circ_{-f,f} \to J^\circ$, which are $N \times N$-orbits.

In more detail: Consider the Lagrangian leaves of the foliation $\mathscr{G}_{\mathrm{ver}} : J^\circ \to A_1$. By Lemma 4.2.7, these are quotients by the (free) $N \times N$-action of Lagrangian subvarieties of M°; we will denote by $\mathscr{G}_{\mathrm{ver},a}$ the leaf of $\mathscr{G}_{\mathrm{ver}}$ over $a \in A_1$, and by $\widetilde{\mathscr{G}}_{\mathrm{ver},a}$ its preimage in M°. The intersection of every $\widetilde{\mathscr{G}}_{\mathrm{ver},a}$ with the leaves of the "vertical" foliation $\mathscr{F}_{\mathrm{ver}} : M \to \mathrm{End}(V)$ is easily seen to be of dimension n ($= \dim A$). The orbits of $N \times N$ provide sections for each quotient

$$\widetilde{\mathscr{G}}_{\mathrm{ver},a} \to \widetilde{\mathscr{G}}_{\mathrm{ver},a}/\mathscr{F}_{\mathrm{ver}}.$$

The elements $\varphi \in \mathcal{D}_{\mathrm{hor}}(M, L)$ are half-densities in the transverse direction to $\mathscr{F}_{\mathrm{ver}}$ (valued in the line bundle L_{ver} of vertically parallel sections of L). The analog of (44), here, is an integral of φ over the quotient $\widetilde{\mathscr{G}}_{\mathrm{ver},a}/\mathscr{F}_{\mathrm{ver}}$. For such an integral to make sense, *we do not multiply φ by a power of $|\omega|$, as in (44), but by our fixed Haar half-density $|d\underline{n}|^{\frac{1}{2}}$ along $N \times N$-orbits*. Using an isomorphism analogous to (45), the product $\varphi \cdot |d\underline{n}|^{\frac{1}{2}}$ can be written as the product of the Haar measure $|d\underline{n}|$ on $\widetilde{\mathscr{G}}_{\mathrm{ver},a}/\mathscr{F}_{\mathrm{ver}}$ by a half-density along $J^\circ/\mathscr{G}_{\mathrm{ver}}$ and by a section of L.

The reader can check, using Remark 4.1.1, that this definition of the pushforward (51) corresponds to the one that we gave in coordinates in (40). The definition depends on the choice of Haar measure on $N \times N$, but this choice applies to both sides of Theorem 4.1.2, and will not affect the Hankel transforms. As we pointed out in that remark, although it is natural, this definition of pushforward is not entirely justified yet – one could imagine varying the $N \times N$-Haar measure along the orbits. Its full justification will appear in Lemma 4.3.6 below.

4.2.10. Jacquet's theorem 4.1.2, now, admits the following simple reformulation:

Theorem 4.2.11. *The diagram*

$$\begin{tikzcd} \mathcal{D}_{\mathrm{hor}}(M,L) \arrow[r, "\mathcal{F}"] \arrow[d] & \mathcal{D}_{\mathrm{ver}}(M,L) \arrow[d] \\ \mathcal{D}_{\mathrm{hor}}(J^\circ,L) \arrow[r, "\mathcal{H}_{\mathrm{Std}}"] & \mathcal{D}_{\mathrm{hor}}(J^\circ,L) \end{tikzcd}$$

commutes, where the horizontal arrows are given by the integrals (44) *along the leaves of the foliations.*

A pleasant feature of this reformulation is that it does not require any arbitrary choices, such as the choice of the subset $wA \subset G$ for representing the orbital integrals (41) of the Kuznetsov quotient $(N,\psi)\backslash G/(N,\psi)$.

4.3. **The case of** GL_2**.** We will use the case $n = 2$, both to explicitly verify that the Hankel transforms $\mathcal{H}_{\mathrm{Std}}$ described by Theorems 4.1.2 and 4.2.11 coincide. The proof of the general case will only be sketched in the next subsection; it uses an inductive argument where every step is almost identical to the case of GL_2. The explicit verification that Theorem 4.2.11 amounts to Jacquet's formula (42) in higher rank is also similar to the case of GL_2, and will be left to the reader; the validity of both theorems proves that, indirectly.

4.3.1. *Verification of* (42) *for* $n = 2$. Taking the description of Theorem 4.2.11 for the operator $\mathcal{H}_{\mathrm{Std}}$, let us see that it is given by (42).

First of all, let us calculate the set $M^\circ_{-f,f}$, and the (free) $N \times N$-action on it. It is not hard to see that this set consists of all elements of the form

$$\left(w\begin{pmatrix} a_1 & \\ & a_2\end{pmatrix}, \begin{pmatrix} b_1 & a_1^{-1} \\ a_1^{-1} & b_2\end{pmatrix} w\right)\cdot\left(\begin{pmatrix} 1 & y \\ & 1\end{pmatrix}\begin{pmatrix} 1 & x \\ & 1\end{pmatrix}\right) =$$
$$\left(w\begin{pmatrix} a_1 & a_1 x \\ -a_1 y & a_2 - a_1 xy\end{pmatrix}, \begin{pmatrix} b_1 - a_1^{-1}(x-y) - b_2 xy & a_1^{-1} - x b_2 \\ a_1^{-1} + y b_2 & b_2\end{pmatrix} w\right), \quad (52)$$

with $a_1, a_2, b_1, b_2 \neq 0$.

