Automorphic Project

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CHAPTER 1

Introduction

1.1. Overview

The Automorphic Project was inspired by the success and incredible utility of the Stacks Project, and necessitated by the lack of a comprehensive introduction to automorphic forms and the Langlands program. While there are, and continue to appear, many good references on many aspects of the subject, we felt that it can be much better presented, in all its vastness, by an evolving and collaborative effort that today's technology enables.

While we will eventually aim at making the Automorphic Project as inclusive as possible, our goal is not to reproduce the existing literature. Rather, and taking into account that the literature in the field is usually considered very technical and difficult to follow for outsiders, our goal is to recast the literature in a way as conceptual as possible, and a language that is not esoteric to the field, but easily accessible to someone who commands the modern language of mathematics.

We expect this material to be read online as a key feature are the hyperlinks giving quick access to internal references spread over many different pages. If you use an embedded pdf or dvi viewer in your browser, the cross file links should work.

This project is a collaborative effort and we encourage you to help out. Please email any typos or errors you find while reading or any suggestions, additional material, or examples you have to automorphic.project@gmail.com. You can download a tarball containing all source files, extract, run make, and use a dvi or pdf viewer locally. Please feel free to edit the LaTeX files and email your improvements.

(We need at least one reference [**Som**] in each chapter.)

1.2. Attribution

Those who contributed to this work are listed on the title page of the book version of this work and online.

1.3. Other chapters

CHAPTER 2

Representation theory: general notions

2.1. Conventions

The definition of representations makes sense over an arbitrary field, but very soon we start working with measures, specializing to the complex numbers as the coefficient field.

2.2. Representations

Let G be a topological group. Topological vector spaces are taken over a topological field k (which we fix). We denote by $\mathrm{End}(V)$, $\mathrm{Aut}(V)$ the sets of *continuous* endomorphisms, resp. automorphisms, of a topological vector space V.

Definition 2.2.1. A *representation* of G is a pair (π, V), where V is a topological vector space V over k, and π is a homomorphism

$$\pi : G \to \mathrm{Aut}(V),$$

with the property that the induced "action" map:

$$G \times V \to V,$$

$$(g, v) \mapsto \pi(g)v$$

is continuous.

Representations of G on topological k-vector spaces form a category, with a morphism

$$(\pi_1, V_1) \to (\pi_2, V_2)$$

being a continuous linear map $V_1 \to V_2$ which commutes with the action of G.

A *subrepresentation* of V is a *closed* subspace of V which is stable under the action of G.

A representation is called *irreducible*, or *simple*, if it does not contain any non-zero, proper subrepresentations.

A representation is called *semisimple* or *(totally) decomposable* if it is equal to the direct sum of irreducible subrepresentations. [1]

Example 2.2.2. Given a space X with a right action of a group G, if $F(X)$ denotes the space of k-valued functions on X, there is a natural representation of G on $F(X)$, sometimes called the *regular representation* of G on X (although the term is more standard for the action of G on itself, see Definition 2.4.4). It is given by

$$R(g)(f)(x) = f(xg).$$

[1] For infinite-dimensional representations, other notions of decomposition, such as by *direct integrals*, are often more useful. They will be discussed later.

If G, X are endowed with topology, so that the map $X \times G \to X$ is continuous, and X is locally compact and Hausdorff, the space $C(X)$ of continuous, complex-valued functions, topologized as the inductive limit over all compact $K \subset X$ of the Banach spaces $C(K)$, becomes a (topological) representation of G, and so does its dual $M_c(X)$ of compactly supported regular measures.

If (X, Ω, μ) is a measure space with a right action by a discrete group G that preserves the σ-algebra Ω and the measure μ, the spaces $L^p(X, \mu)$ become topological representations of G.

If G is any topological group acting continuously on a locally compact Hausdorff topological space X as before, and μ is a regular Borel measure preserved by the action of G, then $L^p(X, \mu)$ is a (continuous) representation of G for $1 \leq p < \infty$. (Exercise!)

Remark 2.2.3. We do not require the map $G \to \mathrm{Aut}(V)$ to be continuous in any of the usual operator topologies (for example, the norm on bounded linear operators, when V is a Banach space), because this would preclude some of the most natural representations. For example, the rotation representation of $G =$ the circle on $V = L^2(\mathbb{R})$ is not a continuous map $G \to \mathrm{Aut}(V)$ with respect to the Hilbert norm on bounded operators. This may seem troubling at first, but it will appear more natural when we talk about the action of group algebras, see Remark 2.5.5.

2.3. Action of measures on the group

A basic principle in representation theory is that we should extend the action of the group to the action of a suitable algebra, because algebras have a lot more structure.

From now on we assume that the field k of coefficients of the representation is the field $\mathbb{C}$ of complex numbers, and that the topological group G is locally compact.

Let $M(G)$ be the Banach space of finite, complex-valued measures on G. ("Measures" will always mean Radon measures.) It is a Banach algebra under convolution. Convolution is, by definition, the push-forward of measures under the multiplication map $G \times G \to G$. We denote by $M_c(G)$ the subalgebra of compactly supported finite measures. If dg is a left Haar measure on G, we will call a measure $\mu = f dg$ continuous, L^1, etc, if f is a function with the same property.

If (π, V) is a topological representation of G, we would like to extend the action of G to an action of the algebra $M(G)$ of measures, or a subalgebra A thereof, in such a way that the action of $g \in G$ will correspond to the action of the delta measure at g; we will keep using the notation π for such an extension.

Example 2.3.1. The convolution algebra of k-valued measures of finite support on G makes sense for an arbitrary commutative ring of coefficients k, and is called the *group algebra* $k[G]$.

If G is discrete, there is an (obvious) equivalence of categories between representations of G on k-vector spaces (without topology, i.e., with the discrete topology) and $k[G]$-modules.

Example 2.3.2. In particular, for the group $G = \mathbb{Z}$, its group algebra is $k[T, T^{-1}]$, and its finitely generated representations (without topology) are classified by the structure theorem for finitely generated modules over principal ideal domains.

We assume throughout that the continuous dual V^* of V separates points. Then, the extension that we seek will be characterized by the property

$$\langle \pi(\mu)(v), v^* \rangle = \int_G \langle \pi(g)v, v^* \rangle \, \mu(g), \tag{2.3.2.1}$$

for every $\mu \in A$, $v \in V$, $v^* \in V^*$.

Proposition 2.3.3. *Assume that V is a locally convex topological vector space, and such that the closure of the convex hull of any compact set is compact. Then, for every $\mu \in M_c(G)$, $v \in V$ the vector $\pi(\mu)(v)$ characterized by (2.3.2.1) is defined.*

The resulting map $M_c(G) \times V \to V$ is continuous, when $M_c(G)$ is topologized as the inductive limit, over compact subsets $K \subset G$, of the Banach spaces $M(K)$.

Moreover, the restriction of this map to any bounded subset of $M_c(G)$ is continuous with respect to the weak- topology, that is, the inductive limit topology of the spaces $M(K)$ endowed with the weak-* topology as duals of the spaces $C(K)$.*

Notice that by [**Rud91**, Theorem 3.20], Fréchet spaces satisfy the conditions of the proposition.

Proof. By [**Rud91**, Theorem 3.27], for a finite Borel measure μ on a compact Hausdorff space K, and a continuous function $f : K \to V$ to a topological vector space V satisfying the conditions of the proposition, the integral $\int_K f \cdot \mu$, characterized by

$$\left\langle \int_K f \cdot \mu, v^* \right\rangle = \int_K \langle f, v^* \rangle \cdot \mu$$

for every $v^* \in V^*$, is defined. (The argument, in brief: First, reduce to the case of positive probability measures. Then, show that the subsets of the closed convex hull of $f(K)$ cut out by a finite number of conditions of the form $\langle v, v^* \rangle = \int_K \langle f, v^* \rangle \cdot \mu$ are nonempty. By compactness, it follows that their intersection is nonempty.)

Here, we take $K \subset G$ to be the support of a given measure, and $f(g) = \pi(g)(v)$, and set $\pi(\mu)(v) := \int_G \pi(g)(v)\mu(g)$.

The same result ensures that, when μ is a positive probability measure, the integral $\int_G \pi(g)(v)\mu(g)$ belongs to the closure of the convex hull of the compact set $\pi(\text{supp}\mu)(v)$ (to be denoted $\overline{\text{co}}\,(\pi(\text{supp}\mu)(v))$.

The continuity statement on (norm) bounded sets with the weak-* topology is stronger than the continuity statement in the norm topology, so it suffices to prove that. Fix a compact $K \subset G$, and let $M(K)_{\leq 1}$ be the unit ball in $M(K)$. It is enough to prove that the preimage of a neighborhood U of $0 \in V$ contains a product $M \times U'$, where M is a weak-* neighborhood of zero in $M(K)_{\leq 1}$, and $U' \subset V$ is a neighborhood of zero.

We may take U to be closed and convex and balanced (i.e., $zU \subset U$ for $|z| \leq 1$). Since every $\mu \in M(K)_{\leq 1}$ can be written as $\mu_1 - \mu_2 + i\mu_3 - i\mu_4$, where the μ_i's are positive measures, also in $M(K)_{\leq 1}$, it's enough to show that if μ_1 is a positive measure, there are neighborhoods as above such that when $\mu_2 \in \mu_1 + M$ and $v \in U'$, we have $\mu_1(v) - \mu_2(v) \in U$.

By continuity of the action map $G \times V \to V$, there is a finite open cover $(K_i)_{i=1}^r$ of K, and an open $U' \subset V$, such that, for each i,

$$\{v_1 - v_2 | v_1, v_2 \in \overline{\text{co}}(\pi(K_i)(U'))\} \subset \frac{1}{2}U.$$

There is a partition of unity $1_K = \sum_i f_i$, subordinate to the cover $(K_i)_i$, where the f_i's are continuous functions and $0 \leq f_i \leq 1$. Let μ_1, μ_2 be positive measures on K, with $\|\mu_i\| \leq 1$, whose difference is contained in the weak-* neighborhood determined by the conditions

$$|\langle \mu_1 - \mu_2, f_i \rangle| < \epsilon,$$

for given $\epsilon > 0$ (to be determined). Let $m_i^j = \int_K f_i \mu_j$ for $j = 1, 2$. Then, for $v \in U'$

$$\pi(\mu_1)(v) - \pi(\mu_2)(v) = \sum_i \int_K f_i(g)\pi(g)(v)(\mu_1 - \mu_2)(g).$$

Since the μ_i's and the f_i's are positive, the integral

$$\int_K f_i(g)\pi(g)(v)\mu_j(g)$$

lies in $m_i^j \cdot \overline{\mathrm{co}}(\pi(K_i)(U'))$. Thus,

$$\int_K f_i(g)\pi(g)(v)(\mu_1 - \mu_2)(g) \in \overline{\mathrm{co}}\left(m_i^1 \pi(K_i)(U')\right) - \overline{\mathrm{co}}\left(m_i^2 \pi(K_i)(U')\right)$$

$$\subset m_i^1 \left[\overline{\mathrm{co}}\left(\pi(K_i)(U')\right) - \overline{\mathrm{co}}\left(\pi(K_i)(U')\right)\right] + \epsilon\overline{\mathrm{co}}\left(\pi(K_i)(U')\right) \subset \frac{m_i^1}{2} \cdot U + \epsilon\overline{\mathrm{co}}\left(\pi(K_i)(U')\right),$$

and therefore

$$\pi(\mu_1)(v) - \pi(\mu_2)(v) \in \sum_i \left(\frac{m_i^1}{2} \cdot U\right) + \sum_i \epsilon \cdot \overline{\mathrm{co}}\left(\pi(K_i)(U')\right)$$

$$\subset \frac{\|\mu_1\|}{2} \cdot U + \sum_i \epsilon \cdot \overline{\mathrm{co}}\left(\pi(K_i)(U')\right) \subset \frac{1}{2} U + \sum_i \epsilon \cdot \overline{\mathrm{co}}\left(\pi(K_i)(U')\right).$$

Choosing ϵ small enough, we can guarantee that this belongs to U, and we are done.

□

2.4. Matrix coefficients

For the moment we are working with an arbitrary topological coefficient field k.

The dual of a topological representation is defined as follows:

Definition 2.4.1. Given a representation (π, V) of a topological group on a locally convex topological vector space V, the dual representation (π^*, V^*) on the dual space of continuous functionals on V is defined by the property

$$\langle \pi^*(g)(v^*), v \rangle = \left\langle v^*, \pi(g^{-1})v \right\rangle.$$

Here, V^* must be endowed with a suitable topology making the action continuous. For example, if G is locally compact, then one can consider the topology of uniform convergence on compact sets:

Lemma 2.4.2. *Let (π, V) be a representation of a localy compact group G. Then, the dual map $G \times V^* \to V^*$ is continuous when V^* is endowed with the topology of uniform convergence on compact subsets of V. If V is a Banach space, this topology is the finest topology which coincides with the weak-* topology on every norm-bounded subset of V^*.*

Proof. Given a compact $U \subset V$, and a compact $K \subset G$, the set $\pi(K)(U)$ is compact. Moreover, if we fix U and a neighborhood V_1 of zero in V, we can find a compact neighborhood K of $1 \in G$ such that $U_K \subset V_1$, where U_K is the set

$$U_K = \{\pi(g)(v) - v | g \in K, v \in U\} \subset V.$$

(These facts follow directly from the continuity of the action of G on V.)

Fix U, $\epsilon > 0$, and a vector $w \in V^*$, and let $V_1 = \{v \in V | \langle w, v\rangle < \epsilon\}$. Choose a compact neighborhood K of $1 \in G$ such that $U_K \subset V_1$. Let $W \subset V^*$ be the neighborhood of w, in the compact-open topology, of all vectors w' with

$$|\langle w - w', \pi(K)U\rangle| < \epsilon.$$

Then, for all $g \in K$ and $w' \in W$, we have

$$|\left\langle \pi^*(g^{-1})(w') - w, U\right\rangle| \leq |\left\langle \pi^*(g^{-1})(w' - w), U\right\rangle| + |\left\langle \pi^*(g^{-1})(w) - w, U\right\rangle| =$$

$$= |\langle (w' - w), \pi(g)U\rangle| + |\langle w, (\pi(g) - 1)U\rangle| < \epsilon + \epsilon.$$

This shows continuity in the compact-open topology on V^*.

In the case of normed spaces, this topology coincides with the *bounded weak-* topology*, the finest topology which coincides with the weak-* topology on every norm-bounded subset of V^* [**Day73**, §II.5, Lemma 2]. (See also [**Meg98a**, §2.7 and 3.4].) $\square$

When V is a Banach space, it is not true, in general, that the action is continuous with respect to the norm topology on V^* — just consider $V^* = M(G)$ as the dual of $V = C(G)$, for a compact group G. However, for locally compact groups and *reflexive* Banach spaces ($V^{**} = V$), this is the case, as we will see in Proposition 2.5.8.

Remark 2.4.3. For two spaces X and Y, not necessarily linear, which are in some sort of duality, in the sense that they come with a map

$$\langle\ ,\ \rangle : X \times Y \to Z,$$

where Z is another space, a *right* action of a group G on X naturally induces a *left* action on Y (when G is assumed to act trivially on the target Z), and vice versa:

$$\langle x \cdot g, y\rangle = \langle x, g \cdot y\rangle .$$

Therefore, we need to replace g by g^{-1} on X, if we want left actions on both spaces.

The remark above is exemplified in the following basic example:

Definition 2.4.4. If k is the field of coefficients, and $F(H)$ denotes the space of k-valued functions on a group H, the left (L) and right (R) *regular representations* of H on $F(H)$ are defined by

$$L \times R(h_1, h_2)(f)(x) = f(h_1^{-1} x h_2).$$

The term applies to any $H \times H$-invariant subspace of $F(H)$, and various $H \times H$-invariant quotients thereof (e.g., $L^p(H)$).

The right and left regular representations are a special case of Example 2.2.2 for $X = H$ and $G = H \times H$ (with left multiplication defined as a right action, $(g, x) \mapsto g^{-1}x$).

Definition 2.4.5. Given a representation (π, V) of a topological group G, the *matrix coefficient* map is the $G \times G$-equivariant map

$$M_\pi : \pi^* \otimes \pi \to F(G)$$

given by

$$v^* \otimes v \mapsto (g \mapsto \langle v^*, \pi(g)v\rangle).$$

Lemma 2.4.6. *The image of the matrix coefficient map (Definition 2.4.5) lies in the space $C(G)$ of continuous functions on G, and the resulting map*

$$V^* \times V \to C(G)$$

is continuous when $C(G)$ is endowed with the seminorms of local uniform convergence.

Proof. This follows immediately from the continuity of the map $G \times V \to V$. $\square$

We return to the case $k = \mathbb{C}$, and G a locally compact topological group. In that case, the spaces $M_c(G)$ and $C(G)$ are in duality, that is, there is a natural continuous map

$$M_c(G) \otimes C(G) \to \mathbb{C}.$$

Lemma 2.4.7. *Given a topological representation (π, V) of a locally compact topological group G on a space satisfying the conditions of 2.3.3 (e.g., a Fréchet space), we have, for every $\mu \in M_c(G)$, $v \in V$, $v^* \in V^*$,*

$$\langle \mu, M_\pi(v^* \otimes v)\rangle = \langle v^*, \pi(\mu)v\rangle, \tag{2.4.7.1}$$

where the first pairing is between measures and continuous functions, while the second is between V and its dual.