Next, we describe the elements of $\mathcal{D}_{\mathrm{hor}}(J^\circ, L)$: Denoting, as before, by $\mathscr{G}_{\mathrm{ver}}$ the "vertical" foliation $J^\circ \to A_1$, and by $\widetilde{\mathscr{G}}_{\mathrm{ver}}$ the corresponding foliation $M^\circ_{-f,f} \to J^\circ \to A$ of Lemma 4.2.7, we have the following description, whose proof will be left to the reader.

Lemma 4.3.2. *The $\mathscr{G}_{\mathrm{ver}}$-parallel sections of L can be described as sections (Φ, Φ^*) of L over $M^\circ_{-f,f}$, such that Φ depends only on the projection to $\mathrm{End}(V)$, and $\Phi(m\cdot(n_1,n_2)) = \psi(n_1^{-1}n_2)\Phi(m)$, for all m. (This completely determines Φ^*, by the* *defining property (47) of the line bundle L.) Thus, the elements of $\mathcal{D}_{\mathrm{hor}}(J^\circ, L)$ are half-densities on A_1 valued in this line bundle. Similarly, elements of $\mathcal{D}_{\mathrm{ver}}(J^\circ, L)$ are half-densities on A_2 valued in the line bundle whose sections are given by pairs (Φ, Φ^*) on M° such that Φ^* depends only on the projection to $\mathrm{End}(V^*)$, and $\Phi^*(m\cdot(n_1,n_2)) = \psi(n_1^{-1}n_2)\Phi^*(m)$. (Again, this completely determines Φ.)*

The elements of $\mathcal{D}^-_{L(\mathrm{Std},\frac12)}(\mathfrak{Y})$, $\mathcal{D}^-_{L(\mathrm{Std}^\vee,\frac12)}(\mathfrak{Y})$ of Theorem 4.1.2 are obtained from the elements of $\mathcal{D}_{\mathrm{hor}}(J^\circ, L)$, $\mathcal{D}_{\mathrm{hor}}(J^\circ, L)$ of Theorem 4.2.11 by evaluating Φ, resp. Φ^*, at the elements of $M^\circ_{-f,f}$ living over representatives $wA \subset G \hookrightarrow \mathrm{End}(V)$ and $wA \subset G \hookrightarrow \mathrm{End}(V^*)$, respectively. Such representatives are given by the pairs

$$A \ni (a_1,a_2) \mapsto \left(w\begin{pmatrix} a_1 & \\ & a_2\end{pmatrix}, \begin{pmatrix} b_1 & a_1^{-1} \\ a_1^{-1} & b_2\end{pmatrix} w\right) \in M^\circ_{-f,f} \text{ (arbitrary } b_1,b_2\in F^\times),$$
$$A \ni (b_1,b_2) \mapsto \left(w\begin{pmatrix} a_1 & b_2^{-1} \\ b_2^{-1} & a_2\end{pmatrix}, \begin{pmatrix} b_1 & \\ & b_2\end{pmatrix} w\right) \in M^\circ_{-f,f} \text{ (arbitrary } a_1,a_2\in F^\times). \tag{53}$$

Now we compute the Hankel tranform of such an element $\varphi \in \mathcal{D}_{\mathrm{hor}}(J^\circ, L)$, according to Theorem 4.2.11.

Writing $(\underline{a},\underline{b}) = (a_1,a_2,b_1,b_2)$ and $\varphi(\underline{a},\underline{b}) = (\Phi,\Phi^*)|d\underline{b}|^{\frac12}$, we have $\mathcal{H}_{\mathrm{Std}}\varphi = (\Psi,\Psi^*)|d\underline{a}|^{\frac12}$, with

$$\Psi^*\left(w\begin{pmatrix} * & b_2^{-1} \\ b_2^{-1} & *\end{pmatrix}, \begin{pmatrix} b_1 & \\ & b_2\end{pmatrix} w\right) =$$
$$\int \Phi^*\left(w\begin{pmatrix} a_1 & b_2^{-1} \\ b_2^{-1} & a_2\end{pmatrix}, \begin{pmatrix} b_1 & \\ & b_2\end{pmatrix} w\right)|da_1\, da_2| =$$
$$\int \Phi\left(w\begin{pmatrix} a_1 & b_2^{-1} \\ b_2^{-1} & a_2\end{pmatrix}, \begin{pmatrix} b_1 & \\ & b_2\end{pmatrix} w\right)\psi(-a_1b_1 - a_2b_2)|da_1\, da_2| =$$
$$\int \Phi\left(w\begin{pmatrix} a_1 & \\ & a_2 - a_1^{-1}b_2^{-2}\end{pmatrix}, \begin{pmatrix} b_1 + b_2^{-1}a_1^{-2} & a_1^{-1} \\ a_1^{-1} & b_2\end{pmatrix} w\right)\cdot$$
$$\psi(-a_1b_1 - a_2b_2 + 2a_1^{-1}b_2^{-1})|da_1\, da_2| =$$
$$\int \Phi\left(w\begin{pmatrix} a_1 & \\ & a_2\end{pmatrix}, \begin{pmatrix} * & a_1^{-1} \\ a_1^{-1} & *\end{pmatrix} w\right)\psi(-a_1b_1 - a_2b_2 + a_1^{-1}b_2^{-1})|da_1\, da_2|.$$