Proof. Just an unfolding of the definitions:

$$\langle \mu, M_\pi(v^* \otimes v)\rangle = \int_G \langle v^*, \pi(g)v\rangle\, \mu(g) = \left\langle v^*, \int_G \pi(g)v\mu(g)\right\rangle = \langle v^*, \pi(\mu)v\rangle.$$

$\square$

2.5. Banach representations of compactly generated groups

Definition 2.5.1. A *radial function* $r : G \to \mathbb{R}_+$, in the language of [**Ber88**][2] is a locally bounded positive function on G such that $r(g_1 \cdot g_2) \le r(g_1) + r(g_2)$.

Two radial functions r, r' are said to be *equivalent* if $(r + 1)$ is comparable to $(r'+1)$, that is, there is a constant $C > 0$ such that $C^{-1}(r+1) \le (r'+1) \le C(r+1)$.

Suppose that G is compactly generated and locally compact. Then there is a canonical equivalence class of radial functions on G:

Definition 2.5.2. The equivalence class of *natural radial functions* on a compactly generated and locally compact group G is the equivalence class of the radial function $r(g) = \min\{k | g \in B^k\}$, where B is a compact generating neighborhood of the identity in G.

We will be working with natural radial functions, unless otherwise stated.

[2]Except that we do not impose the condition $r(g) = r(g^{-1})$, as Bernstein does for convenience.

Lemma 2.5.3. *Let G be a compactly generated group, and (π, V) a Banach representation of G. There is a constant $C \geq 1$, depending on V and the choice of radial function r, such that $\|\pi(g)\| \leq C^{r(g)}$.*

Proof. From the definitions, if K is a compact generating subset, defining the scale function r, then there is a constant C such that $\|\pi(g)\| \leq C$ for every $g \in K$, and therefore $\|\pi(g)\| \leq C^{r(g)}$ for every $g \in G$. $\square$

Proposition 2.5.4. *Let G be a compactly generated, locally compact group, and (π, V) a Banach representation of G. Endow $End(V)$ with the operator norm. The map: $M_c(G) \to End(V)$ is continuous, and bounded by the norm $\mu \mapsto \|\mu \cdot C^r\|$ on $M_c(G)$, for some natural radial function r and constant $C \geq 1$.*

Proof. By [**Rud91**, Theorem 3.29],

$$\left\| \int_G \pi(g)(v)\mu(g) \right\| \leq \int_G \|\pi(g)\| |\mu|(g),$$

and by Lemma 2.5.3 this is $\leq \int_G C^{r(g)} |\mu|(g)$. $\square$

Remark 2.5.5. Returning to Remark 2.2.3, this proposition shows why it is not natural to require from the map $G \to \mathrm{Aut}(V) \hookrightarrow \mathrm{End}(V)$ to be continuous in the norm topology for $\mathrm{End}(V)$: We can identify the action of elements $g \in G$ with the action of the corresponding delta measures, but in the space of measures we do not have $\delta_{g_n} \to \delta_g$ when $g_n \to g$.

Definition 2.5.6. Let G be a compactly generated, locally compact group. The algebra $M_{rd}(G)$ of *rapidly decaying* measures on G is the Fréchet subalgebra of $M(G)$ defined by the norms $\|\mu \cdot C^r\|$, for a natural radial function r and all $C \geq 1$.

Proposition 2.5.7. *Every Banach representation extends to a continuous homomorphism $M_{rd}(G) \to End(V)$.*

Proof. Follows immediately from Proposition 2.5.4. $\square$

Finally, a result that was mentioned earlier, which shows that the dual of a representation on a reflexive Banach space is continuous in the norm topology:

Proposition 2.5.8. *Let (π, V) be a representation of a locally compact group on a reflexive Banach space. Then the dual representation (π^*, V^*) is continuous in the norm topology.*

Proof. Let $W \subset V^*$ denote the subspace of elements w for which the map

$$G \ni g \mapsto \pi^*(g)(w) \in V^*$$

is continuous in the norm topology for V^*. It will suffice to prove that $W = V^*$. Indeed, we also know (as a special case of Lemma 2.5.3) that the operators $\pi(g)$, and hence their adjoints $\pi^*(g)$, are uniformly bounded for g in any compact set. Thus, for every g in a fixed compact neighborhood of the identity, and any $\epsilon > 0$, there is a $\delta > 0$ such that

$$\|w' - w\| < \delta \Rightarrow \|\pi^*(g)(w') - \pi^*(g)(w)\| < \epsilon.$$

Hence, if we can guarantee that $\|\pi^*(g)(w) - w\|$ is sufficiently small, for $w \in W$ and g sufficiently close to 1, the same holds for every w' in a small neighborhood of w.

The set W is clearly a vector subspace, in particular convex. We will also show below that it is weakly-* dense. Since V is reflexive, this is the same as weakly dense. *The weak and strong closures of convex sets coincide* (as a corollary of the Hahn–Banach theorem), and therefore we will conclude that $W = V$.

There remains to show that W is weakly-* dense. Fix a left Haar measure dg, and consider the map $C_c^\infty(G) \otimes V^* \to V^*$ given by

$$f \otimes v^* \mapsto \pi^*(f dg)(v^*). \tag{2.5.8.1}$$

If we fix v^* and a compact $K \subset G$, this becomes a continuous map

$$C(K) \to V^*$$

with respect to the norm topology on V^*; indeed, we have

$$\|\pi^*(f dg)(v^*)\| = \sup_{\|v\|=1} |\langle v, \pi^*(f dg)(v^*)\rangle| =$$

$$= \sup_{\|v\|=1} |\langle \pi(f^* dg)(v), (v^*)\rangle| \ll_K \|f\| \cdot \|v^*\|,$$

by an application of Proposition 2.5.4, where we have set $f^*(g) = f(g^{-1})$, and the symbol $\ll_K a$ means $< c \cdot a$, for a constant c depending on K. But the map (2.5.8.1), for a fixed v^*, is equivariant with respect to the left regular action of G on $C_c(G)$, which is continuous. Therefore, the image of this map belongs to W.

Moreover, v^* is in the weak-* closure of this map: it is enough to consider a sequence of positive continuous functions f_n with $\int f_n dg = 1$ and f_n converging weakly to the delta measure at the identity, and apply Proposition 2.3.3 to the representation π to deduce that $\pi^*(f_n dg)(v^*) \to v^*$ in the weak-* topology. Thus, we have shown that W is weakly-* dense, completing the proof.

$\square$

Remark 2.5.9. Proposition 2.5.8 holds even without the assumption on local compactness of G, see [**Meg98b**, Corollary 6.9].

If the Banach space V is not reflexive, then V^* may fail to be a continuous representation with the norm topology. In this case, we can work with the subspace of V^* for which this is true:

Definition 2.5.10. The *contragredient* of a Banach representation (π, V) of a topological group G is the natural representation $(\tilde{\pi}, \tilde{V})$ of G on the subspace $\tilde{V} \subset V^*$ consisting of those vectors v^* such that the orbit map $G \ni g \mapsto g \cdot v^* \in V^*$ is (norm) continuous.

Lemma 2.5.11. *The subspace $\tilde{V} \subset V^*$ of the contragredient representation is closed.*

Proof. See [**BK14**, §3.1] for references. $\square$

Example 2.5.12. If G is a compact group, and $V = L^1(G)$ under the regular representation, then $\tilde{V} = C(G)$, but $V^* = L^\infty(G)$. Moreover, $\tilde{\tilde{V}} = V$, while $(\tilde{V})^* = M(G)$, the space of Radon measures.

2.6. Fréchet representations of moderate growth

We continue assuming that the coefficient field is $\mathbb{C}$. We also assume that the topological group G is locally compact, and compactly generated.

Definition 2.6.1. An *F-representation* of G (in the language of [**BK14**]), or *Fréchet representation of moderate growth* is a countable (inverse) limit of Banach representations, that is, a representation on a Fréchet space V, such that V admits an equivariant topological isomorphism

$$V = \lim\nolimits_{\leftarrow} V_n,$$

where the V_n's are Banach representations of G.

Note that this is *stronger* than just a Fréchet representation of G: In a Fréchet representation, for every (continuous) seminorm p, and for every compact $K \subset G$, there is a seminorm q such that

$$p(\pi(g)(v)) \le q(v)$$

for $g \in K$, $v \in V$. For an F-representation, there is a complete system of seminorms p_n such that we can take $q_n = c_{K,n} \cdot p_n$, where $c_{K,n}$ is a scalar that depends on K and n.

Note that by Proposition 2.5.7, the action of $M_c(G)$ on V extends to the measures $M_{rd}(G)$ of rapid decay.

Fix a (natural) radial function r, and let $\|g\| := e^{r(g)}$. The following definition is due to [**Cas89**]:

Definition 2.6.2. A Fréchet representation (π, V) of G is said to be of *moderate growth* if for any (continuous) seminorm p on V there exists a seminorm q and an integer $N > 0$ such that

$$p(\pi(g)v) \le \|g\|^N q(v) \tag{2.6.2.1}$$

for all $g \in G$.

Lemma 2.6.3. *Let (π, V) be a Fréchet representation of the Lie group G. Then the following statements are equivalent:*

1. *(π, V) is of moderate growth;*
2. *(π, V) is an F-representation.*

Proof. We follow [**BK14**, Lemma 2.10].

If (π, V) if of moderate growth, p is any seminorm, and q, N are a seminorm and a positive number satisfying (2.6.2.1), setting

$$\tilde{p}(v) = \sup_{g \in G} \frac{p(\pi(g)v)}{\|g\|^N}$$

we have inequalities $p \le \tilde{p} \le q$, and $\tilde{p}(\pi(g)) \le \|g\|^N \tilde{p}(v)$. The former implies that the seminorms of the form $\tilde{p}$ define the topology, and the latter implies that they are G-continuous.

Vice versa, if (π, V) is an F-representation, it suffices to prove the moderate growth condition for a system of G-continuous seminorms defining the topology. But then it reduces to the case of Banach representations, Lemma 2.5.3. $\square$

Remark 2.6.4. Bernstein and Krötz introduce a more general notion of F-representations, which allows for more general scale functions on the group than the one defined in 2.5.

Definition 2.6.5. Let V be an F-representation of G, and let $V = \lim_{\leftarrow} V_n$ be a presentation as a limit of Banach representations. The *contragredient representation* is the direct limit $\tilde{V} = \lim_{\rightarrow} \tilde{V}_n$ of contragredient Banach representations (Definition 2.5.10).

Lemma 2.6.6. *The contragredient representation does not depend on the chosen fundamental system of seminorms.*

Proof. Indeed, if q_n denotes the chosen system of seminorms corresponding to the spaces V_n, and q is any other G-continuous seminorm, then there is an n with $q \ll q_n$, hence a continuous map of Banach completions $V_n \to V_q$. On contragredients, this induces $\tilde{V}_q \to \tilde{V}_n \to \tilde{V}$. Thus, for another system of seminorms q'_n with completions V'_n, we have continuous maps between the spaces $\lim_{\rightarrow} \tilde{V}_n$ and $\lim_{\rightarrow} \tilde{V}'_n$ in both directions, which are clearly inverse to each other. □

[To be added: a discussion of the concept of *distinction* of a Fréchet space, i.e., the property that the inductive limit dual coincides with the "strong dual" of uniform convergence on bounded sets. True for reflexive Fréchet spaces, which includes nuclear spaces and, more generally, limits of Hilbert spaces.]

2.7. Unitary representations

We continue assuming that the coefficient field is $\mathbb{C}$, and that the topological group G is locally compact.

Definition 2.7.1. A *unitary* representation of G is a representation of G on a Hilbert space[3] V (over $\mathbb{C}$) which preserves the norm (i.e. π has image in the subgroup of unitary transformations, $U(V) \subset \mathrm{Aut}(V)$).

A representation (π, V) of G on a topological vector space V is *unitarizable* if V admits a (continuous, positive definite) inner product such that the corresponding Hilbert space completion is unitary.

If V is a Hilbert space, the algebra $B(V)$ of bounded operators on V is a C^*-algebra, with the hermitian adjoint corresponding to the $*$-operation. Before we proceed, let us recall the relevant definitions.

Definition 2.7.2. A *C^*-algebra* is a Banach *-algebra, that is, a Banach algebra with a conjugate-linear anti-involution $T \mapsto T^*$, which has the property that $\|T^*T\| = \|T\|^2$. A *morphism of C^*-algebras* is a continuous algebra map that commutes with the $*$ operation. (Note that we are not requiring the norm be preserved; it is only used for the continuity.)

A *representation of a C^*-algebra* (or, more generally, of a Banach *-algebra; sometimes called a *-representation for emphasis) A on a Hilbert space V is a morphism of C^*-algebras $\pi : A \to B(V)$. Morphisms of representations are morphisms of Hilbert spaces commuting with the action of A. The representation is called *nondegenerate* if the closure of the image, $\overline{\pi(A)V}$, is equal to V, and *irreducible* if the only closed, invariant subspaces are 0 and V.

[3] We will assume throughout that Hilbert spaces are separable.

C^*-algebras play an important role in the analysis of unitary representations of a group. The map $M_c(G) \to B(V)$ defines, by pullback, a seminorm on $M_c(G)$. Moreover, this is a *-morphism, with the $*$-operation on $M_c(G)$ given by:

$$\mu^*(g) = \overline{\mu(g^{-1})}. \tag{2.7.2.1}$$

Assume that G is a locally compact group, with right Haar measure dg. The restriction of all those seminorms to $C_c(G)dg \subset M_c(G)$ defines a norm:

$$\|fdg\|_{C^*} = \sup_{(\pi,V)} \|\pi(fdg)\|,$$

where (π, V) ranges over all unitary representations of G.

Considering just the right regular representation R of G on $L^2(G, dg)$, we obtain another norm

$$\|fdg\|_{C_r^*} = \|R(fdg)\|.$$

Definition 2.7.3. The completion of $C_c(G)dg$ with respect to the norm $\|\bullet\|_{C^*}$ is the *C^*-algebra of G*. Its completion with respect to $\|\bullet\|_{C_r^*}$ is the *reduced C^*-algebra of G*.

Proposition 2.7.4. *Let G be a locally compact group. Given a unitary representation π of G, the restriction to $C_c(G)dg$ of the induced action of $M_c(G)$ (Proposition 2.3.3) extends continuously to a representation of $C^*(G)$, and gives rise to an equivalence of categories between unitary representations of G, on one hand, and non-degenerate representations of $C^*(G)$, on the other.*

Proof. Let π be a unitary representation of G, and denote by the same symbol the induced action of $M_c(G)$, and its restriction to $C_c(G)dg$. Notice that, by the definition of the norm $\|\bullet\|_{C^*}$, we must have $\|\pi(\mu)\| \le \|\mu\|$ for any $\mu \in C_c(G)dg$. Since V is complete and Hausdorff, it follows that we get a unique continuous extension of π to $C^*(G)$, which is immediately seen to be a *-representation. Choosing an approximation of the identity $(\mu_n)_n \in C_c(G)dg$, by the weak-* convergence statement of Proposition 2.3.3, we see that $\pi(\mu_n)v \to v$ for every $v \in V$, therefore the representation is nondegenerate.

To show the reverse direction, we start with a nondegenerate representation (π, V) of $C^*(G)$, and let $V' = \pi(C_c(G)dg)V$, a dense subspace of V. Choose an approximation $(\mu_n)_n \in C_c(G)dg$ of the identity, and let δ_g denote the delta measure at $g \in G$. Then, for every $g \in G$ and $v \in V'$, the limit $\pi(g) := \lim_n \pi(\mu_n \star \delta_g)v$ exists, since v can be written as $\pi(f)v'$, and $\pi(\mu_n \star \delta_g)\pi(f) = \pi(\mu_n \star \delta_g \star f)$, and $\mu_n \star \delta_g \star f \to \delta_g \star f$ in $L^1(G)$, hence also in $C^*(G)$. Since $\|\pi(\mu_n \star \delta_g)\| \le \|\mu_n \star \delta_g\|_{L^1(G)} = 1$, the resulting operator $\pi(g)$ on V' has bounded norm, hence extends to an element of $B(V)$. Finally, for any fixed n and any net $g_i \to g \in G$, we have that $\mu_n \star \delta_{g_i} \to \mu_n \star \delta_g$ in $L^1(G)$, and this shows continuity of the resulting map $G \times V \to V$.

We have $(\mu_n \star \delta_g)^* = \delta_{g^{-1}} \star \mu_n^*$, which is an approximation of $\delta_{g^{-1}}$, and this shows that $\pi(g^{-1}) = \pi(g)^*$, i.e., the representation is unitary. It is easy to see that we have constructed two inverse functors between unitary representations of G and nondegenerate representations of $C^*(G)$, and that they are faithful, hence give rise to an equivalence of categories. $\square$

2.8. The Plancherel decomposition

In this section, we collect results from the theory of C^*-algebras (which, in turn, rely on results on the theory of Von Neumann-, or W^*-, algebras). We point the reader to [**Dix77**] for complete definitions and proofs.

Definition 2.8.1. The *spectrum* $\hat{A}$ of a C^*-algebra A is the set of isomorphism classes of irreducible representations of A. The *unitary dual* $\hat{G}$ a locally compact group G is the spectrum of $C^*(G)$, or equivalently (by Proposition 2.7.4), the set of isomorphism classes of its irreducible unitary representations.

Definition 2.8.2. A linear form ϕ on a C^*-algebra A is called *positive* if $\phi(aa^*) \geq 0$ for all $a \in A$. A *state* of A is a positive linear functional ϕ such that $\|\phi\| = 1$.

For a representation (π, V) of a C^*-algebra A, the diagonal matrix coefficients $\omega_v : A \ni a \mapsto \langle \pi(a)v, v\rangle$ (where $v \in V$) are called the *positive forms associated to the representation*, and the *states associated to the representation* if $\|\omega_v\| = 1$.

States can be used to describe a topology on the spectrum:

Definition 2.8.3. Let π be a representation of a C^*-algebra A, and S a set of representations of A. We say that π is *weakly contained* in S if every positive form (or every state) associated to S is a weak-* limit of linear combinations of positive forms associated to elements of S.

We endow the spectrum $\hat{A}$ with the topology where $\pi \subset \overline{S} \iff \pi$ is weakly contained in S. In the case of $\hat{A} = \hat{G}$, the unitary dual of a locally compact group G, this is called the *Fell topology*, and convergence $\pi_i \to \pi$ is equivalently described by the requirement that, for all $v \in \pi$, there exists a collection $(v_i)_i, v_i \in \pi_i$ such that

$$\langle v_i, \pi_i(g)v_i\rangle \ \to \ \langle v, \pi(g)v\rangle \,,$$

uniformly on compact subsets of G.

For the equivalence of this definition with other definitions of the topology on $\hat{A}$, see [**Dix77**, Theorem 3.4.10]. For this description of the Fell topology, notice that it is enough to check weak-* convergence of bounded functionals on $C^*(G)$ on elements of the dense subspace $L^1(G)$, and if these are positive-definite functionals represented by continuous functions ϕ_i with $\phi_i(1) = 1$, by [**Dix77**, Theorem 13.5.2] weak-* convergence is equivalent to locally uniform convergence of the ϕ_i's.

The Fell topology is not a very nice topology; for example, it is not Hausdorff, in general. In any case, notice, for later use, that it allows us to view $\hat{G}$ as a Borel space.

We will now define direct integrals of Hilbert spaces. The rough idea is to bring these spaces together as a bundle sitting over some parameter space, and then to take a collection of measurable sections of this bundle. Of course, in order to discuss measurability, we will need some assumptions.

Definition 2.8.4. Let X be a Borel space with a positive measure μ. For each point $x \in X$, let V_x be a Hilbert space. A section will refer to a collection of elements $(v_x)_{x\in X}$, such that $v_x \in V_x$. Assume we are given a subset F of all of the sections, which we will call the *measurable sections*. F is required to have two properties:

(1) A section $(v_x)_x$ is in F iff $\langle v_x, u_x\rangle$ is a measurable function for every section $(u_x)_x \in F$.

(2) There is a countable subset of F so that their restrictions to each V_x are dense.

Two measurable sections are considered equivalent if they are the same almost everywhere.

On the set of equivalence classes of measurable sections, define the inner product

$$\langle u, v\rangle = \int_X \langle u_x, v_x\rangle \, d\mu(x).$$

The *direct integral* $\int_X^{\oplus} V_x d\mu(x)$ is the Hilbert space of equivalence classes of measurable square-integrable sections.

When referring to a direct integral of Hilbert spaces, we will often suppress F in the notation and leave it implicit that such an F has been chosen.

Example 2.8.5. Assume that we may partition $X = \bigsqcup_{n=1}^{\omega} X_n$ as a union of measurable subsets, such that if $x \in X_n$, then we have an identification of V_x with the standard complex Hilbert space of dimension n. Then, a natural choice of F is to let a section be measurable iff its restriction to each X_n is measurable.

Definition 2.8.6. Continue to let V be the direct integral of the Hilbert spaces $(V_x)_{x\in X}$. Assume we are given a family of operators $(T_x)_{x\in X}$ that essentially bounded norms (that is, bounded except possibly on a set of measure zero). Furthermore, assume this family is measurable in the sense that if $(v_x)_x$ is a measurable section then so is the pointwise image $(T_x v_x)_x$. Then, we form the *direct integral of operators*, which will be a bounded linear operator that acts on the direct integral Hilbert space. In particular, let the action be defined pointwise:

$$\left(\int_x^{\oplus} T_x d\mu(x)\right)\left(\int_x^{\oplus} v_x d\mu(x)\right) = \int_x^{\oplus} T_x(v_x) d\mu(x)$$

The measurability of the family $(T_x)_x$ ensures that we get a well defined action on V. The linearity and boundedness then follow easily from the definitions.

Remark 2.8.7. In the context of Definition (2.8.4), if $\mu' = f\cdot\mu$ is another measure, where f is a measurable function that is μ-almost-everywhere positive, there is a canonical isomorphism $\int_X^{\oplus} V_x d\mu'(x) \simeq \int_X^{\oplus} V_x d\mu(x)$ given by the operators $T_x = \sqrt{f(x)}I$. We will say that μ and μ' are *equivalent measures*.

Definition 2.8.6 will be applied to families of representations, as follows: Let A be a C^*-algebra and let $((\pi_x, V_x))_{x\in X}$ be a family of representations of A. Assume that the Hilbert spaces $(V_x)_x$ are endowed with a measurable structure as in Definition 2.8.4, and that for every element $a \in A$ the family of operators $(\pi_x(a))_{x\in X}$ is measurable. In this case, we can form the *direct integral representation* (π, V). It is given by

$$V = \int_X^{\oplus} V_x d\mu(x) \qquad \pi(a) = \int_X^{\oplus} \pi_x(a) d\mu(x)$$

The only concern with the above definition is whether the family $(\pi_x(a))_x$ is essentially bounded in norm. The following lemma assures us that this is the case.

Lemma 2.8.8. *Let A and B be C^*-algebras, and $\pi : A \to B$ a morphism. Then, for all $a \in A$, $\|\pi(a)\| \leq \|a\|$.*

Proof. See [Dix77, 1.3.7]. Although we will not need so much generality, we note that this lemma still holds when A is merely an involutive Banach algebra. $\square$

We now come to the Plancherel decomposition for unitary representations. Although fairly general, one must place an important restriction on the group, that $C^*(G)$ be *postliminal*. Defining postliminal would take us a bit afield, so instead we will state as fact that as long as G is a reductive group, then $C^*(G)$ is postliminal. For our purposes, a reductive group means any group of two types. The first type is a reductive real Lie group. For the second type, let F be a p-adic field. Then, G can be the group of F points of a reductive algebraic group over F. See [**Dix77**, 18.9.4], [**Dix77**, 4.7.11] for how to show reductive implies liminal, and [**Dix77**, 4.2.4] for liminal implies postliminal.

Proposition 2.8.9. *If A is a postliminal C^*-algebra, there is a decomposition $\hat{A} = \bigsqcup_{n=1}^{\omega} \hat{A}_n$ and, for every $\pi \in A_n$, an identification of the standard n-dimensional Hilbert space with the Hilbert space of π, so that, with respect to the standard measurable structure of Example 2.8.5, the sections of operators $A \ni \pi \mapsto \pi(a)$ are measurable.*

Proof. See [**Dix77**, Lemma 8.6.2]. $\square$

Theorem 2.8.10 (Plancherel Decomposition)**.** *Let G be a reductive, separable, locally compact group, and (ρ, V) a unitary representation of G. Then, on the unitary dual $\hat{G}$, there exists a mutually singular*[4] *family of measures $(\mu_n)_{1\leq n\leq\omega}$ such that we can form a decomposition*

$$V \simeq \bigoplus_{n=1}^{\omega} \int_{\pi\in\hat{G}}^{\oplus} \pi^n d\mu_n(\pi) \tag{2.8.10.1}$$

This decomposition is unique in the sense that if another such family of measures $(\mu'_n)_x$ exists, then μ_n and μ'_n are equivalent for all n. (See Remark 2.8.7.)

Proof. Consider [**Dix77**, 8.6.6] in light of [**Dix77**, 18.7.6]. $\square$

Example 2.8.11. In the case that G is abelian, all of the irreducible representations are 1-dimensional, so $\hat{G}$ can be identified with the set of unitary characters of G. Letting $\mathbb{C}_\chi$ be the one dimensional representation where the group acts by χ, we see that the Plancherel Decomposition just becomes "simultaneous diagonalization":

$$V = \bigoplus_{n=1}^{\omega} \int_{\chi\in\hat{G}}^{\oplus} \mathbb{C}_\chi^n d\mu_n,$$

where $\mathbb{C}_\chi^n$ denotes an n-dimensional Hilbert space where the group acts by χ.

In practice, the isomorphism (2.8.10.1) of the Plancherel decomposition is often described by identifying the integrands π^n as Hilbert completions of certain dense subspaces of V. [Also related to the construction of the bundle of representations in Proposition 2.8.9 — to be added.] For this, we need the following notion:

[4] Two measures μ_1 and μ_2 on a space X are mutually singular if we can partition $X = X_1 \cup X_2$, so that μ_1 is 0 on subsets of X_2, and μ_2 is 0 on subsets of X_1. This may be symbolically denoted by $\mu_1 \perp \mu_2$.

Definition 2.8.12. Let V be the direct integral of the Hilbert spaces $(V_x)_{x\in X}$, and let U be a topological vector space. A morphism $T: U \to V$ is called *pointwise defined* if there exists a family of morphisms $T_x : U \to V_x$ such that for each $u \in U$, the section $(T_x(u))_{x\in X}$ is equivalent to $T(u)$.

A basic example is the following:

Proposition 2.8.13. *Let G be a Lie or p-adic group, and $X = H\backslash G$ a homogeneous space for G. Assume, for simplicity, that X has a G-invariant measure dx, that we fix, or else replace functions by half-densities in what follows. Let $S = C_c^\infty(X)$, with its natural topology. Then, for any direct integral decomposition*

$$L^2(X) = \int_{\hat{G}} V_\pi \mu(\pi),$$

the embedding $S \to L^2(X)$ is pointwise defined.

Proof. See [**Ber88**, 2.3] for the "natural topology" and the proof. □

Since S is also dense in $L^2(X)$, and by assumption (Definition 2.8.4) the sections corresponding to elements of $L^2(X)$ are dense in (almost) every V_π, this means that the image of $T_\pi : S \to V_\pi$ is dense for (almost) every π. Hence, V_π is equal to the *Hilbert space completion* of S with respect to the seminorm $\| \bullet \|_\pi$ on S pulled back from V_π. Since $V_\pi \simeq \pi^n$ for some $n \leq \omega$, this seminorm factors through the π-*coinvariants*, or *maximal* π-*isotypic quotient* of S, that is, the quotient S_π : $S/\bigcap_l \mathrm{Ker}(l)$, where l ranges over all morphisms $S \to \pi$. If $(H_\pi)_\pi$ denotes the corresponding family of Hermitian forms on S, the product $H_\pi\mu(\pi)$ can be thought of as a measure valued in positive semi-definite Hermitian forms on S, with the property that for (almost) every π, the corresponding form factors through S_π. In the context of Theorem 2.8.10, this Hermitian-form-valued-measure on $\hat{G}$ is unique, and specifying this measure is equivalent to describing the Plancherel decomposition (2.8.10.1).

2.9. Other chapters

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CHAPTER 3

Representations of compact groups

The (complex) representation theory of compact groups, in a nutshell:

(1) All representations are unitarizable, by averaging an inner product against Haar measure.
(2) All unitary representations decompose into (Hilbert space, i.e., orthogonal and completed) direct sums of finite-dimensional irreducibles (the Peter–Weyl theorem), because operators of convolution by continuous measures are Hilbert–Schmidt (hence compact).

In this chapter, G (or H) is a compact group, and we fix throughout the probability Haar measure dg. Multiplication of functions by dg turns them into measures which can act on the space of a representation π, and we feel free to write $\pi(f)$ for $\pi(f dg)$.

3.1. Unitarity

Proposition 3.1.1. *Let (π, V) be a representation of G on a space admitting an inner product (positive definite hermitian form). Then, it is unitarizable.*

Proof. Take any positive definite hermitian form $\langle\, ,\, \rangle'$, and integrate it over the action of the group in order to make it invariant:

$$\langle v, v\rangle := \int_G \langle \pi(g)v, \pi(g)v\rangle' \, dg.$$

$\square$

3.2. Hilbert–Schmidt property of continuous convolution operators

Given a Hilbert space V, its linear dual V^* is identified with its complex conjugate $\bar{V}$. We denote by $B(V) = \mathrm{End}(V)$ the space of bounded linear operators on V, and by $\hat{\otimes}$ the Hilbert space tensor product of two Hilbert spaces, i.e., the completion of the algebraic tensor product with respect to the Hilbert norm characterized by $\|v \otimes w\| = \|v\| \cdot \|w\|$.

There is a natural embedding $\bar{V}\hat{\otimes}V \hookrightarrow B(V)$, with $\bar{v}_1 \otimes v_2$ mapping to the operator $w \mapsto \langle v_1, w\rangle \cdot v_2$. (Let us take inner products to be complex-linear in the second variable.) The image is the space of *Hilbert–Schmidt operators*, and the Hilbert norm on $\bar{V}\hat{\otimes}V$ is called the Hilbert–Schmidt norm of an operator. Explicitly:

$$\|T\|_{HS}^2 = \sum_i \|Te_i\|^2,$$

where e_i runs over an orthonormal basis of V.

Hilbert–Schmidt operators are *compact*: they map bounded sets to precompact sets.

Proposition 3.2.1. *Let L (resp. R) denote the left (resp. right) regular representation of G on $L^2(G)$. For every continuous measure μ on G the operator $L(\mu)$ (resp. $R(\mu)$) is Hilbert–Schmidt and, hence, compact.*

Recall that we call a measure "continuous" if it is the product of a continuous function by Haar measure, see Section 2.3.

Proof. Let $\mu = hdg$. Then the operator $L(\mu)$ has the integral expression:

$$L(\mu)(f)(x) = \int_H K_h(x,y)f(y)dy,$$

where the kernel is given by:

$$K_h(x,y) = h(xy^{-1}).$$

The Hilbert–Schmidt norm of an integral operator T with kernel K on a measure space (X, dx) is given by:

$$\|T\|^2_{HS} = \|K\|^2_{L^2(X\times X)}.$$

In particular, $L(\mu)$ is Hilbert–Schmidt. (It was important here that the group was compact for the L^2-norm of K_h to be finite.) $\square$

3.3. Recollection of spectral theorems

We recall the following spectral theorems from functional analysis.

Let V be a Hilbert space. The *adjoint* of a (bounded) operator T on V is the operator T^* with $\langle T^*v, w\rangle = \langle v, Tw\rangle$. An operator T is called *normal* if $T^*T = TT^*$, and *self-adjoint* if $T = T^*$. The *spectrum* $\sigma(T)$ of an operator T is the (closed) subset of all $\lambda \in \mathbb{C}$ such that $T - \lambda I$ is not invertible.

The idea of the spectral theorem is that the space V decomposes as an "integral" of "eigenspaces" of T. A familiar case of an integral of eigenspaces is when $V = L^2(\mathbb{R})$, in which case the theory of Fourier transform says:

$$V = \int_{i\mathbb{R}} \langle e^{sx}\rangle \, ds,$$

with the spaces $\langle e^{sx}\rangle$ being eigenspaces for all translation operators.

Theorem 3.3.1 (Spectral theorem for normal operators)**.** *If T is a normal operator on a Hilbert space V then there is a measure space $(X, \mathcal{B}, \mu)$, a measurable function $\lambda : X \to \mathbb{C}$ and a unitary isomorphism: $V \simeq L^2(X, \mu)$ which carries the operator T to "multiplication by λ".*

Moreover, for every measurable $\omega \subset \mathbb{C}$, let $E(\omega)$ be the projection (restriction) $L^2(X, \mu) \to L^2(\lambda^{-1}(\omega), \mu)$. Then, $E(\sigma(T)) = I$, $E(\omega) \neq 0$ for every relatively open nonempty subset of $\sigma(T)$, and $E(\omega)$ commutes with the commutator of T in $B(V)$.

Proof. See [**Rud91**, Theorem 12.23]. The measure space (X, μ) in our formulation is not canonical, but can be obtained from the more canonical formulation of *loc.cit.* by considering subspaces of V of the form $\overline{\mathbb{C}[T]v}$, for $v \in V$. $\square$

An operator on a Banach space is *compact* if it maps bounded sets to precompact sets (i.e. sets whose closure is compact). Compact operators form a closed Banach subalgebra $K(V)$ of the algebra of bounded operators $B(V)$, and for a large class of Banach spaces, that includes Hilbert spaces, $K(V)$ is the closure of the set $V^* \otimes V$ of operators with finite-dimensional range, i.e., compact operators are those that can be approximated in the operator norm by operators with finite-dimensional range.

Theorem 3.3.2 (Spectral theorem for compact self-adjoint operators)**.** *Let T be a compact self-adjoint operator on a Hilbert space V, then there is a sequence of eigenvectors v_n with (real) eigenvalues $0 \neq \lambda_n \to 0$ (or a finite number of such eigenvalues, if the operator is of finite range) such that*

$$V = \ker T \oplus \hat{\bigoplus_n} \langle v_n \rangle ,$$

a Hilbert space (i.e., orthogonal and completed) direct sum.

Proof. See [**Rud91**, Theorem 12.30]. □

In particular, all eigenspaces with nonzero eigenvalues are finite-dimensional.