This is (43), taking into account that the embedding $\mathrm{GL}_n \hookrightarrow \mathrm{Mat}_n^* \xrightarrow{\sim} \mathrm{Mat}_n$ takes $\begin{pmatrix} b_1 & \\ & b_2\end{pmatrix} w$ to $w\begin{pmatrix} b_1^{-1} & \\ & b_2^{-1}\end{pmatrix}$.

4.3.3. *Proof for $n=2$.* The proof of Theorem 4.2.11 for GL_2 will occupy the rest of § 4.3. Let us rephrase what needs to be proven: We have Lagrangian foliations $\mathscr{F}_{\mathrm{ver}}$, $\mathscr{F}_{\mathrm{hor}}$ on M, and also two Lagrangian foliations $\widetilde{\mathscr{G}}_{\mathrm{ver}}$, $\widetilde{\mathscr{G}}_{\mathrm{hor}}$ on its subspace $M^\circ_{-f,f}$. Our goal is to show that, starting from a "horizontal" half-density φ (i.e., parallel along $\mathscr{F}_{\mathrm{ver}}$), we obtain the *same* half-density that is parallel along $\widetilde{\mathscr{G}}_{\rm hor}$, either by integrating φ directly[4] over the leaves of $\widetilde{\mathscr{G}}_{\rm hor}$, or by first integrating it over the leaves of $\mathscr{F}_{\rm hor}$ (= Fourier transform), and then integrating it over the leaves of $\widetilde{\mathscr{G}}_{\rm hor}$.

Such relations would be immediate, by the theory of the Weil representation (up to a certain factor that has to do with the 8-fold metaplectic cover), if the foliations denoted by $\widetilde{\mathscr{G}}$ were *linear*, by which we mean each fiber to be open dense in an affine subspace of M. The fact that it remains true for the nonlinear foliations of the theorem is remarkable, but the proof will take advantage of such statements ("Weil's formula") in the linear case. The main observation behind the proof is that *there is another, linear Lagrangian foliation* $\widetilde{\mathscr{G}}$ of $M^\circ_{-f,f}$, that will serve as an intermediary between $\widetilde{\mathscr{G}}_{\rm ver}$ and $\widetilde{\mathscr{G}}_{\rm hor}$.

Let us recall once more that the fibers of $M^\circ_{-f,f} \to J^\circ \simeq A_1 \times A_2$ are $N \times N$-orbits. We will write $(\underline{a}, \underline{b})$ for an element of $A_1 \times A_2$ (the underline because we will sometimes explain things in coordinates, and think of $\underline{a}$ as the pair (a_1, a_2), and similarly for $\underline{b}$), and $\mathcal{O}_{\underline{a},\underline{b}}$ for its fiber. Using $\approx$ for two sets whose intersection is open dense in both, we have, in coordinates,

$$\bigcup_{a_2,b_1} \mathcal{O}_{\underline{a},\underline{b}} \approx M_{a_1,b_2},$$

where M_{a_1,b_2} denotes the affine subspace

$$\left\{ \left(w \begin{pmatrix} a_1 & a_1 x \\ -a_1 y & a_2 \end{pmatrix}, \begin{pmatrix} b_1 & a_1^{-1} - x b_2 \\ a_1^{-1} + y b_2 & b_2 \end{pmatrix} w \right),\ (a_2, b_1, x, y) \in F^4 \right\}.$$

Coordinate-independently, we have made a choice of unipotent subgroup N, hence of the line $L = V^N \subset V$, and these affine subspaces are the fibers of the composition

$$M^\circ_{-f,f} \hookrightarrow \mathrm{End}(V) \times \mathrm{End}(V^*) \to \mathrm{Hom}(L, V/L) \times \mathrm{Hom}(L^\perp, V^*/L^\perp),$$

over pairs of nonzero morphisms in codomain. In what follows, we also use (a_1, b_2) to denote such a pair in $\mathrm{Hom}(L, V/L) \times \mathrm{Hom}(L^\perp, V^*/L^\perp)$. These spaces M_{a_1,b_2} (or rather, their open intersections $M^\circ_{a_1,b_2}$ with $M^\circ_{-f,f}$) form a Lagrangian foliation $\tilde{\mathscr{G}}$ of $M^\circ_{-f,f}$. We will write $\mathscr{G}$ for the corresponding foliation of the quotient $J^\circ = M^\circ_{-f,f}/(N \times N)$.