Theorem 3.3.3 (Schur's lemma)**.** *If V is an irreducible unitary representation of any group G, and $S \in End^G(V)$, then S is a scalar multiple of the identity.*

If V, V' are two irreducible unitary representations of a group G, and $T \in \mathrm{Hom}^G(V, V')$, then T is a scalar multiple of an isometry.

Proof. For the first claim, by unitarity, the Hilbert space adjoint S^* of S also commutes with G; hence, we may assume that S is self-adjoint, by replacing it with $S + S^*$ and $i(S - S^*)$. Then, we claim that the spectrum $\sigma(S)$ is a singleton, making S a scalar operator by the Spectral Theorem 3.3.1. Otherwise, by the same theorem, for two non-empty, disjoint open subsets of $\sigma(S)$, the corresponding projections $E(\omega_1)$ and $E(\omega_2)$ are non-zero, and have orthogonal images. But these projections commute with the action of G, which commutes with S, hence V cannot be irreducible.

For the second claim, it is enough to show that the self-adjoint bounded operators T^*T and TT^* are scalars. Again, by unitarity, the Hilbert space adjoint T^* of T also commutes with G. Let S be one of these operators, and apply the first claim. □

3.4. Peter–Weyl theorems

Let H be a compact group, and consider the space $V = L^2(H)$. It is a unitary representation for $G = H \times H$.

Let (π, V) be a finite-dimensional representation of H, and consider the matrix coefficient map (see 2.4)

$$M_\pi : \pi^* \otimes \pi \to C(H).$$

Lemma 3.4.1. *For π irreducible, the matrix coefficient map is an embedding. For π, σ irreducible and non-isomorphic, the images of their matrix coefficient maps are orthogonal in $L^2(H)$.*

Proof. The matrix coefficient map is an embedding (injection), because it is clearly non-zero, and $\pi^* \otimes \pi$ is an irreducible representation of $G = H \times H$.

If π, σ are irreducible and non-isomorphic, the orthogonal projection from the image of M_π to the image of M_σ is G-equivariant, and since $\pi^* \otimes \pi$ is not isomorphic to $\sigma^* \otimes \sigma$ (already as an $1 \times H$-representation), it has to be zero. □

Notice that the image of the matrix coefficient map consists of *(left and right) finite* vectors (since $\pi \otimes \pi^*$ is finite dimensional). Our goal is to prove:

Theorem 3.4.2 (Peter–Weyl theorem)**.** *The matrix coefficient maps give rise to a canonical isomorphism*

$$L^2(H) \simeq \hat{\bigoplus} \pi^* \otimes \pi \tag{3.4.2.1}$$

(orthogonal, completed direct sum), where π runs over representatives for the isomorphism classes of irreducible, finite dimensional representations of H.

Proof. We start with the following assertions:

- Let μ_n be a sequence of positive probability measures on G, supported on a fixed compact neighborhood of the identity, which converge to δ_1 in the weak-* topology of $M_c(G)$; we will refer to such a sequence as an approximation of the identity. Then, for any Banach representation (π, V) and vector $v \in V$ we have $\pi(\mu_n)v \to v$.

  This follows from Proposition 2.5.4.
- For any subrepresentation V of $L^2(H)$ under the right (or left) regular action, continuous functions are dense in V.

  Indeed, it is enough to choose an approximation $(\mu_n)_n$ of the identity by continuous measures (i.e., continuous functions times a Haar measure). By the above, $L(\mu_n)(f) \to f$ for every $f \in V$, but $L(\mu_n)(f)$ is simply the convolution $\mu_n \star f$, which is continuous. Thus, continuous functions are dense.
- Right-finite (or left-finite) functions are dense in any closed, invariant subspace of $L^2(H)$.

  This is the most important step of the proof. Assume to the contrary that there is a non-zero closed subspace V without a dense subspace of right-finite functions, which (by taking orthogonal complement of the subspace of right-finite functions) reduces to the case where V does not have any right-finite vectors. We can find a continuous, self-adjoint measure μ on H such that $L(\mu)V \neq 0$. Here, by self-adjoint we mean that the operator $L(\mu)$ is self-adjoint, which is equivalent to $h(g^{-1}) = \overline{h(g)}$ if $\mu = hdg$ — exercise! The existence of such a measure follows by approximating the identity by positive, continuous self-adjoint measures μ_n, and then using the fact that $\mu_n(v) \to v$ for every vector v. We know (Proposition 3.2.1) that $L(\mu)$ is compact, hence by the Spectral Theorem 3.3.2 there is a non-zero (real) eigenvalue λ of $L(\mu)$, and the λ-eigenspace is finite-dimensional. But the λ-eigenspace for $L(\mu)$ is stable under the right action of H, hence there are right-finite vectors, a contradiction.

Now let π be a finite-dimensional irreducible representation of H; by the previous point, such representations exist. We have a tautological map $T : \mathrm{Hom}^H(\pi, L^2(H)) \otimes \pi \to L^2(H)$, where we only consider the right regular representation. Moreover, the image of T lies in the subspace $C(H)$ of continuous functions. If we endow the space $\mathrm{Hom}^H(\pi, L^2(H))$ with the action induced from the left regular representation of H on $L^2(H)$, the map T is equivariant. Evaluation at the identity defines a morphism $\mathrm{Hom}(\pi, L^2(H)) \to \pi^*$, whose kernel is (tautologically) trivial. We conclude that the π-isotypic component of $L^2(H)$ under the right representation (the image of T) is isomorphic to $\pi^* \otimes \pi$. Those subspaces, as the isomorphism class of π varies, are mutually orthogonal by Lemma 3.4.1, and they span a dense subspace, by the points above. This proves the L^2-part of the theorem.

□

Remark 3.4.3. If π is endowed with an invariant Hilbert norm, so $\pi^* = \bar{\pi}$, the decomposition (3.4.2.1) is not an isometry. This will be the subject of the Plancherel formula, Theorem 3.5.1.

Theorem 3.4.4. *Every Fréchet representation (π, V) of H contains a dense subspace of finite vectors. In particular, every irreducible Fréchet represenation is finite dimensional. Every Hilbert representation of H is the Hilbert space direct sum of irreducibles.*

Proof. Let $f_n dh$ be an approximation of the identity by positive, continuous probability measures. Then, given a convex neighborhood $U \subset V$ of zero, and a vector $v \in V$, we have $\pi(f_n dh) \in v + \frac{1}{2}U$, for large n, by Proposition 2.3.3.

Now fix such a large n, and choose a sequence h_j of finite functions such that $h_j \to f_n$ in $L^2(H)$. In particular, $h_j \to f_n$ in $L^1(H)$, and the measures $h_j dh$ converge strongly to $f_n dh$. Again by Proposition 2.3.3, we have $\pi(h_j dh)(v) \to \pi(f_n dh)(v)$, hence $\pi(h_j dh)(v) \in v + U$, for large j.

But, for a vector $v \in V$, $\pi(\mu_n)v$ is finite since μ_n is left-finite. This proves the first claim, and the others follow easily.

□

Proposition 3.4.5. *Assume that H is a compact Lie group (or just a compact group, ignoring mentions of smooth vectors below). We have a sequence of dense inclusions of Fréchet spaces:*

$$L^2(H)_{fin} \subset C^\infty(H) \subset C(H) \subset L^2(H), \tag{3.4.5.1}$$

where $_{fin}$ denotes left and right finite functions.

Proof. If H is a Lie group, we can show as in the proof of Theorem 3.4.2, by choosing a smooth approximation of the identity, that any subrepresentation of $L^2(H)$ contains a dense subspace of smooth vectors, and that $L^2(H)_{\mathrm{fin}}$ belongs to the space of smooth functions.

We then apply Theorem 3.4.4 to any of these Fréchet spaces, viewed as a representation of the group $H \times H$. □

3.5. The Plancherel formula

The Plancherel formula expresses a function $f \in L^2(H)$ (using probability Haar measure for the L^2-norms throughout) in terms of its *spectral transforms*, which have to be defined as explicit projections onto the summands $\pi^* \otimes \pi$ of the Peter–Weyl theorem. Of course, f decomposes as a convergent sum of its orthogonal projections to those subspaces, but these are not the most natural projections to consider in practice. Instead, the "natural" projection is the map

$$f \mapsto \pi(f dh) \in \mathrm{End}(\pi) = \pi^* \otimes \pi$$

(where dh is the probability Haar measure).

Hence, the content of the Plancherel formula is the comparison of these "natural" projections with the orthogonal ones.

Theorem 3.5.1. *For any $f \in L^2(H)$, we have*

$$\|f\|^2 = \sum_\pi \|\pi(f dh)\|_{HS}^2 \cdot d(\pi), \tag{3.5.1.1}$$

where π ranges over all isomorphism classes of irreducible unitary representations of H, $d(\pi)$ denotes the dimension of π, and $\|\cdot\|_{HS}$ denotes the Hilbert–Schmidt norm on End(π), i.e., the Hilbert norm on $\pi^* \otimes \pi$.

If $f \in C(H)$ can be written as the convolution of two functions in $L^2(H)$ (or a linear combination thereof), then we have

$$f(1) = \sum_\pi tr\,(\pi(fdh)) \cdot d(\pi), \tag{3.5.1.2}$$

with the right hand side being absolutely convergent.

Proof. Assume that $f \in L^2(H)$. The right regular action $R(fdh)$ is represented by the kernel $K_f(x,y) = f(x^{-1},y)$, i.e.,

$$R(fdh)\Phi(x) = \int \Phi(y)f(x^{-1}y)dy.$$

Calculating the Hilbert–Schmidt norm of this operator we get, on one hand,

$$\|R(fdh)\|^2_{HS} = \|K_f\|^2_{L^2(H\times H)} = \|f\|^2,$$

and on the other, by the Peter–Weyl theorem 3.4.2,

$$\|R(fdh)\|^2_{HS} = \sum_\pi \dim(\pi^*)\|\pi(fdh)\|^2_{HS} = \sum_\pi d(\pi)\|\pi(fdh)\|^2_{HS}.$$

This proves (3.5.1.1).

Now let f be the convolution of two L^2-functions: $f = f_1 \star f_2$. Then, $f(1) = \langle f_1^*, f_2\rangle$, where $f_1^*(g) = \overline{f_1(g^{-1})}$, and $\star$ denotes convolution (=pushforward by the multiplication map $H \times H \to H$). (We take the linear factor to be the second one in the hermitian inner products.)

By the Plancherel formula just proven, we have (with all sums absolutely convergent)

$$\langle f_1^*, f_2\rangle = \sum_\pi d(\pi)\,\langle \pi(f_1^* dh), \pi(f_2 dh)\rangle_{HS} =$$

$$= \sum_\pi d(\pi)\mathrm{tr}\,(\pi(f_1dh)\circ\pi(f_2dh)) = \sum_\pi d(\pi)\mathrm{tr}\,(\pi(fdh))\,,$$

where we have used the fact that, for two Hilbert–Schmidt operators T_1, T_2, we have $\mathrm{tr}(T_1 \circ T_2) = \langle T_1^*, T_2\rangle_{HS}$. □

Remark 3.5.2. If $f = f_1 \star f_2$, so $R(f) = R(f_1)\circ R(f_2)$, the operator $R(f)$, being the composition of two Hilbert–Schmidt operators, is a *trace class operator*, and its trace can be computed in any orthogonormal basis of the Hilbert space.

Hilbert–Schmidt and trace class operators form ideals in the algebra $B(V)$ of bounded operators on a Hilbert space V, let us denote them by $B(V)_2$ and $B(V)_1$, respectively. We have inclusions

$$B(V)_1 \subset B(V)_2 \subset K(V) \subset B(V),$$

where $K(V)$ is the subspace of compact operators. The first two inclusions are dense, and the last one is closed. The space of trace class operators admits a norm, under which

$$K(V)^* = B(V)_1,$$
$$B(V)_1^* = B(V).$$

The first equality, restricted to the dense subspace of Hilbert–Schmidt operators in $K(V)$, can be taken as the definition of trace class operators, i.e., an operator T is trace class if

$$\sup_{S \in B(V)_2, \|S\| \leq 1} |\langle T, S \rangle_{HS}| \leq \infty,$$

where $\|S\|$ is the *operator norm*, not the Hilbert–Schmidt norm of S, which ensures that the Hilbert–Schmidt pairing extends to a pairing between $B(V)_1$ and the space of compact operators. It can be shown that trace class operators are precisely the compositions of two Hilbert–Schmidt operators.

In practice, there are many situations where a continuous function can be shown to be a convolution of two functions. For example, it is a corollary of the *Dixmier–Malliavin theorem* that if G is a compact Lie group then every smooth function of G is a convolution of two smooth functions, hence the pointwise Plancherel formula applies to smooth functions.

On the other hand, pointwise convergence for general continuous functions fails, even for the example of Fourier series when $H = S^1$!

3.6. Example: Spherical harmonics

We finish this chapter with a classical example, the decomposition of the space $L^2(S^n)$ into irreducibles for the action of $\mathrm{SO}(n+1)$. Here, S^n is the unit sphere in Euclidean space $\mathbb{R}^{n+1}$, and $G = \mathrm{SO}(n+1)$ is the special orthogonal group of length-preserving linear transformations of determinant $+1$. We follow the notes [**Gal**].

Here, we work with real-valued functions and real Hilbert spaces; the translation to complex-valued functions is immediate, after tensoring by $\mathbb{C}$. We let G act on the right on $\mathbb{R}^{n+1}$, and by fixing a base point we have an isomorphism $S^n = \mathrm{SO}(n)\backslash\mathrm{SO}(n+1)$, which proves (inductively) that $\mathrm{SO}(n)$ is compact.

Lemma 3.6.1. *Restrictions of polynomials on $\mathbb{R}^{n+1}$ are dense in $L^2(S^n)$.*

Proof. Apply the Stone–Weierstrass theorem. □

The restriction map $\mathbb{R}[\mathbb{R}^{n+1}] \to L^2(S^n)$ is not injective. For example, on $S^1 = \exp(i\mathbb{R})$ we have $x^2 + y^2 = 1$.

One proves that it is enough to restrict to *harmonic polynomials*, that is, eigenvalues for the Laplacian Δ on $\mathbb{R}^{n+1}$. The proof goes as follows: We define an isomorphism $P \mapsto \partial(P)$ between the algebra of polynomials and the algebra of linear differential operators with constant coefficients on $V = \mathbb{R}^{n+1}$, given in an orthonormal set of coordinates $(x_i)_i$ by $x_i \mapsto \frac{\partial}{\partial x_i}$. (Up to a scalar, this is simply the Fourier transform of a differential operator, when we use the inner product to identify the space V with its dual V^*.) Then one easily sees that the pairing

$$\langle P, Q \rangle = \partial(P)\overline{Q} \tag{3.6.1.1}$$

is an inner product on the space $\mathbb{R}[V]_k$ of homogeneous polynomials of degree k, or a hermitian inner product on $\mathbb{C}[V]_k$.

Lemma 3.6.2. *The Laplace operator*

$$\Delta : \mathbb{R}[V]_{k+2} \to \mathbb{R}[V]_k$$

is surjective for every $k \geq -1$ (setting $\mathbb{R}[V]_{-1} = 0$).

Proof. If $Q \in \mathbb{R}[V]_k$ is orthogonal to the image of Δ then for every $P \in \mathbb{R}[V]_{k+2}$ we have

$$0 = \langle Q, \Delta P \rangle = \left\langle \|x\|^2 Q, P \right\rangle$$

(by basic properties of Fourier transform), hence $\|x\|^2 Q = 0$, and therefore $Q = 0$. □

The same argument shows

Lemma 3.6.3. *If $H_{k+2} \subset \mathbb{R}[V]_{k+2}$ denotes the subspace of harmonic polynomials ($\Delta P = 0$), we have an orthogonal decomposition*

$$\mathbb{R}[V]_{k+2} = H_{k+2} \oplus \|x\|^2 R[V]_k.$$

Proof. Indeed, if $P \in H_{k+2}$ and $Q \in R[V]_k$, we have

$$\left\langle \|x\|^2 Q, P \right\rangle = \langle Q, \Delta P \rangle = 0,$$

which shows that the two subspaces are orthogonal. By 3.6.2, their dimensions are complementary; this proves the lemma. □

This shows:

Proposition 3.6.4. *We have an orthogonal decomposition*

$$R[V]_k = H_k \oplus \|x\|^2 H_{k-2} \oplus \|x\|^4 H_{k-4} \oplus \dots.$$

The restriction of every polynomial to S^n is equal to the restriction of a linear combination of harmonic polynomials.

Proof. The first statement follows by induction from Lemma 3.6.3, and the second because $\|x\| = 1$ on the unit sphere. □

Now, if Δ_S is the Laplacian on S^n, and $r = \|x\|$ is the radial coordinate, the Laplacian on $\mathbb{R}^{n+1}$ can be written

$$\Delta = \frac{1}{r^n} \frac{\partial}{\partial r} \left(r^n \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \Delta_S. \tag{3.6.4.1}$$

Hence,

Lemma 3.6.5. *If $P \in H_k$, then $P|_{S^n}$ is an eigenvector of the Laplacian Δ_S with eigenvalue $-k(n+k-1)$.*

Proof. We write $P(r,\theta) = r^k f(\theta)$, with θ the coordinate on S^n, and the result follows from (3.6.4.1). □

This leads to the main result:

Theorem 3.6.6. *If H_k denotes the space of harmonic polynomials, homogeneous of degree k, on $\mathbb{R}^{n+1}$, the restriction maps*

$$H_k \to L^2(S^n)$$

are injective (allowing us to identify H_k as a subspace of $L^2(S^n)$, and we have an orthogonal direct sum decomposition

$$L^2(S^n) = \hat{\bigoplus}_{k=0}^{\infty} H_k.$$

This is the decomposition of $L^2(S^n)$ into irreducible representations for the group $SO(n+1)$.