4.3.4. *Integrals over Lagrangians.* We now come to one of the finest, albeit elementary, part of the argument, which will also answer the question that we posed in Remark 4.1.1: why is the chosen way to define pushforward of half-densities the "correct" one? We start by asking the question: *If $\mathscr{F}$ is a Lagrangian foliation on M, what does it mean to average an element of $\mathcal{D}_{\mathscr{F}}(M, L)$ over a Lagrangian subspace $\mathscr{L}$?*

[4]It is immediate to see that the composition $\hookrightarrow$ in the diagram of Theorem 4.2.11 is the same as directly integrating over the leaves of $\widetilde{\mathscr{G}}_{\rm hor}$.

In the context where $\mathscr{L}$ is part of another Lagrangian foliation, the answer is given by formula (44) and Remark 4.2.2. *It is important to observe that these do not make sense with knowledge of a single Lagrangian subspace $\mathscr{L}$, without the foliation.* Indeed, the foliation allows the identification of *all* conormal spaces to points of $\mathscr{L}$ with the cotangent fiber of their image in $M/\mathscr{F}$, and in particular gives a *uniform* (up to scalar) way to write the factor $|\omega|^{\frac{1}{2}\dim(\mathscr{F}_{2,x}/\mathscr{F}_1)}$ of (44) as a product of a half-density along $\mathscr{F}_{2,x}/\mathscr{F}_1$ and a half-density in the "transverse" direction.

However, in our context, we need to integrate over Lagrangians which are not part of a foliation of the entire space M, but just of its subspace $M^\circ_{-f,f}$. Let us call such a foliation of a (non-dense) subspace a "partial" foliation. In this context, as we have seen, *the integrals only make sense after we pick a Haar half-density on each $N \times N$-orbit*, in order to first push forward to $M^\circ_{-f,f}/(N \times N) = J^\circ$, where our Lagrangians do form a foliation. The choice of Haar half-densities on the $N \times N$-orbits affects the answer; understanding which choice is right is the question we posed in Remark 4.1.1. That remark gave an interpretation to our choice, but it remains to be seen why this is the correct choice.

For the proof of Theorem 4.2.11, it is particularly important to compare our integrals over the fibers M_{a_1,b_2} of the partial foliation $\widetilde{\mathscr{G}}$ with the corresponding integrals for the linear foliations containing them. Note that these affine subspaces are not parallel to each other – they belong to different linear foliations. But if we fix a pair (a_1, b_2), we can consider the linear foliation $\mathscr{F} = \mathscr{F}_{a_1,b_2}$ of parallel Lagrangians to it. We then have two versions of the "integral over M_{a_1,b_2}" of an element of $\mathcal{D}_{\mathrm{hor}}(M, L)$ (or of $\mathcal{D}_{\mathrm{ver}}(M, L)$ – but we present the former):

(a) The integral corresponding to the foliation $\mathscr{F}$. Since $\dim M_{a_1,b_2}/\mathscr{F}_{\mathrm{ver}} = 3$, this is given by (44), with the exponent of $|\omega|$ being $\frac{3}{2}$. It produces a section of the line bundle multiplied by a half-density on the 4-dimensional quotient space $M/\mathscr{F}$.
(b) The integral arising as the pushforward to $\mathcal{D}_{\mathrm{hor}}(J^\circ, L)$ ("integration over $N \times N$-orbits," depending on our choices of Haar half-densities on those), followed by the integral (44) along the fibers of the foliation $\mathscr{G}$. Here, since $\mathscr{G}_{\mathrm{ver},x}/\mathscr{G}$ has dimension 1, the exponent of $|\omega|$ is $\frac{1}{2}$. It produces a section of the line bundle multiplied by a half-density on the quotient $J^\circ/\mathscr{G}$.

How can we compare the outcomes of the two integrals? It clearly does not make sense to say that their restrictions on M_{a_1,b_2} are "equal," since the meaning of these restrictions is different: Both have a factor that can be intepreted as a parallel section of L along this Lagrangian, but in the first case this is multiplied by a "transverse" half-density for the quotient $M/\mathscr{F}$ (which is 4-dimensional), while the latter is multiplied by a "transverse" half-density for the quotient $J^\circ/\mathscr{G}$ (which is 2-dimensional). These half-densities, restricted to M_{a_1,b_2}, live in the line bundle $|\det \mathcal{N}^*|^{\frac{1}{2}}$, where $\mathcal{N}^*$ denotes the conormal bundle in each case.

The quotient of the normal bundle of $M^\circ_{a_1,b_2}$ in M by its normal bundle in $M^\circ_{-f,f}$ is identified with the tangent space of (f,f) in $\mathfrak{n}\times\mathfrak{n}$ via the moment map, hence the cokernel of the map of normal bundles can be identified with $\mathfrak{n}^*\times\mathfrak{n}^*$. It follows that *it makes sense to compare the restriction of an element of $\mathcal{D}_{\mathscr{F}}(M,L)$ to M_{a_1,b_2} with the restriction of an element of $\mathcal{D}_{\mathscr{G}}(J^\circ,L)$ multiplied by an element of* $|\det(\mathfrak{n}^*\times\mathfrak{n}^*)|^{\frac{1}{2}} = |\det\mathfrak{n}^*|$. I reformulate this fact as a lemma:

Lemma 4.3.5. *Let $D_{M/\mathscr{F}}$ be the line bundle of half-densities on $M/\mathscr{F}$, pulled back to the leaf M_{a_1,b_2}. Let $D_{J^\circ/\mathscr{G}}$ be the line bundle of half-densities on $J^\circ/\mathscr{G}$, pulled back to the same leaf. Then, we have a canonical isomorphism of line bundles:*

$$D_{M/\mathscr{F}} = D_{J^\circ/\mathscr{G}} \otimes |\det\mathfrak{n}^*|. \tag{54}$$

Here is a more precise version: Fix a point $o\in M^\circ_{a_1,b_2}$, and consider the following filtrations of its tangent space in M:

$$\begin{tikzcd}
 & T_o(M_{a_1,b_2}\cap\mathscr{F}_{\mathrm{ver},o}) & & & \\
0 \arrow[ur, "1"] \arrow[dr, "2"'] & \bigoplus \arrow[r, "\sim"] & T_o(M_{a_1,b_2}\cap\mathscr{G}_{\mathrm{ver},o}) \arrow[r, "1"] & T_oM_{a_1,b_2} \arrow[r, "1"] & {} \\
 & \mathfrak{n}\times\mathfrak{n} & & &
\end{tikzcd}$$

$$\begin{tikzcd}
 & & T_o(M_{a_1,b_2}\cap\mathscr{F}_{\mathrm{ver},o})^\perp \arrow[dr, "1"] & \\
{} \arrow[r] & T_o(M_{a_1,b_2}\cap\mathscr{G}_{\mathrm{ver},o})^\perp \arrow[ur, "2"] \arrow[dr, "1"'] & & T_oM. \\
 & & T_oM^\circ_{-f,f}(\mathfrak{n}\times\mathfrak{n})^\perp \arrow[ur, "\mathfrak{n}^*\times\mathfrak{n}^*"'] &
\end{tikzcd}$$

All the maps here are inclusions, and the labels on the arrows give a description or designate the dimension of the quotient space.

As o varies over $M_{a,b}$, this sequence of subspaces becomes a sequence of vector bundles, that we denote by their ranks (and by underline for a vector bundle with canonically constant fibers – i.e., a flat connection with trivial monodromy):

$$\begin{tikzcd}
 & S_1 & & & \\
0 \arrow[ur, "1"] \arrow[dr, "2"'] & \bigoplus \arrow[r, "\sim"] & S_3 \arrow[r, "1"] & S_4 = T_o M_{a_1,b_2} \arrow[r, "1"] & {} \\
 & \underline{\mathfrak{n}\times\mathfrak{n}} & & &
\end{tikzcd}$$

$$\begin{tikzcd}
 & & S_7 \arrow[dr, "1"] & \\
{} \arrow[r] & S_5 \arrow[ur, "2"] \arrow[dr, "1"'] & & S_8 = T_o M. \\
 & & \underline{(\mathfrak{n}\times\mathfrak{n})^\perp} \arrow[ur, "\underline{\mathfrak{n}^*\times\mathfrak{n}^*}"'] &
\end{tikzcd}$$

With regard to the integrals (a) and (b) above, the factor ω which appears, via its square root, in (b) trivializes the density bundle (absolute value of the determinant) of the (symplectic) quotient S_5/S_3. On the other hand, the factor ω^3 appearing in (a) trivializes the density bundle of the (symplectic) quotient S_7/S_1. This is not the whole story, however: As mentioned, to make sense of those integrals, we need a *factorization* of these trivializations into a "parallel" and a "transversal" direction. For example, in case (a), we need to write ω^3 as the product of a section of $|\det S_4/S_1|$ and a section of an element of (its dual line) $|\det S_7/S_4|$. Equivalently, we need to turn these density bundles into local systems (with trivial monodromy); in other words, they have a canonical line of nonvanishing sections.

Such a structure is provided by the Lagrangian foliations and (45). For S_5/S_4, this comes by applying (45) to the (tangent spaces of) the quotients $J^\circ/\mathscr{G}_{\rm ver}$, $J^\circ/\mathscr{G}$, and S_5/S_3. For S_7/S_5, we use the "linear" quotients $M/\mathscr{F}_{\rm ver}$, $M/\mathscr{F}$, together with S_7/S_3. Colloquially, we will refer to the resulting canonical (up to scalar) section of $|\det S_7/S_4|$ as the "linear" factorization of ω^3.

The following lemma, now, will be the basis of the proof of Theorem 4.2.11, and justifies the definition of pushforwards (40).

Lemma 4.3.6. *In the context above, the canonical isomorphisms of line bundles*

$$|\det S_7/S_4| \simeq |\det S_5/S_4| \otimes |\det(\underline{\mathfrak{n}^*\times\mathfrak{n}^*})|$$

are isomorphisms of local systems; i.e., they preserve the canonical lines of sections.

Stated differently, the linear factorization of ω^3 over M_{a_1,b_2} coincides with the product of the $(\mathscr{G},\mathscr{G}_{\rm ver})$-factorization of ω by a *constant* element of $|\det(\mathfrak{n}^*\times\mathfrak{n}^*)|$.