Proof. The restriction of a homogeneous polynomial to S^n determines the polynomial, therefore the maps $\mathbb{R}[\mathbb{R}]_k \to L^2(S^n)$ (and, a fortiori, their restriction to harmonic polynomials) are injective for every k.

The Laplacian Δ_S is a self-adjoint operator; therefore, its eigenspaces corresponding to distinct eigenvalues are mutually orthogonal. This applies to the spaces H_k, by Lemma 3.6.5.

By Lemma 3.6.1, restrictions of polynomials to S^n are dense, and by Proposition 3.6.4, those are the same as the restrictions of harmonic polynomials.

The group $G = \mathrm{SO}(n+1)$ preserves the metric on S^n, therefore its action on $C^\infty(S^n)$ commutes with the Laplacian. In particular, eigenspaces for Δ_S are stable under G.

[Proof of irreducibility to be added.] □

3.7. Other chapters

CHAPTER 4

Lie groups and Lie algebras: general properties

In this chapter we begin studying representations of compact, and non-compact, Lie and algebraic groups. The focus will be on the representation theory of *reductive algebraic groups* over $\mathbb{R}$, and over the p-adic numbers. All compact Lie groups are reductive algebraic, and most of the interesting non-compact Lie groups are such.

The study of *continuous representations* of *compact Lie groups* goes in parallel with the study of *algebraic representations* of their *complexifications*, and with *finite-dimensional representations* of their *Lie algebras*. We will introduce these topics a little more generally, in order to be able to use them later for *non-compact Lie (algebraic) groups* and their *infinite-dimensional representations*.

4.1. Lie groups, group schemes, algebraic groups

Definition 4.1.1. A *Lie group* is a group in the category of differentiable manifolds.

Remark 4.1.2. As a corollary of the Baker–Campbell–Hausdorff formula that we will prove later, any Lie group is automatically real-analytic. See Proposition 4.4.8. In many references it is defined from the outset as a group in the category of analytic manifolds.

Definition 4.1.3. A *group scheme* (over a base scheme S) is a group in the category of (S-)schemes.

If $S = \mathrm{Spec}(k)$, where k is a field in *characteristic zero*, then a k-group scheme is automatically *smooth* over k, see Theorem 4.6.7. This is not the case in positive characteristic, as the following example shows:

Example 4.1.4. Consider the (smooth) additive group scheme over $k = \mathbb{F}_p$:

$$\mathbb{G}_a = \mathrm{Spec} k[T]$$

with the obvious group structure. For instance, addition $\mathbb{G}_a \times \mathbb{G}_a \to \mathbb{G}_a$ is given by the morphism induced by:

$$k[T] \ni f(T) \mapsto f(T_1, T_2) \in k[T] \otimes_k k[T] = k[T_1, T_2].$$

Now consider the "Frobenius" homomorphism:

$$\begin{array}{rcl} \mathbb{G}_a & \to & \mathbb{G}_a \\ k[T] \ni f(T^p) & \leftarrow & f(T) \in k[T]. \end{array}$$

The *kernel* K of this homomorphism is, as a scheme, isomorphic to $k[T]/(T^p)$, with the embedding $K \to \mathbb{G}_a$ given by the quotient map:

$$k[T] \to k[T]/(T^p)$$

and the inherited addition morphism:

$$k[T]/(T^p) \ni f(T) \mapsto f(T_1, T_2) \in k[T]/(T^p) \otimes_k k[T]/(T^p) = k[T_1, T_2]/(T_1^p, T_2^p).$$

Notice that this is a k-group scheme with *a unique closed point* (the identity), but it is *not* the trivial k-group scheme $\mathrm{Spec}(k)$, as it has non-trivial tangent space (=Lie algebra), i.e. it is not reduced (hence not smooth).

Other examples of group schemes that are not smooth can be obtained, e.g. over $\mathbb{Z}_p$, for instance by taking the subgroup of GL_2 (defined over $\mathbb{Z}$) which stabilizes the quadratic form $Q(x,y) = p(x^2+y^2)$. The fiber of this over the generic point $\mathrm{Spec}\mathbb{Q}$ is an orthogonal group in two variables (hence of dimension 1), while the fiber over the special point $\mathrm{Spec}\mathbb{F}_p$ is GL_2 (of dimension 4) – in particular, this is not a smooth group scheme.

Definition 4.1.5. An *algebraic group* over a field k is a smooth group scheme over k. If an algebraic group is affine, it is called a *linear algebraic group*.

The following is a very basic theorem about quotients:

Theorem 4.1.6. *Let G be a Lie or linear algebraic group over a field k, and H a closed subgroup. In the first case, the quotient G/H exists as a smooth manifold. In the second case, there is a linear representation $G \to GL(V)$ such that H is the stabilizer of a line, and the quotient G/H is isomorphic to a locally closed subset of $\mathbf{P}(V)$, hence quasiprojective.*

Proof. Omitted, together with the definitions of quotients. Notice that the quotient in the case of algebraic groups is taken in the fpqc topology; i.e., the maps $G(R)/H(R) \to (G/R)(R)$ are not surjective for any k-algebra R, but they are surjective over some faithfully flat, quasi-compact cover. $\square$

4.2. Lie algebras; the Lie algebra of a Lie or algebraic group

Definition 4.2.1. A *Lie algebra* over a ring k is a k-module $\mathfrak{g}$ with a bilinear, antisymmetric operation

$$[\bullet,\bullet] : \mathfrak{g} \wedge \mathfrak{g} \to \mathfrak{g},$$

satisfying the *Jacobi identity*:

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0. \tag{4.2.1.1}$$

A *morphism of Lie algebras* $T : \mathfrak{g} \to \mathfrak{h}$ over k is a k-linear map T from $\mathfrak{g}$ to $\mathfrak{h}$, which is compatible with the Lie bracket: $[T(X), T(Y)] = T([X,Y])$.

There is a functor from associative algebras to Lie algebras, mapping an associative algebra A to the Lie algebra $\mathrm{Lie}(A)$, with the same underlying set and Lie bracket $[X,Y] = XY - YX$. We will often write simply A for the Lie algebra $\mathrm{Lie}(A)$.

Definition 4.2.2. A *representation* of a Lie algebra $\mathfrak{g}$ on a vector space V is a morphism of Lie algebas $\mathfrak{g} \to \mathrm{End}(V)$.

Definition 4.2.3. The *adjoint representation* of a Lie algebra $\mathfrak{g}$ is the homomorphism $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ given by $\mathrm{ad}(X)(Y) = [X,Y]$.

Remark 4.2.4. These notions explain the meaning of the Jacobi identity (4.2.1.1): It simply says that the adjoint map is, indeed, a representation.

Example 4.2.5. Let A be an associative algebra, and consider the submodule $\mathrm{Der}(A) \subset \mathrm{End}(A)$ of *derivations*, i.e., endomorphisms D satisfying the Leibniz rule $D(fg) = fDg + D(f)g$. It is a Lie subalgebra of $\mathrm{End}(A)$.

For a manifold or algebraic variety M, we will be denoting by $\mathrm{Der}(M)$ the derivations on M, i.e., sections of the tangent bundle of M.

Given a Lie group G, or an algebraic group over a field k, its tangent space $\mathfrak{g} = T_1 G$ at the identity can be endowed with the structure of a Lie algebra. This works as follows: First, evaluation of a vector field at the identity defines a linear maps

$$\mathrm{Der}(G) \to \mathfrak{g}.$$

Let $\mathrm{Der}(G)^{G\text{-left}}$ and $\mathrm{Der}(G)^{G\text{-right}}$ denote, respectively, the subspaces of left- and right- invariant derivations. For example, a left-invariant vector field V has the property that $L(g)_* V = V$ for every $g \in G$, where $L(g)$ is the left action of g on G. Then

Lemma 4.2.6. *Evaluation at the identity gives bijections*

$$Der(G)^{G\text{-}left} \xrightarrow{\sim} \mathfrak{g},$$

$$Der(G)^{G\text{-}right} \xrightarrow{\sim} \mathfrak{g}.$$

These bijections are mapped to each other under the inversion map $g \mapsto g^{-1}$ on G, which acts by -1 on $\mathfrak{g}$.

Proof. This is clear from the definitions. ☐

Derivations satisfy the Jacobi identity (think of them, locally, as a subalgebra of the Lie algebra associated to the endomorphism algebra of smooth/algebraic functions), which gives rise to a Lie algebra structure on $\mathfrak{g}$:

Definition 4.2.7. The space $\mathfrak{g} = T_1 G$, endowed with the Lie bracket of its identification with left- invariant derivations according to Lemma 4.2.6, is the *Lie algebra of the group G*.

Notice that the identification with right-invariant derivations would give the opposite Lie bracket.

Definition 4.2.7 makes sense for an algebraic group over a field k, as well, producing a Lie algebra over k. In positive characteristic, this Lie algebra has extra structure:

Definition 4.2.8. Let k be a field of characteristic $p > 0$. A *restricted Lie algebra* over k is a Lie algebra $\mathfrak{g}$ together with an operation $X \mapsto X^{[p]}$ such that:

(1) $\mathrm{ad}(X^{[p]}) = \mathrm{ad}(X)^p$;
(2) $(tX)^{[p]} = t^p X^{[p]}$ (for $t \in k$, $X \in \mathfrak{g}$);
(3) $(X+Y)^{[p]} = X^{[p]} + Y^{[p]} + \sum_{i=1}^{p-1} i^{-1} s_i(X,Y)$, where $s_i(X,Y)$ is the coefficient of t^i in $\mathrm{ad}(tX+Y)^{p-1}(X)$; in particular, if $[X,Y]=0$, $(X+Y)^{[p]} = X^{[p]} + Y^{[p]}$.

Example 4.2.9. If A is an associative algebra over k, then $\mathrm{Lie}(A)$ is a restricted Lie algebra, with $A^{[p]} = A^p$. In particular, if G is an algebraic group over k, the p-th power of a left-invariant vector field, viewed as a differential operator, is also a left-invariant vector field, and endows the Lie algebra $\mathfrak{g}$ with the structure of a restricted Lie algebra.

4.3. The universal enveloping algebra and the Poincaré–Birkhoff–Witt theorem

The functor $A \mapsto \mathrm{Lie}(A)$ from associative to Lie algebras has a left adjoint.

Definition 4.3.1. Given a Lie algebra $\mathfrak{g}$, the initial object $U(\mathfrak{g})$ of the category of associative algebras A with a homomorphism of Lie algebras: $\mathfrak{g} \to A$ is called the *(universal) enveloping algebra* of $\mathfrak{g}$.

Equivalently, the association $\mathfrak{g} \to U(\mathfrak{g})$ is left adjoint to the natural functor from associative to Lie algebras, i.e.

$$\mathrm{Hom}_{\mathrm{Lie}}(\mathfrak{g}, \mathrm{Lie}(A)) = \mathrm{Hom}_{\mathrm{Assoc}}(U(\mathfrak{g}), A)$$

for every associative algebra A.

In other words, $U(\mathfrak{g})$, together with the homomorphism $\mathfrak{g} \to U(\mathfrak{g})$ is defined by the universal property that any other homomorphism of Lie algebras $\mathfrak{g} \to A$ factors uniquely through $U(\mathfrak{g})$.

Proposition 4.3.2. *The universal enveloping algebra of any Lie algebra $\mathfrak{g}$ exists.*

Proof. One can construct it as the quotient of the tensor algebra of $\mathfrak{g}$,

$$T\mathfrak{g} := \bigoplus_{n \geq 0} \mathfrak{g}^{\otimes^n},$$

by the two-sided ideal generated by all elements of the form:

$$X \otimes Y - Y \otimes X - [X, Y], \; X, Y \in \mathfrak{g}.$$

(The rest of the proof is left to the reader.)

□

Example 4.3.3. Let X be a smooth manifold, or an algebraic variety (over a field k). Let $\mathfrak{g} = \mathrm{D}er(X)$ be the vector space of *derivations*, or *vector fields* on X: these are sections of the tangent bundle of X.[1] Then, $\mathfrak{g}$ is a Lie algebra, and $U(\mathfrak{g})$ is, by definition, the algebra of (smooth/algebraic) differential operators on X.

The universal enveloping algebra is ($\mathbb{N}$-)*filtered*: We have $F^0U(\mathfrak{g}) = k$, $F^1U(\mathfrak{g}) = \mathfrak{g} \oplus k$, and $F^nU(\mathfrak{g})$ = the subspace generated by n-fold products of elements of $F^1U(\mathfrak{g})$.

Definition 4.3.4. Let A be an algebra with an increasing $\mathbb{N}$-filtration, and write F^i simply for F^iA: $A = \sum_{n\geq 0} F^nA$, $F^i \subset F^{i+1}$, and $F^i \cdot F^j \subset F^{i+j}$. Set, by convention, $F^{-1} = 0$.

The *associated graded algebra* of A is the $\mathbb{N}$-graded algebra

$$\mathrm{gr}A = \bigoplus_{i\geq 0} \mathrm{gr}^i A = \bigoplus_{i \geq 0} F^iA/F^{i-1}A.$$

The *Rees algebra* of A is the $\mathbb{N}$-graded algebra

$$\mathcal{A} = \bigoplus_{i\geq 0} F^iA \cdot t^i \subset A \otimes k[t].$$

[1] In algebraic geometry, derivations are defined as certain endomorphisms of the structure sheaf, and are used to define the tangent bundle.

Lemma 4.3.5. *In the setting of Definition 4.3.4, the Rees algebra $\mathcal{A}$ is a free $k[t]$-module, its fiber over any $t = a \neq 0$ (i.e., the quotient $\mathcal{A}/(t-a)\mathcal{A}$) is canonically isomorphic, through the evaluation map $t \mapsto a$, to the original filtered algebra A, and its fiber over $t = 0$ (i.e., the quotient $\mathcal{A}/t\mathcal{A}$) is canonically isomorphic to its associated graded grA.*

Proof. The fact that it is free over $k[t]$ is obvious.

Away from $t = 0$, that is, if we tensor with $k[t^{-1}]$, we get an isomorphism

$$\mathcal{A} \otimes k[t^{-1}] = A \otimes k[t^{-1}, t],$$

so the fiber at $t = a \neq 0$ is isomorphic to A through the evaluation map.

On the other hand, the element t is homogeneous, so the quotient $\mathcal{A}/(t)$ is also a graded algebra, with i-th graded piece equal to $F^iA \cdot t^i / t \cdot F^{i-1}A \cdot t^{i-1} = \mathrm{gr}^i A$. □

The structure of the universal enveloping algebra is described by the Poincaré–Birkhoff–Witt theorem:

Theorem 4.3.6 (Poincaré-Birkhoff-Witt)**.** *Let $\mathfrak{g}$ be a Lie algebra over a field k. Then there is a canonical isomorphism*

$$grU(\mathfrak{g}) \simeq S(\mathfrak{g}), \tag{4.3.6.1}$$

where $S(\mathfrak{g})$ denotes the symmetric algebra in $\mathfrak{g}$.

In particular, if we choose a linearly ordered vector space basis $(X_i)_{i \in I}$ (possibly with infinite indexing set I), then the monomials of the form $X_{i_1}^{r_1} X_{i_2}^{r_2} \cdots X_{i_k}^{r_k}$, with $i_1 < i_2 < \cdots < i_k$, form a vector space basis for $U(\mathfrak{g})$.

Proof. First, we construct a natural surjection:

$$S(\mathfrak{g}) \to \mathrm{gr}U(\mathfrak{g}).$$

The symmetric algebra $S(\mathfrak{g})$ is the *homogeneous quadratic algebra* $T(\mathfrak{g})/(R)$, where $T\mathfrak{g}$ is the tensor algebra $T = \bigoplus T^i$ with $T^i(\mathfrak{g}) = \mathfrak{g} \otimes \cdots \otimes \mathfrak{g}$ (i times), and R the subspace of $T^2(\mathfrak{g})$ (hence "homogeneous quadratic") generated by elements of the form $x \otimes y - y \otimes x$. The notation (R) denotes the two-sided ideal generated by R.

On the other hand, $U(\mathfrak{g})$ is the *inhomogeneous quadratic algebra* $T(\mathfrak{g})/(P)$, where $P \subset T^{\leq 2}(\mathfrak{g})$ the subspace generated by elements of the form $x \otimes y - y \otimes x - [x, y]$.

The image of P under the quotient $T^{\leq 2}(\mathfrak{g}) \to T^2(\mathfrak{g})$ is equal to R, and this implies that the ideal (R) is in the kernel of the natural surjective map

$$S(\mathfrak{g}) = \mathrm{gr}T(\mathfrak{g}) = \bigoplus_i T^{\leq i}(\mathfrak{g})/T^{\leq i-1}(\mathfrak{g}) \to \mathrm{gr}U(\mathfrak{g}).$$

The hard part of the proof is to show that the resulting map $S(\mathfrak{g}) = T(\mathfrak{g})/(R) \to \mathrm{gr}U(\mathfrak{g})$ is an isomorphism. The proof uses the Jacobi identity in an essential way:

The idea of the proof is to construct a representation ρ of $\mathfrak{g}$ (equivalently: of $U(\mathfrak{g})$) on the free vector space V generated by the monomials $X_{i_1}^{r_1} X_{i_2}^{r_2} \cdots X_{i_k}^{r_k}$. This representation will have the property that $Y = X_{i_1}^{r_1} X_{i_2}^{r_2} \cdots X_{i_k}^{r_k}$, considered as an element of $U(\mathfrak{g})$, takes $1 \in V$ to $X_{i_1}^{r_1} X_{i_2}^{r_2} \cdots X_{i_k}^{r_k}$ (a posteriori, it is just left multiplication on $U(\mathfrak{g})$). In particular, the map $U(\mathfrak{g}) \ni Y \mapsto \rho(Y)(1) \in V$ is injective, which proves the theorem.

Define the *structure constants* of the Lie algebra, c_{ij}^k, by $[X_i, X_j] = \sum_k c_{ij}^k X_k$. We write every monomial as above in the form Y_M, where $M = (j_1 \leq j_2 \leq \cdots \leq j_n)$ is a finite ordered sequence of elements of the indexing set I (with repetitions). This includes the element $Y_\emptyset = 1 \in k$. Notice that we will be using the letter X for elements of $\mathfrak{g}$, and the letter Y for elements of V, to distinguish them. We define the representation as a filtered map

$$\mathfrak{g} \times V \to V,$$

with respect to the filtration of V by the subspaces V_n spanned by monomials of length $\leq n$ (with the elements of $\mathfrak{g}$ in degree 1, of course). First of all, we set

$$X_i \cdot Y_\emptyset = Y_{(i)}.$$

Assume now that we have defined a map $\mathfrak{g} \times V_{n-1} \to V_n$, $n \geq 1$, satisfying the following three properties:

- $X_i \cdot Y_M = Y_{(i,M)}$ when $i \leq$ the smallest (first) element of M (which we will denote by $i \leq M$;
-
$$X_i \cdot Y_{(j_1 \leq \cdots \leq j_n)} = Y_{(j_1 \leq \cdots \leq i \leq \cdots j_n)} + \text{ lower order terms;} \tag{4.3.6.2}$$
-
$$X_i \cdot (X_j \cdot Y_M) - X_j \cdot (X_i \cdot Y_M) = [X_i, X_j] \cdot Y_M. \tag{4.3.6.3}$$

We define the map on $\mathfrak{g} \times V_n$, inductively on the basis elements (i.e., assuming it has been defined for X_j with $j < i$) by

$$X_i \cdot Y_{(j,M)} = \begin{cases} Y_{(i,j,M)}, & \text{if } i \leq j \\ X_j \cdot (X_i \cdot Y_M) + \sum_k c_{ij}^k X_k \cdot Y_M, & \text{otherwise.} \end{cases}$$

Then, the first of the three properties above holds by definition. For (4.3.6.2), also by the definition, if the ordering is $j_1 \leq \cdots \leq j_m < i \leq \ldots$, also by the definition we have

$$\begin{aligned} X_i \cdot Y_{(j_1 \leq \cdots \leq j_n)} &= X_{j_1} \cdots X_{j_m} \cdot X_i \cdot Y_{(j_{m+1}, \ldots)} + \text{ lower order terms} \\ &= Y_{(j_1 \leq \cdots \leq i \leq \cdots j_n)} + \text{ lower order terms.} \end{aligned}$$

Finally, for (4.3.6.3), if $i = j$, both sides are zero. Also, since both sides are anti-symmetric in i and j, we may assume that $i > j$. In $j \leq M$, then the property holds by definition. Assume now that $M = (k, N)$ with $j > k$. Then, by definition,

$$X_j \cdot Y_{(k,N)} = X_j \cdot (X_k \cdot Y_N) = X_k \cdot (X_j \cdot Y_N) + [X_j, X_k] \cdot Y_N,$$

and similarly for $X_i \cdot Y_{(k,N)}$. By (4.3.6.2), the element $X_j \cdot Y_N$ can be written as $Y_{(j,N)_{\mathrm{ord}}} +$ lower order terms, where $(j,N)_{\mathrm{ord}}$ denotes the ordering of the multiset obtained by appending j to N. Since $k \leq N$ and $k < N$, by the above and by the induction hypothesis we have

$$X_i \cdot (X_k \cdot (X_j \cdot Y_N)) = X_k \cdot (X_i \cdot (X_j \cdot Y_N)) + [X_i, X_k] \cdot (X_j \cdot Y_N).$$

Thus,

$$\begin{aligned} &X_i \cdot (X_j \cdot Y_M) - X_j \cdot (X_i \cdot Y_M) = X_i \cdot (X_j \cdot (X_k \cdot Y_N)) - X_j \cdot (X_i \cdot (X_k \cdot Y_N)) = \\ &= X_k \cdot (X_i \cdot (X_j \cdot Y_N)) + [X_i, X_k] \cdot (X_j \cdot Y_N) + [X_j, X_k] \cdot (X_i \cdot Y_N) + [X_i, [X_j, X_k]] \cdot Y_N \\ &- X_k \cdot (X_j \cdot (X_i \cdot Y_N)) - [X_j, X_k] \cdot (X_i \cdot Y_N) - [X_i, X_k] \cdot (X_j \cdot Y_N) - [X_j, [X_i, X_k]] \cdot Y_N. \end{aligned}$$

By the Jacobi identity, this is equal to

$$X_k \cdot (X_i \cdot X_j - X_j \cdot X_i) \cdot Y_N + [X_k, [X_i, X_j]] \cdot Y_N,$$

and again by the induction hypothesis this is is

$$X_k \cdot [X_i, X_j] \cdot Y_N + [X_k, [X_i, X_j]] \cdot Y_N,$$

and once more by the induction hypothesis this is

$$[X_i, X_j] \cdot X_k \cdot Y_N = [X_i, X_j] \cdot Y_M,$$

as desired.

□

Remark 4.3.7. A different and more general proof by Braverman and Gaitsgory interprets the Jacobi identity in terms of *Hochschild cohomology*, see [**BG96**]. We summarize the ideas: In this proof, instead of starting from $U(\mathfrak{g})$, we start from the symmetric algebra $S(\mathfrak{g})$, and construct the Rees algebra of $U(\mathfrak{g})$ as a deformation of that.

Let $A = S(\mathfrak{g})$, considered as a graded algebra. An *i-th level graded deformation* of A will be a graded $k[t]/k[t]t^{i+1}$-algebra A_i (where $\deg(t) = 1$), which is free as a $k[t]/k[t]t^{i+1}$-module, together with an isomorhism of $A_i/tA_i \simeq A$. A *graded deformation* $\mathcal{A}$ of A will be a graded algebra over the polynomial ring $k[t]$, which is free as a module over this ring, together with an isomorphism $\mathcal{A}/t\mathcal{A} \simeq A$.

Suppose we are given a first-level deformation $A_1 \to A$, and choose a splitting $A \to A_1$ as a graded $k[t]$-module, so that $A_1 = A \oplus tA$. Then, the multiplication on A_1 is described by a k-linear map $f : A \otimes A \to A$, homogeneous of degree -1, such that $(a + t \cdot 0) \cdot (b + t \cdot 0) = ab + tf(a, b)$. The associativity condition is rewritten in terms of f as:

$$f(a, b)c - f(ab, c) + f(a, bc) - af(b, c) = 0, \tag{4.3.7.1}$$

for any a, b, $c \in A$.

It turns out that this condition defines a *Hochschild cocycle*. The *Hochschild cohomology* of A is the derived functor of $\mathrm{Hom}_{A\otimes A^{\mathrm{op}}}(A, A)$, the endomorphisms of A as an A-bimodule. It can be computed using the *bar resolution* by free bimodules $B^i(A) = A^{\otimes^{i+2}}$, with the boundary map $B^i(A) \to B^{i-1}(A)$ given as the alternating sum of replacements $a \otimes b \mapsto ab$, over all identifications of $A \otimes A$ with the $(j, j+1)$-st factor of B^i (where $j = 0 \dots i$). So, the derived functor $\mathrm{Ext}^i_{A\otimes A^{\mathrm{op}}}(A, A)$ can be computed in terms of the complex consisting of

$$\mathrm{Hom}_{A\otimes A^{\mathrm{op}}}(B^i(A), A),$$

which is the same as

$$\mathrm{Hom}(A^{\otimes^i}, A)$$

with appropriate boundary maps. For $i = 2$, the cocycle condition is precisely the equation (4.3.7.1). One checks that the choice of splitting $A \to A_1$ changes the 2-cocycle by a coboundary, so the first-level deformations of A correspond uniquely to classes in $H^2(A)$ (second Hochschild cohomology group). In fact, since A is graded, so is the Hochschild cohomology, and we get a bijection between isomorphism classes of first-level deformations, and the -1-graded piece $H^2_{-1}(A)$.

There is a similar description of extensions of an i-th level deformation to an $(i+1)$-st level deformation by $H^2_{-i-1}(A)$, *provided such deformations exist.* The obstruction to the existence of such a deformation is an element of $H^3_{-i-1}(A)$.

Now, it so happens that $A = S(\mathfrak{g})$ is a *Koszul algebra*. One of the equivalent definitions of this notion for $\mathbb{N}$-graded algebras is that $A_0 = k$, and $A_0 = A/A_{>0}$, as a graded A-module, has a graded projective resolution

$$\cdots \to P^{(2)} \to P^{(1)} \to P^{(0)} \to A/A_{>0} \to 0$$

where $P^{(i)}$ is generated by homogeneous elements in degree i. This turns out to be equivalent, for a homogeneous quadratic algebra of the form $Q(V, R)$, to the statement that the bar resolution can be replaced by a resolution by the subspaces $\tilde{K}^i = A \otimes K^i \otimes A$, where K^i is the intersection of the spaces $V^{\otimes^j} \otimes R \otimes V^{\otimes^{i-j-2}}$, $0 \le j \le i-2$. Then, the following four conditions on the generator P of the non-homogeneous quadratic ideal:

(1) $P \cap F^1(T(V)) = 0$; hence, we can write every element of P as $r + \alpha(r) + \beta(r)$, with $r \in R$, $\alpha(r) \in T^1(V)$, $\beta(r) \in T^0(V) = k$;
(2) $\mathrm{Im}(\alpha \otimes I - I \otimes \alpha) \subset R$; (this map is defined on $K^3 = R \otimes V \cap V \otimes R$);
(3) $\alpha \circ (\alpha \otimes I - I \otimes \alpha) = -(\beta \otimes I - I \otimes \beta)$;
(4) $\beta \circ (\alpha \otimes I - I \otimes \alpha) = 0$

(where the second, third, and fourth condition follow from the Jacobi identity, in our case) have, correspondingly, the following cohomological interpretations:

(1) this is just saying, as remarked, that we can write every element of P as $r + \alpha(r) + \beta(r)$;
(2) $d\alpha = 0$; thus, α defines a cohomology class in $H^2(A)$, which can be checked to belong to $H^2_{-1}(A)$, thus defining a first-degree deformation of A;
(3) the cocycle representing the obstruction to a second-level deformation is trivial;
(4) the cocycle representing the obstruction to a third-level deformation is trivial.

Then, it turns out that for Koszul algebras every third-level graded deformation extends uniquely to a graded deformation $\mathcal{A}$ over $k[t]$.

The PBW theorem has several corollaries:

Proposition 4.3.8. *The universal enveloping algebra $U(\mathfrak{g})$ is Noetherian.*

Proof. This follows from the Noetherian property of $\mathrm{gr}U(\mathfrak{g}) = S(\mathfrak{g})$ by the following standard argument: if $J_1 \subset J_2 \subset \cdots$ is an increasing sequence of ideals, then so is $\mathrm{gr}J_1 \subset \mathrm{gr}J_2 \subset \cdots$, where $\mathrm{gr}J = \oplus_n (J \cap F_n / F_{n-1})$. Notice that the map $J \mapsto \mathrm{gr}J$ is not injective on ideals: two different ideals of $U(\mathfrak{g})$ can have the same image in its graded. However, the map *is* injective on chains, i.e. if $J_1 \subset J_2$ and their graded ideals coincide, then $J_1 = J_2$. From the Noetherian property of $S(\mathfrak{g})$, the sequence of graded ideals stabilizes, therefore so does the original sequence. ☐

Another corollary is the following:

Proposition 4.3.9. *If $\mathfrak{h} \subset \mathfrak{g}$ is a Lie subalgebra then $U(\mathfrak{g})$ is a free $U(\mathfrak{h})$-module, and hence the* induction *functor:*

$$M \mapsto U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} M$$

(where M is an $\mathfrak{h}$-module) is exact.

4.4. Exponential map and the Baker–Campbell–Hausdorff formula

Now we work differential-geometrically in the setting of a real Lie group, following [**Ste09**].

Definition 4.4.1. A *one parameter subgroup* is a homomorphism of Lie groups: $\gamma : \mathbb{R} \to G$.

Lemma 4.4.2. *The map $\gamma \mapsto \gamma'(0)$ is a bijection between one-parameter subgroups and elements of the Lie algebra.*

Proof. Locally around any point x, any vector field is uniquely integrable (this is a basic result from ODEs), namely: if $\mathbf{v}$ is a vector field then there is an interval $(-\epsilon, \epsilon)$ and a curve $\gamma : (-\epsilon, \epsilon) \to G$ such that $\gamma(0) = x$ and $\gamma'(t) = \mathbf{v}(\gamma(t))$, and any two such curves coincide in a neighborhood of 0.

For a left-invariant vector field, we can use left translations by the group to show that this local existence and uniqueness statement becomes global. $\square$

Definition 4.4.3. The *exponential map*

$$\mathfrak{g} \to G$$

is defined by

$$\exp(X) = \gamma_X(1),$$

where γ_X is the unique one-parameter subgroup with $\gamma'_X(0) = X$.

Lemma 4.4.4. *The exponential map is a local diffeomorphism around $0 \in \mathfrak{g}$.*

Proof. Its differential, if well defined, is the identity on $\mathfrak{g} = T_eG$, so we only need to show that it is a smooth map. The flow on $G \times \mathfrak{g}$ associated to the smooth vector field $(g, X) \mapsto (X(g), 0)$ is given by: $\mathbb{R} \times G \times \mathfrak{g} \ni (t, g, X) \mapsto (g \cdot \exp(tX), X)$, and the flow of a smooth vector field is smooth. Therefore, the exponential map is smooth. $\square$

The exponential map is not a group homomorphism, except if G is abelian (but, by definition, it is a group homomorphism when restricted to any one-dimensional subspace of $\mathfrak{g}$). Its failure to be a homomorphism is addressed by the so-called Baker–Campbell–Hausdorff formula (which goes back to Schur). Before we state and prove the BCH formula, we prove an important formula that will be used in the proof, the Maurer–Cartan equation.

Theorem 4.4.5 (Maurer–Cartan equation)**.** *Let θ be the unique left-invariant, $\mathfrak{g}$-valued differential 1-form on G which at the identity (e) is equal to the canonical ("identity") element of $T_e^*G \otimes \mathfrak{g} = \mathfrak{g}^* \otimes \mathfrak{g} = End(\mathfrak{g})$. Then its differential is given by*

$$d\theta = -\frac{1}{2}[\theta, \theta]. \tag{4.4.5.1}$$

The convention here is that for two $\mathfrak{g}$-valued 1-forms θ_0, θ_1, and two vector fields v_0, v_1, we have $[\theta_0, \theta_1](v_0, v_1) = [\theta_0(v_0), \theta_1(v_1)] - [\theta_0(v_1), \theta_1(v_0)]$, hence $\frac{1}{2}[\theta, \theta](v_0, v_1) = [\theta(v_0), \theta(v_1)]$.

Proof. Since θ is left-invariant, so will be its differential $d\theta$, which is a section of the exterior square of the cotangent bundle of G, valued in $\mathfrak{g}$. Thus, $d\theta$ is determined by its value at the identity, and it therefore suffices to verify the formula when $d\theta$

is applied to a pair (v_0, v_1) of left-invariant vector fields (identified with elements of $\mathfrak{g}$). By the definition of exterior derivative,

$$d\theta(v_0, v_1) = v_0\theta(v_1) - v_1\theta(v_0) - \theta([v_0, v_1]).$$

Since the v_i's are left-invariant, $\theta(v_i)$ is the constant $v_i \in \mathfrak{g}$, and therefore $v_j\theta(v_i) = 0$. Thus, $d\theta(v_0, v_1) = -\theta([v_0, v_1]) = -[\theta(v_0), \theta(v_1)]$, as claimed. □

Theorem 4.4.6 (Baker–Campbell–Hausdorff formula)**.** *If G is a Lie group with Lie algebra $\mathfrak{g}$, then for sufficiently small $X, Y \in \mathfrak{g}$ we have*

(4.4.6.1)
$$\exp(X)\exp(Y) = \exp(X+Y+\frac{1}{2}[X,Y]+\frac{1}{12}([X,[X,Y]]+[Y,[Y,X]])+P_3(X,Y)+\dots),$$

where $P_i(X,Y)$ is a Lie polynomial of order *i, i.e., $P_i(X,Y)$ is a linear combination of $i-1$ nested commutators in the variables X, Y.*

Remark 4.4.7. There is a precise formula for the Lie polynomials P_i:

(4.4.7.1)
$$P_i(X,Y) = \sum_n \frac{(-1)^{n-1}}{n} \sum_{\substack{r_1+s_1+r_2+s_2+\cdots+r_n+s_n=i \\ r_j,s_j\geq 0,\ r_j+s_j>0}} \frac{[X^{(r_1)}Y^{(s_1)}X^{(r_2)}Y^{(s_2)}\dots X^{(r_n)}Y^{(s_n)}]}{\sum_{j=1}^n (r_j+s_j)\cdot \prod_{j=1}^n r_j!s_j!},$$

where $[X^{(a)}Y^{(b)}]$ denotes the Lie polynomial $[X,[X,\dots,[X,[Y,[Y,\dots,Y]\cdots]$, with X appearing a times and Y appearing b times (and similarly for more "factors").