Proof. After this abstract discussion, the reader will probably appreciate an explicit calculation. In coordinates $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \times \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix}$ for M, coordinates for the linear quotient $M/\mathscr{F}$ are given by $(A, D', b_2 B + a_1 B', b_2 C + a_1 C')$, with the differential of dA vanishing on S_7. (Here, S_7 is the tangent space to the hyperplanes A =constant.) Coordinates for $J^\circ/\mathscr{G}$ are given by (A, D') (recall that now we are restricting to the subset $M_{-f,f}$), with the differential of dA again vanishing on S_5. Thus, the canonical section for the density bundle associated to S_5/S_4 is (up to a scalar) $|dD'|$, and the section for the density bundle associated to S_7/S_4 is given (again up to a scalar) by $|dD' \wedge d(b_2 B + a_1 B') \wedge d(b_2 C + a_1 C')|$. We could express the latter as the wedge of dA with differentials factoring through the $\mathfrak{n}^* \times \mathfrak{n}^*$-moment map, but instead let us use the symplectic duality to translate these sections to densities on the appropriate quotients of $M_{a,b}$: Using the symplectic form (50) on J°, the dual of $|dD'|$ is the density $|dD|$ on $(M_{a_1,b_2}/N \times N)/\mathscr{G}_{\mathrm{ver}}$. Similarly, using (46), the dual of $|dD' \wedge d(b_2 B + a_1 B') \wedge d(b_2 C + a_1 C')|$ is the density $|dD \wedge a_1^{-1} dB \wedge a_1^{-1} dC|$ on $M_{a_1,b_2}/\mathscr{F}_{\mathrm{ver}}$. Given the action (52) of $N \times N$, this can be written as the product of $|dD|$ by the Haar measure $|dx \wedge dy|$ on $N \times N$. The latter is constant on M_{a_1,b_2} (does not depend on the chosen point), hence, dually, the quotient of the canonical sections is a constant section of $|\det(\underline{\mathfrak{n}^* \times \mathfrak{n}^*})|$. ☐

The proof already demonstrated the duality between "transverse" and "parallel" half-densities. In particular, the flat connection afforded by the foliation on the "transverse" bundles of densities induces one on the "parallel" ones, and we have the following immediate corollary:

Corollary 4.3.7. *The canonical isomorphism* (54) *of line bundles of half-densities on $M_{a,b}/\mathscr{F}_{\mathrm{ver}}$ is an isomorphism of local systems, i.e., it preserves the canonical lines of sections.*

4.3.8. *Completion of the proof of Theorem 4.2.11 for $n = 2$.* We keep fixing a pair (a_1, b_2), and denoting by $\mathscr{F}$ the corresponding "linear" foliation of M.

Proposition 4.3.9. *The integrals* (44) *along the leaves of the following Lagrangian foliations give rise to a commutative diagram*

$$\begin{tikzcd} \mathcal{D}_{\mathrm{hor}}(M, L) \arrow[rr, "\mathcal{F}"] \arrow[dr] & & \mathcal{D}_{\mathrm{ver}}(M, L) \arrow[dl] \\ & \mathcal{D}_{\mathscr{F}}(M, L). & \end{tikzcd}$$

Proof. In the theory of the Weil representation, such diagrams of integrals along "linear" foliations commute up to an 8-th root of unity, see [Li08, § 3.5]. For the case at hand, this root of unity is trivial; this is "Weil's formula," whose (very simple) proof is recalled in [Jac03, Proposition 2], and we leave the verification to the reader. ☐

We can now complete the proof of Theorem 4.2.11 for the case of GL_2. Using a fixed Haar measure on $N \times N$ to define the pushforwards (51), we have a diagram

$$\begin{tikzcd} \mathcal{D}_{\mathrm{hor}}(M, L) \arrow[rr, "\mathcal{F}"] \arrow[d] \arrow[dr] & & \mathcal{D}_{\mathrm{ver}}(M, L) \arrow[d] \arrow[dl] \\ \mathcal{D}_{\mathrm{hor}}(J^\circ, L) \arrow[r, "\alpha"] & \mathcal{D}_{\mathscr{G}}(J^\circ, L) & \mathcal{D}_{\mathrm{ver}}(J^\circ, L), \arrow[l, "\beta"'] \end{tikzcd} \tag{55}$$

where all the arrows are given by integrals along the Lagrangian leaves of each foliation. In particular, the maps to $\mathcal{D}_{\mathscr{G}}(J^\circ, L)$ are given by the integrals over the leaves of the foliation $\widetilde{\mathscr{G}}$ on $M^\circ_{-f,f}$. We discussed in § 4.3.4 how to make sense of these integrals. Corollary 4.3.7 implies that the evaluations of those on each leaf M_{a_1,b_2} are *equal*, up to a fixed half-density on $|\mathfrak{n}^* \times \mathfrak{n}^*|$ that is dual to the chosen Haar half-density on $N \times N$, to the "linear" integral along M_{a_1,b_2} corresponding to the linear foliation $\mathscr{F} = \mathscr{F}_{a_1,b_2}$. Proposition (4.3.9), now, implies that the upper triangle commutes. The two lower triangles also commute, by construction; i.e., the diagram (55) is commutative.