This precise formula can be worked out inductively from the differential equation (4.4.7.2) below. What is important (and difficult) is the existence of such a series.

Proof. We outline two proofs, following [**Ste09**], and point the reader to Sternberg's notes for details. The second proof, which is algebraic, assumes that the group is analytic (as Lie groups are often defined to be, from the outset, e.g., in Bourbaki). The first, which is analytic, proves the analyticity of Lie groups (defined in the differentiable category; see Proposition 4.4.8 below).

For the first proof, the main idea is to express the product $\exp(X)\exp(Y)$ in terms of the elements $\mathrm{ad}(X)$, $\mathrm{ad}(Y)$ in the concrete associative (and Lie) algebra of endomorphisms of $\mathfrak{g}$.

We consider the former power series $\psi(1+u) = (1+u)\frac{\log(u)}{u} = 1+\frac{u}{2}-\frac{u^2}{6}+\dots$; the BCH formula, with the precise terms (4.4.7.1), is equivalent to the statement that

$$\log(\exp(X)\exp(Y)) = X + \int_0^1 \psi(\exp(\mathrm{ad}(X))\exp(t\cdot\mathrm{ad}(Y)))(Y)dt \tag{4.4.7.2}$$

for sufficiently small elements X, Y. Notice, first of all, that the formal power series defining the operator $\psi(\exp(t\cdot\mathrm{ad}(X))\exp(\mathrm{ad}(Y))) \in \mathrm{End}(\mathfrak{g})$ on the right hand side converges for small X, Y. The logarithm on the left hand side is, by definition, the inverse of the exponential map on $\mathfrak{g}$, defined in a small neighborhood of the origin.

This, in turn, will be proven by proving the following formula about the "logarithmic derivative" of any smooth curve $C(t)$ on $\mathfrak{g}$:

$$\exp(C(t))^{-1}\frac{d}{dt}\exp(C(t)) = \phi(-\mathrm{ad}C(t))C'(t), \tag{4.4.7.3}$$

where $\phi(z)$ is the power series

$$\frac{e^z - 1}{z} = \sum_{n \geq 0} \frac{1}{(n+1)!} z^n.$$

Both sides of the last equation are valued in $\mathfrak{g}$, identified with the tangent space at the identity of G. Applying this relation to the curve $C(t) = \exp(X)\exp(tY)$, we get

$$B = \phi(-\log(\exp(\mathrm{ad}(X)) \cdot \exp(t \cdot \mathrm{ad}(B))))C'(t),$$

and, using the fact that

$$\psi(z)\phi(-\log(z)) = 1,$$

we get

$$C'(t) = \psi(\exp(\mathrm{ad}(X)) \cdot \exp(t \cdot \mathrm{ad}(B)))(B),$$

which is equivalent to (4.4.7.2).

There remains to prove the formula (4.4.7.3), about the logarithic derivative of a smooth curve.

Setting $f(s,t) = \exp(sC(t))$, and with θ the Maurer–Cartan form, the left hand side of (4.4.7.3) is equal to $f^*\theta(\frac{\partial}{\partial t})(1,t)$, while it is immediate to compute

$$f^*\theta(\frac{\partial}{\partial s})(s,t) = \exp(sC(t))^{-1} \frac{\partial}{\partial s} \exp(sC(t)) = C(t)$$

(for any s, t). We let $\kappa(s,t) = f^*\theta(\frac{\partial}{\partial t})(s,t)$.

The differential of $f^*\theta$, applied to these vector fields, is

$$df^*\theta(\frac{\partial}{\partial s}, \frac{\partial}{\partial t}) = \frac{\partial}{\partial s} f^*\theta(\frac{\partial}{\partial t}) - \frac{\partial}{\partial t} f^*\theta(\frac{\partial}{\partial s}) =$$
$$= \frac{\partial}{\partial s}\kappa(s,t) - C'(t).$$

Applying the Maurer–Cartan formula (4.4.5.1) to the pullback of θ, we get that this is equal to

$$-[f^*\theta(\frac{\partial}{\partial s}), f^*\theta(\frac{\partial}{\partial t})] = -[C(t), \kappa(s,t)].$$

Thus, fixing t, the $\mathfrak{g}$-valued function $\kappa(s) = \kappa(s,t)$ satisfies the ordinary differential equation

$$\kappa'(s) = -\mathrm{ad}(C(t))\kappa(s) + C'(t),$$

with initial value $\kappa(0) = 0$.

This is now easily seen to have the unique solution

$$\kappa(s) = \frac{e^{-s\mathrm{ad}(C(t))} - 1}{\mathrm{ad}(C(t))} C'(t),$$

where the fraction is a formal expression for the series

$$\sum_{n \geq 0} \frac{1}{(n+1)!} s^{n+1} z^n$$

in the operator $-\mathrm{ad}(C(t))$.

Setting $s = 1$, the proof is now complete.

[TO ADD: ALGEBRAIC PROOF]

□

Immediate corollaries of the BCH theorem include:

Proposition 4.4.8. *Every Lie group has a unique structure of a group in the category of real analytic spaces with the property that the exponential map is an analytic isomorphism in a neighborhood of the identity.*

Proof. Fix a sufficiently small neighborhood U of zero in $\mathfrak{g}$, and use it to define an analytic chart in the neighborhood $g\exp(U)$ of any element $g \in G$. The Baker–Campbell–Hausdorff theorem 4.4.6 implies that the transition maps between these charts are analytic, so we have a well-defined analytic structure. The same theorem shows that multiplication is analytic. □

Proposition 4.4.9. *Given a Lie group G and a sub-Lie algebra $\mathfrak{h} \subset \mathfrak{g}$, there is a unique connected immersed Lie subgroup $H \subset G$ whose Lie algebra is $\mathfrak{h}$.*

By an immersed Lie subgroup we mean an immersed submanifold: $H \to G$ such that H is a subgroup of G.

Proof. The left translations of $\mathfrak{h}$ give rise to a *distribution* $D_{\mathfrak{h}}$, i.e. a subbundle of TG. It is known from the theory of differential equations that a distribution D is (uniquely) *integrable* if and only if for any two vector fields which lie in it, their commutator also lies in it. This is easily seen to be the case for $D_{\mathfrak{h}}$, since $\mathfrak{h}$ is a Lie subalgebra. By the Baker–Campbell–Hausdorff Theorem 4.4.6, the leaf through zero of the corresponding foliation is an immersed subgroup. □

Proposition 4.4.10. *Let G_1, G_2 be Lie groups with G_1 connected and simply connected, then every morphism between their Lie algebras*

$$f' : \mathfrak{g}_1 \to \mathfrak{g}_2$$

lifts to a unique morphism

$$f : G_1 \to G_2.$$

Proof. The pair (f', I) defines an embedding $\mathfrak{g}_1 \to \mathfrak{g} := \mathfrak{g}_1 \oplus \mathfrak{g}_2$ which, by Proposition 4.4.9, corresponds to a unique connected immersed Lie subgroup: $H \to G_1 \times G_2$ whose Lie algebra is $\mathfrak{g}_1$. Composing with projection to G_1 we get: $H \to G_1$ which is an isomorphism on tangent spaces, hence a covering map. Since G_1 is simply connected, $H = G_1$. □

Remark 4.4.11. In a following chapter, 5.7, we will discuss Ado's theorem, which states that every finite-dimensional Lie algebra over a field in characteristic zero has a faithful representation; hence, Proposition 4.4.9 implies that, given a finite-dimensional Lie algebra over $\mathbb{R}$, it is the Lie algebra of a Lie group. We may assume that this Lie group is connected and simply connected by passing to the universal cover, in which case Proposition 4.4.10 implies that it is uniquely determined, up to unique isomorphism, by the Lie algebra.

4.5. Open and closed subgroups of Lie groups

For any Lie group G we will be denoting by G^0 the connected component of the identity. It is a normal subgroup (exercise!).

Lemma 4.5.1. *Any open subgroup of G contains G^0.*

Proof. Let H be an open subgroup. Its complement is a union of (left, let's say) H-cosets, and since right multiplication takes open sets to open sets, those cosets are open. Hence, the complement of H is open, therefore H is both open and closed, and therefore it contains the connected component of the identity. □

It is not true that every subgroup of a Lie group is closed. For instance, any one-parameter subgroup in the torus $(\mathbb{R}/\mathbb{Z})^2$ with non-rational slope is dense, but not closed.

On the other hand, every closed subgroup is a Lie subgroup:

Theorem 4.5.2 (Cartan)**.** *Every closed subgroup of a Lie group is a smooth manifold, hence a Lie subgroup.*

Proof. Let $H \subset G$ be a closed subgroup of a Lie group. Let $\mathfrak{g}$ denote the Lie algebra of G, i.e. the tangent space at the identity. We will define a subspace of $\mathfrak{g}$ which will be the candidate for the tangent space of the identity for H. Then we will show that it is indeed so.

Choose a Euclidean metric on $\mathfrak{g}$ and let $\exp : \mathfrak{g} \to G$ be the exponential map. In a neighborhood of the idenity in $\mathfrak{g}$, it is a diffeomorphism onto a neighborhood of the identity in G, and let log denote its inverse in that neighborhood.

Let $W \subset \mathfrak{g}$ be the set of all tX, where $t \in \mathbb{R}$ and $X \in \mathfrak{g}$ is the limit of a sequence: $\frac{h_n}{|h_n|}$ with $h_n \to 0 \in \mathfrak{g}$ and $\exp(h_n) \in H$. We claim:

(1) $\exp(W) \subset H$;
(2) W is a linear subspace of $\mathfrak{g}$.

For the first, if $\frac{h_n}{|h_n|} \to X$ and $|h_n| \to 0$ we can choose, for given $t \in \mathbb{R}$, integers $m_n \in \mathbb{Z}$ such that $m_n|h_n| \to t$, so $\exp(m_n \cdot h_n) \to \exp(tX)$ as $n \to \infty$.

Here we will use the following fact: for an one-dimensional subspace of $\mathfrak{g}$ the exponential map is a homomorphism of groups. Therefore, $\exp(m_n \cdot h_n) = \exp(h_n)^{m_n}$, therefore it belongs to H. Since H is closed, the limit $\exp(tX)$ is also in H.

For the second claim, if $X, Y \in W$ set $h(t) = \log(\exp(tX)\exp(tY))$. We claim that $\lim_{t\to 0} h(t)/t = X + Y$. Indeed, the differential at the identity of the multiplication map: $G \times G \to G$ is $\mathfrak{g} \times \mathfrak{g} \ni (X, Y) \mapsto X + Y$. Hence, $h(t)/|h(t)| = h(t)/t \cdot t/|h(t)| \to \frac{X+Y}{|X+Y|}$ as $t \to 0$, $t > 0$, therefore $X + Y \in W$.

Having proven the two claims, and given that the exponential map is a diffeomorphism in a neighborhood of the identity, it now suffices to show that $\exp(W)$ is a neighborhood of the identity in H. Let D be the orthogonal complement of W in $\mathfrak{g}$ with respect to the above norm. For a sequence $h_n \in H$ with $h_n \to e$, we can eventually write $h_n = \exp(x_n + y_n)$ with $x_n \in W$ and $y_n \in D$, $(x_n, y_n) \to 0$. We claim that

$$\lim_{n\to\infty} \frac{\log(h_n \exp(-x_n))}{|y_n|} = \lim_{n\to\infty} \frac{y_n}{|y_n|}$$

if one of the two limits exists.

Indeed, by the Baker–Campbell–Hausdorff formula (4.4.6.1), the left hand side can be written as

$$\lim_{n\to\infty} \frac{y_n + P_2(x_n + y_n, y_n) + P_3(x_n + y_n, y_n) + \dots}{|y_n|},$$

where P_i is a homogeneous Lie polynomial of order i. When both x_n and y_n tend to zero, the quotient

$$\frac{P_i(x_n + y_n, y_n)}{|y_n|}$$

tends to zero, for every $i \geq 2$. This proves the claim.

But then, we must have $y_n = 0$ for large n, for otherwise a subsequence of the $\frac{y_n}{|y_n|}$'s will have a limit point $y \in D$, $|y| = 1$, which should then belong to W, a contradiction. This completes the proof of the theorem.

□

4.6. Algebraic groups in characteristic zero

4.6.1. The functor from schemes to topological spaces. If k is a topological field (e.g., $\mathbb{R}$, $\mathbb{C}$), and $X = \mathrm{Spec}(A)$ is an affine k-scheme of finite type, the set

$$X(k) = \mathrm{Hom}(A, k)$$

acquires a natural topology, the *open compact topology* when A is viewed as a discrete ring, i.e., the restricted topology under the embedding

$$\mathrm{Hom}(A, k) \hookrightarrow k^A.$$

There is a unique way to extend this definition to any scheme of finite type over k, in such a way that open embeddings of schemes give rise to open embeddings of topological spaces, and this gives rise to a functor

$$\mathrm{Top} : \text{Schemes of finite type over } k \to \text{topological spaces}.$$

For these facts, we point the reader to Brian Conrad's expository article [**Con12**].

4.6.2. Smooth schemes and manifolds. A morphism $X \to S$ of algebraic schemes, locally of finite presentation, is said to be *smooth of relative dimension r* if it is given, locally on the source X, by equations which in differential geometry would satisfy the conditions of the *implicit funcion theorem*, namely: restricting to sufficiently small open neighborhoods, we have $X = \mathrm{Spec}(B)$, $S = \mathrm{Spec}(A)$, with $A \to B$ a map of rings which can be presented as $B = A[x_1, \dots, x_{m+r}]/(f_1, \dots, f_m)$, with the Jacobian

$$\det \left(\frac{\partial f_i}{\partial f_j} \right)_{i,j=1}^{m}$$

being invertible in A.

This condition on the Jacobian can be checked locally at every point of X. In particular, if $S = \mathrm{Spec}(k)$ with k a field, it is a condition on the local rings $\mathcal{O}_x$ for every $x \in X$, and in this case it is known to be equivalent to *regularity*, see [**Sta19**, Tag 00TV]: namely, to the condition that

$$\dim_{\mathrm{Krull}}(\mathcal{O}_x) = \dim_{\mathcal{O}_x/\mathfrak{m}_x} \mathfrak{m}_x/\mathfrak{m}_x^2. \tag{4.6.2.1}$$

When k is a topological field that is complete with respect to an absolute value, we can upgrade the functor from schemes to topological spaces to a functor from smooth k-schemes to analytic k-manifolds; those are, by definition, topological spaces endowed with a complete k-analytic class of charts or, equivalently, locally ringed spaces that are locally isomorphic to open subsets of k^n with their sheaf of k-analytic functions. We point the reader to [**Ser65**] for a more detailed discussion.

Proposition 4.6.3. *If X is a smooth scheme over a topological field k, there is a unique structure of analytic k-manifold on the topological space $Top(X) = X(k)$, such that for every open affine $U = Spec(B)$ with presentation*

$$B = k[x_1, \dots, x_{m+r}]/(f_1, \dots, f_m),$$

the open subset $U(k)$ is an analytic submanifold of k^{m+r}.

Proof. We may give ourselves such a presentation, with the Jacobian det $\left(\frac{\partial f_i}{\partial f_j}\right)_{i,j=1}^m$ being nonzero everywhere on $U(k)$. Then, $U(k)$ is the fiber over zero of a map $k^{m+r} \to k^m$ which is submersive at every point of that fiber, and the impicit function theorem implies that this fiber is an analytic submanifold of k^{m+r}. The resulting analytic structure is independent of the choice of (smooth) presentation. □

4.6.4. Weil restriction of scalars. For every finite-type scheme X' over $\mathbb{C}$, the set of $\mathbb{C}$-points $X'(\mathbb{C})$ can also be thought of as the set of $\mathbb{R}$-points $X(\mathbb{R})$ of a scheme X over $\mathbb{R}$.

More generally, let $S' \to S$ be a morphism of schemes, and $X' \to S'$ a scheme. The *Weil restriction of scalars*

$$\mathrm{Res}_{S'/S}(X')$$

is a S-scheme X representing the functor which assigns to any S-scheme T the set

$$\mathrm{Hom}_{S'}(T \times_S S', X')$$

of $T \times_S S'$-points on X'.

If such a scheme X exists, it is unique up to unique isomorphism, by Yoneda's lemma.