Finally, the inverse of the arrow labeled by β also the "integral (44) over Lagrangians." This can be reduced to a usual, linear Fourier transform in dual $\underline{a}$ and $\underline{b}$ variables, using representatives as in (53) and the symplectic form (50). Therefore, the composition $\beta^{-1} \circ \alpha$ is the map $\mathscr{H}_{\mathrm{Std}}$ of Theorem 4.2.11.

4.4. **Sketch of the proof in the general case.** If one understands how the argument of Jacquet translates to the argument we presented in § 4.3 for the case of $n = 2$, it is straightforward to adapt it for general n. The only additional idea needed, already present in [Jac03], is to apply induction in n. I will give a vague and impressionistic summary of the argument; most of the details remain to be filled in by the interested reader, who will also need to consult Jacquet's paper. This summary is not meant as a standalone account of the argument; its goal is simply to convince the reader that a translation to the setting of Theorem 4.2.11 is possible, even straightforward, given the argument for $n = 2$.

The inductive step needed corresponds to the following factorization of the Kuznetsov orbital integrals (41)

$$O_a(\Phi) = \int_{U_n' \times U_n} \int_{N_{n-1}' \times N_{n-1}} \Phi(u_1 n_1 w a n_2 u_2) \psi^{-1}(u_1 n_1 n_2 u_2) d(n_1, n_2) d(u_1, u_2), \tag{56}$$

where, if N is the unipotent subgroup of the Borel stabilizing a flag $V_1 \subset V_2 \subset \cdots \subset V_n = V$ (which, in coordinates, to take to be the standard flag of F^n, with N upper triangular), U_n is the unipotent radical of the stabilizer of V_1, and N_{n-1} is the corresponding group for the $(n-1)$-dimensional space V/V_1, identified with a subgroup of N by choosing a splitting of the quotient (which, in coordinates, we will do using the standard basis of F^n. The groups U_n' and N_{n-1}' are defined similarly, in terms of the dual filtration on the dual space.

The inner integral of (56) is then the Kuznetsov orbital integral for a function Φ_{a_1} on GL_{n-1}, defined as

$$\Phi_1(g) = \int_{U_n' \times U_n} \Phi\left(u_1 w_n \begin{pmatrix} a_1 & \\ & g \end{pmatrix} u_2\right) \psi^{-1}(u_1 u_2) d(u_1, u_2),$$

where w_n is the permutation matrix $\begin{pmatrix} & I_{n-1} \\ 1 & \end{pmatrix}$. An inductive application of the theorem, then, relates the Kuznetsov orbital integrals of Φ_1 (appropriately normalized – i.e., we need to work with half-densities again) to those of its Fourier transform, and applies the same argument, using Weil's formula (a direct generalization of Proposition 4.3.9) to relate the Fourier transform of Φ_1 to the Fourier transform of Φ.

It is clear that this argument directly translates to prove our reformulation 4.2.11 of Jacquet's theorem. I will only make a few comments on the inductive step: Starting with the same foliations $\mathscr{F}_{\mathrm{ver}} : M \to \mathrm{End}(V)$ and $\mathscr{F}_{\mathrm{hor}} : M \to \mathrm{End}(V)$ as before, we can interpret the definition of Φ_1 as an integral of an element of $\mathcal{D}_{\mathrm{hor}}(M, L)$ over appropriate Lagrangians corresponding to a foliation $\mathscr{G}_1$ of the symplectic reduction $U_n' \backslash\!\!\backslash\!\!\backslash_f M /\!\!/\!\!/_f U_n$ (or rather, of its "open Bruhat cell"). These Lagrangians are parametrized by pairs (a_1, g) as above, which, coordinate-independently, can be thought of as invertible elements $a_1 \in \mathrm{Hom}(V_1, V_n/V_{n-1})$ and $g \in \mathrm{Hom}(V/V_1, V_{n-1})$. Moreover, fixing a_1, these Lagrangians map to a similar to $\mathscr{F}_{\mathrm{ver}}$ linear foliation of (an open dense subset of) $T^* \mathrm{Hom}(V/V_1, V_{n-1})$, in a way that allows for an inductive application of the theorem to compute the Kuznetsov orbital integrals of the original function in terms of the Kuznetsov orbital integrals of the Fourier transform of Φ_1. Finally, the relation between the Fourier transform of Φ_1 and the Fourier transform of Φ (and its proof) can be thought of as the statement that a diagram of the form

$$\begin{tikzcd}
\mathcal{D}_{\mathrm{hor}}(M, L) \arrow[rr, "\mathcal{F}"] \arrow[d] \arrow[dr] & & \mathcal{D}_{\mathrm{ver}}(M, L) \arrow[d] \arrow[dl] \\
\mathcal{D}_{\mathscr{G}_1}(U_n' \backslash\!\!\backslash\!\!\backslash_f M /\!\!/\!\!/_f U_n, L) \arrow[r, "\alpha"] & \mathcal{D}_{\mathscr{G}}(U_n' \backslash\!\!\backslash\!\!\backslash_f M /\!\!/\!\!/_f U_n, L) & \mathcal{D}_{\mathscr{G}_2}(U_n' \backslash\!\!\backslash\!\!\backslash_f M /\!\!/\!\!/_f U_n, L) \arrow[l, "\beta"']
\end{tikzcd} \tag{57}$$

commutes. This diagram is the analog of (55), with $\mathscr{G}_2$ a foliation parametrized by pairs (b_1, g'), with b_1, g' in the duals $\mathrm{Hom}(V_n/V_{n-1}V_1)$, $\mathrm{Hom}(V_{n-1}, V/V_1)$ of V_n/V_{n-1}, $\mathrm{Hom}(V/V_1, V_{n-1})$, respectively. The argument for the proof of this remains essentially the same, with an intermediate "linear" foliation $\mathscr{G}$, parametrized by pairs (a_1, g') (notation as before), where one can apply Weil's formula.

References

[Bri01] Michel Brion. On orbit closures of spherical subgroups in flag varieties. *Comment. Math. Helv.*, 76(2):263–299, 2001. doi:10.1007/PL00000379.

[Car97] Jacques Carmona. Terme constant des fonctions tempérées sur un espace symétrique réductif. *J. Reine Angew. Math.*, 491:17–63, 1997. doi:10.1515/crll.1997.491.17.

[Del73] P. Deligne. Les constantes des équations fonctionnelles des fonctions L. In *Modular functions of one variable, II (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972)*, Lecture Notes in Math., Vol. 349, pages 501–597. Springer, Berlin, 1973.

[DHS21] Patrick Delorme, Pascale Harinck, and Yiannis Sakellaridis. Paley-Wiener theorems for a p-adic spherical variety. *Mem. Amer. Math. Soc.*, 269(1312):v+102, 2021. doi:10.1090/memo/1312.

[DKSBP22] Patrick Delorme, Bernhard Krötz, Sofiane Souaifi, and Raphaël Beuzart-Plessis. The constant term of tempered functions on a real spherical space. *Int. Math. Res. Not. IMRN*, (12):9413–9498, 2022. doi:10.1093/imrn/rnaa395.

[GJ72] Roger Godement and Hervé Jacquet. *Zeta functions of simple algebras*. Lecture Notes in Mathematics, Vol. 260. Springer-Verlag, Berlin-New York, 1972.

[GW21] Wee Teck Gan and Xiaolei Wan. Relative character identities and theta correspondence. In Werner Müller, Sug Woo Shin, and Nicolas Templier, editors, *Relative Trace Formulas*, pages 101–186, Cham, 2021. Springer International Publishing. doi:10.1007/978-3-030-68506-5_4.

[Jac03] Hervé Jacquet. Smooth transfer of Kloosterman integrals. *Duke Math. J.*, 120(1):121–152, 2003. doi:10.1215/S0012-7094-03-12015-3.

[Kno95] Friedrich Knop. On the set of orbits for a Borel subgroup. *Comment. Math. Helv.*, 70(2):285–309, 1995. doi:10.1007/BF02566009.

[Li08] Wen-Wei Li. The Weil representation and its character. Master's thesis, Universiteit Leiden, 2008.

[Sak] Yiannis Sakellaridis. Spherical varieties, functoriality, and quantization. To appear in the *Proceedings of the 2022 International Congress of Mathematicians*, 44pp. arXiv:2111.03004.

[Sak08] Yiannis Sakellaridis. On the unramified spectrum of spherical varieties over p-adic fields. *Compos. Math.*, 144(4):978–1016, 2008. doi:10.1112/S0010437X08003485.

[Sak13] Yiannis Sakellaridis. Spherical functions on spherical varieties. *Amer. J. Math.*, 135(5):1291–1381, 2013. doi:10.1353/ajm.2013.0046.

[Sak16] Yiannis Sakellaridis. The Schwartz space of a smooth semi-algebraic stack. *Selecta Math. (N.S.)*, 22(4):2401–2490, 2016. doi:10.1007/s00029-016-0285-3.

[Sak19] Yiannis Sakellaridis. Relative functoriality and functional equations via trace formulas. *Acta Math. Vietnam.*, 44(2):351–389, 2019. doi:10.1007/s40306-018-0295-7.

[Sak21] Yiannis Sakellaridis. Functorial transfer between relative trace formulas in rank 1. *Duke Math. J.*, 170(2):279–364, 2021. doi:10.1215/00127094-2020-0046.

[Sak22a] Yiannis Sakellaridis. Transfer operators and Hankel transforms between relative trace formulas, I: Character theory. *Adv. Math.*, 394:Paper No. 108010, 75pp, 2022. doi:10.1016/j.aim.2021.108010.

[Sak22b] Yiannis Sakellaridis. Transfer operators and Hankel transforms between relative trace formulas, II: Rankin-Selberg theory. *Adv. Math.*, 394:Paper No. 108039, 104pp, 2022. doi:10.1016/j.aim.2021.108039.

[SV17] Yiannis Sakellaridis and Akshay Venkatesh. Periods and harmonic analysis on spherical varieties. *Astérisque*, (396):360, 2017.

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