Theorem 4.6.5. *Assume that $S' \to S$ is finite and locally free, and X' is affine or, more generally, has the property that for every $s \in S$, any finite set of points P in the fiber of X' over s is contained in an affine open $U' \subset X'$. Then, the Weil restriction $Res_{S'/S}(X')$ exists.*

Proof. See [**BLR90**, Theorem 7.6.4]. We just explain how to write down equations when everything is affine, and $S' \to S$ is free:

Let $S = \mathrm{Spec} R$, $S' = \mathrm{Spec} R'$, where R' is free and of finite type as an R-module. Choose free generators:

$$R' = Re_1 \oplus \cdots \oplus Re_n.$$

Assume that $X' = \mathrm{Spec} R'[\underline{t}]/(f_1, \ldots, f_r)$. Here, $\underline{t}$ denotes an m-tuple $(t_1, \ldots, t_m)$, but we won't explicitly write the indices $1, \ldots, m$, in order to avoid confusion, as we are about to clone the m-tuple. Namely, consider the linear combination

$$e_1\underline{t}_1 + e_2\underline{t}_2 + \cdots + e_n\underline{t}_n \in R'[\underline{t}_1, \ldots \underline{t}_n],$$

where each $\underline{t}_k$ denotes an m-tuple. For each j, write

$$f_j(e_1\underline{t}_1 + e_2\underline{t}_2 + \cdots + e_n\underline{t}_n) = \sum_{k=1}^{n} c_{jk}(\underline{t}_1, \ldots \underline{t}_n)e_k,$$

where the $c_{jk} \in R[\underline{t}_1, \ldots \underline{t}_n]$.

Then, the restriction of scalars X can be presented as the spectrum of the ring

$$R[\underline{t}_1, \ldots \underline{t}_n]/(c_{jk})_{1\le k\le n}^{1\le j\le r}.$$

□

4.6.6. Smoothness of group schemes in characteristic zero.

Theorem 4.6.7. *If G is a group scheme of finite type over a field k of characteristic zero, then G is smooth over k.*

Proof. A summary of the proof: By homogeneity, and the fact that every algebraic variety contains a regular point, the reduced group scheme associated to G is smooth. Thus, a group is smooth iff it is reduced, which again by homogeneity reduces to the local ring at the identity.

Let R be the local ring $\mathcal{O}[G]_{\mathfrak{m}}$, where $\mathfrak{m} = \mathfrak{m}_e$ is the maximal ideal of the structure sheaf at the identity of G. We need to show that it contains no nilpotents. The comultiplication

$$\Delta : R \to R \times R$$

induced by the multiplication map $G \times G \to G$ sends any $a \in \mathfrak{m}$ to $1 \otimes a + a \otimes 1$ module $\mathfrak{m} \otimes \mathfrak{m}$. (Exercise in Hopf algebras!) For a nilpotent element a with $a^n = 0$ and n minimal such, we will have

$$0 = \Delta(a^n) = (\Delta(a))^n \equiv na^{n-1} \otimes a \mod \left(a^{n-1}\mathfrak{m} \otimes A + A \otimes \mathfrak{m}^2\right),$$

and since $na^{n-1} \notin a^{n-1}\mathfrak{m}$ (characteristic zero plus minimality of n!), we must have $a \in \mathfrak{m}^2$.

But then, $\mathfrak{m}_e/\mathfrak{m}_e^2$ coincides with the corresponding quotient for the reduction of G, which by regularity has dimension equal to the dimension of the ring. Thus, the local ring $\mathcal{O}[G]_{\mathfrak{m}_e}$ is regular. See [**Mil12**, §VI.9] for details on this proof, and [**Sta19**, Tag 047N] for a more abstract thread of arguments, which boils down to essentially the same calculation. □

4.7. Compact Lie groups are algebraic

An amazing fact is that the passage from real algebraic groups to Lie groups also works the other way in the case of compact Lie groups: they can all be realized as the points of a real algebraic group, as was proven by Weyl.

Proposition 4.7.1. *Every compact Lie group has a faithful (i.e. trivial kernel), finite-dimensional representation.*

Proof. Let $\pi_1, \pi_2, \dots$ be an enumeration of the irreducible representations of G. We already know from the Peter–Weyl theorem that they are finite-dimensional. For every n, let G_n be the kernel of the map: $G \to \mathrm{GL}(\pi_1 \oplus \cdots \oplus \pi_n)$. Hence, we have a sequence of closed subgroups:

$$G = G_0 \supset G_1 \supset G_2 \supset \dots .$$

We claim that every such sequence terminates. Indeed, by Cartan's theorem 4.5.2, we know that all G_n are Lie groups, therefore the dimension of G_n has to stabilize after some n. But then, the induced map of Lie algebras $\mathfrak{g}_{n+1} \hookrightarrow \mathfrak{g}_n$ will be an isomorphism, which means that the identity components G_n^0, G_{n+1}^0 are eventually equal, and since connected components in Lie groups are both open and closed, by compactness each G_n has a finite number of connected components, so the sequence has to terminate.

On the other hand, the intersection of the G_i's is (again by Peter–Weyl) the kernel of the left regular representation of G on $L^2(G)$, hence trivial. □

The second element is of invariant-theoretic nature. For this, let $G \to \mathrm{GL}(V)$ be a (complex), finite-dimensional representation of G and consider it as a real representation by regarding V as a real vector space. (This is the baby case of "restriction of scalars".) Accordingly, $\mathrm{GL}(V)$ is considered as an algebraic group over $\mathbb{R}$ (by restriction of scalars). Notice that the Zariski closure[2] of the image of G is a real algebraic subgroup. We need to show that it coincides with G. One thing that G and its Zariski closure have in common is the set of invariants on the polynomial ring $\mathbb{R}[V]$. Recall that the polynomial ring $\mathbb{R}[V]$ is (essentially, by definition) the symmetric algebra on the dual space $S^\bullet V^*$.

Proposition 4.7.2. *For each orbit X of a compact group G on the space V of a finite-dimensional real representation, there is a canonical real algebraic subset Y, defined as the fiber over the image of X under the map $V \to V//G := \mathit{Spec}\mathbb{R}[V]^G$, such that $X = Y(\mathbb{R})$.*

The compact group in the proposition is not required to be a Lie group.

Proof. We consider the map $V \to V//G := \mathrm{Spec}\mathbb{R}[V]^G$, and the induced map on $\mathbb{R}$-points: $V(\mathbb{R}) \to V//G(\mathbb{R})$. Clearly, the preimage of any point is a union of G-orbits. We claim:

> The preimage of every $\mathbb{R}$-point contains at most one G-orbit on $V(\mathbb{R})$.

This will be enough to prove the first claim: Since the preimage is an algebraic variety over $\mathbb{R}$, it means that G-orbits are the $\mathbb{R}$-points of algebraic varieties (maybe empty, because the preimage of an $\mathbb{R}$-point does not need to contain any $\mathbb{R}$-points – for instance, consider the quotient of $\mathbb{C}^\times$ by the circle group).

To prove the claim we must show that if Y_1, Y_2 are two distinct G-orbits on $V(\mathbb{R})$, then there is a G-invariant polynomial which takes different values on Y_1 and Y_2 (i.e. the ring of invariant polynomials separates G-orbits).

Notice that $\mathbb{R}[V]$ is a locally finite representation of G (this follows by its identification with $S^\bullet V^*$), and therefore by the Peter–Weyl theorems it is completely reducible. If we fix two points $y_1 \in Y_1$ and $y_2 \in Y_2$, then the integrals:

$$\int_G f(y_i \cdot g)dg$$

represent two G-invariant functionals ℓ_1, ℓ_2 on the space of continuous functions on V. They obviously factor through restriction to the compact subset $Y_1 \cup Y_2$, and by the Stone–Weierstrass theorem the restriction of polynomials is dense in the space of continuous functions on $Y_1 \cup Y_2$. Therefore, ℓ_1 and ℓ_2, when restricted to $\mathbb{R}[V]$, are linearly independent, i.e. ℓ_2 is non-zero on the kernel W of ℓ_1.

Hence, ℓ_2 defines a G-invariant functional: $W \to \mathbb{C}$, and by complete reducibility this splits; in particular, *there is a G-invariant element $f \in W$ with $\ell_2(f) \neq 0$.* That is, there is a G-invariant polynomial on V whose integral over Y_1 is zero and whose integral over Y_2 is non-zero. But this means that its value on Y_1 is zero and its value on Y_2 is non-zero, which is what we wanted to prove. ☐

[2]It is important here that we have restricted scalars to $\mathbb{R}$, because the Zariski closure depends on whether we consider $\mathrm{GL}(V)$ as a complex or as a real variety; for example, the Zariski closure of the circle group S^1 in $\mathbb{C}^\times$ is S^1 or $\mathbb{C}^\times$, according as $\mathbb{C}^\times$ is considered as a real or complex variety.

Remarks 4.7.3. (1) A similar argument works to establish the following important result: Let G be a reductive algebraic group over an algebraically closed field k in characteristic zero. We have not defined "reductive", but in characteristic zero this is equivalent to the statement that every algebraic representation of G is completely reducible. Let X be an affine variety on which G acts. Then the closed points of $X//G := \mathrm{Spec} k[X]^G$ are in bijection with (Zariski) closed orbits of G on X.

Here is the proof: Let Y_1, Y_2 be two closed orbits and consider the G-stable ideal $I \subset k[X]$ of regular functions vanishing on Y_1. Restriction to Y_2 gives a map: $I \to k[Y_2]$, and the image I' has to be non-zero because otherwise Y_2 would be in the Zariski closure of Y_1. But since Y_2 is a Zariski-closed orbit, a non-zero ideal coincides with the whole ring, therefore the image I' of I contains constant functions. By reductivity, there is a G-invariant quotient of I', hence a G-invariant quotient of I. By reductivity, again, I has a G-invariant element whose image in I' is non-zero. In other words, Y_1 and Y_2 are separated by G-invariant regular functions.

(2) Proposition 4.7.2 is not true for non-compact groups. For instance, not only is the subgroup:

$$\left\{ \begin{pmatrix} 1 & & \\ x & t & \\ y & & t^\alpha \end{pmatrix} : x, y \in \mathbb{R}, t \in \mathbb{R}_+^\times \right\}$$

of $\mathrm{GL}_3(\mathbb{R})$ (where α is an irrational number) not an algebraic subgroup of GL_3, but it is not isomorphic to (the $\mathbb{R}$-points of) *any* real algebraic group.[3]

Given, now, a compact group G, denote by $\mathbb{R}[G]$ the space of $\mathbb{R}$-valued functions which are *finite* under left (or, equivalently, right) translation by G.

Proposition 4.7.4. $\mathbb{R}[G]$ *is a finitely generated, commutative Hopf algebra.*

Recall that a Hopf algebra is an algebra A which also has structures which correspond to the axioms of a group (if A were an algebra of functions on the group): a *comultiplication* $A \to A \otimes A$, a *counit* $A \to \mathbb{R}$, and an *antipode* $A \to A$ satisfying certain natural axioms.

Proof. The structure of a commutative algebra follows once we observe that left-finiteness is preserved under tensor products, and the multiplication map $C(G) \otimes C(G) \to C(G)$ is left-equivariant, hence preserves finiteness.

The structure of a commutative Hopf algebra is obvious for the space $C(G)$ of continuous functions; moreover, since left- and right-finiteness are equivalent, the comultiplication $C(G) \to C(G \times G)$, which sends f to the function $f(g_1, g_2) = f(g_1 g_2)$, preserves finiteness, and the finite vectors of $C(G \times G)$ are $\mathbb{R}[G] \otimes \mathbb{R}[G]$.

There remains to argue about finite generation. Let $G \to \mathrm{GL}(V)$ be a faithful representation. By the Stone–Weierstrass theorem, the restriction of polynomial functions on $\mathrm{GL}(V)$ (viewed here as a real algebraic variety, that is, as $\mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathrm{GL}_V$) is dense in $C(G)$. On the other hand, it is a $G \times G$-invariant subspace of $\mathbb{R}[G]$. The (real!) representation $\mathbb{R}[G]$ being semisimple, with its irreducible components

[3]For details, cf.
http://terrytao.wordpress.com/2011/06/25/two-small-facts-about-lie-groups/.

being orthogonal in the (real-valued) $L^2(G)$ the only way that the restriction of polynomials on $\mathrm{GL}(V)$ be dense is that it is equal to $\mathbb{R}[G]$. Hence, $\mathbb{R}[G]$ is of finite type. $\square$

Let, now $\mathbf{G} = \mathrm{Spec}\mathbb{R}[G]$. Evaluation at the points of G gives rise to a natural map $G \to \mathbf{G}(\mathbb{R}) = \mathrm{Hom}(\mathbb{R}[G], \mathbb{R})$. We have arrived at Weyl's theorem:

Theorem 4.7.5. *For every compact Lie group G, setting $\mathbf{G} = Spec\mathbb{R}[G]$, the natural map $G \to \mathbf{G}(\mathbb{R})$ is an isomorphism of Lie groups. Every continuous, finite-dimensional (complex) representation $G \to GL(V)$ factors through an algebraic representation $\mathbf{G}_{\mathbb{C}} \to GL(V)$.*

Proof. We start with the second claim: Polynomial functions on $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_V$ restrict to finite function on G, which gives rise to a morphism $\mathbf{G} \to \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_V$. By the universal property of Weil restriction, this is the same as a morphism $\mathbf{G}_{\mathbb{C}} \to \mathrm{GL}_V$.

For the first claim, choose a faithful, finite-dimensional representation $G \to \mathrm{GL}(V)$. Replacing the space V of Proposition 4.7.2 with $\mathrm{End}(V) = V^* \otimes V$, and using the embedding $\mathrm{GL}_V \hookrightarrow \mathrm{End}(V)$, there is a canonical real algebraic subvariety $Y \subset V^* \otimes V$, namely, the fiber of the image of G under the map $V^* \otimes V \to (V^* \otimes V)//G$, such that $G = Y(\mathbb{R})$. On the other hand, the map to $V^* \otimes V$ factors through $\mathbf{G}$, which clearly has to belong to Y, therefore at the level of points $G = \mathbf{G}(\mathbb{R})$. The map $G \to \mathbf{G}(\mathbb{R})$ is also a homomorphism of Lie groups, so it is an isomorphism. $\square$

We will later see a strengthening of this theorem, due to Chevalley (Theorem 8.5.6).

4.8. Other chapters

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(2) Basic Representation Theory
(3) Representations of compact groups
(4) Lie groups and Lie algebras: general properties
(5) Structure of finite-dimensional Lie algebras
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CHAPTER 5

Structure of finite-dimensional Lie algebras

Good references for the structure of Lie algebras include Humphreys' book [**Hum78**] and Sternberg's notes [**Ste09**].

5.1. Nilpotency, solvability, semisimplicity

In this chapter, all Lie algebras are taken to be *finite-dimensional* over a field k.

5.1.1. Definitions.

Definition 5.1.2. An *ideal* of a Lie algebra $\mathfrak{g}$ is an $\mathrm{ad}(\mathfrak{g})$-stable subspace (automatically a Lie subalgebra) of $\mathfrak{g}$.

The *quotient* of a Lie algebra $\mathfrak{g}$ by an ideal $\mathfrak{h}$ is the vector space $\mathfrak{g}/\mathfrak{h}$, equipped with the Lie algebra structure descending from $\mathfrak{g}$.

Definition 5.1.3. The *lower central series* of a Lie algebra $\mathfrak{g}$ is the descending sequence of ideals defined by

$$C^0\mathfrak{g} = \mathfrak{g},$$

$$C^{i+1}\mathfrak{g} = [\mathfrak{g}, C^i\mathfrak{g}].$$

A Lie algebra is called *nilpotent* if its lower central series terminates, i.e., if $C^n\mathfrak{g} = 0$ for some n.

The *derived series* of a Lie algebra $\mathfrak{g}$ is the descending sequence of ideals

$$D^0\mathfrak{g} = \mathfrak{g},$$

$$D^{i+1}\mathfrak{g} = [D^i\mathfrak{g}, D^i\mathfrak{g}].$$

A Lie algebra is called *solvable* if its derived series terminates, i.e., if $D^n\mathfrak{g} = 0$ for some n.

A Lie algebra is *semisimple* if it does not have any nonzero solvable ideals, and *simple* if it is non-abelian, and has no nonzero proper ideals.

Example 5.1.4. The Lie algebra of strictly upper triangular $n \times n$ matrices (i.e., with zeroes on the diagonal) is nilpotent. The Lie algebra of upper triangular $n \times n$ matrices is solvable.

Lemma 5.1.5. *The center of a (nontrivial) nilpotent Lie algebra is always nontrivial.*

Proof. The last nontrivial element of its lower central series belongs to the center. □

Lemma 5.1.6. *A Lie algebra $\mathfrak{g}$ is nilpotent if and only if there exists a finite decreasing filtration by ideals*

$$\mathfrak{g} = \mathfrak{g}^0 \supset \mathfrak{g}^1 \cdots \supset \mathfrak{g}^n = 0$$

such that $[\mathfrak{g}, \mathfrak{g}^i] \subset [\mathfrak{g}^{i+1}]$.

The sum of two nilpotent ideals in a Lie algebra $\mathfrak{g}$ is also a nilpotent ideal.

Subalgebras, quotient algebras, and extensions of solvable Lie algebras by solvable Lie algebras are solvable.

A Lie algebra $\mathfrak{g}$ is solvable if and only if there exists a finite decreasing filtration by subalgebras

$$\mathfrak{g} = \mathfrak{g}^0 \supset \mathfrak{g}^1 \supset \cdots \supset \mathfrak{g}^n = 0$$

such that $\mathfrak{g}^{i+1}$ is an ideal in $\mathfrak{g}^i$, and the quotient algebra $\mathfrak{g}^i/\mathfrak{g}^{i+1}$ is abelian.

Proof. Left to the reader. □

Definition 5.1.7. The *radical* of a Lie algebra is its largest solvable ideal.

The *nilradical*, or *nilpotent radical*, of a Lie algebra is its largest nilpotent ideal.

The definition of the nilradical and radical makes sense in view of Lemma 5.1.6: For the nilradical, the lemma ensures that the sum of all nilpotent ideals is a nilpotent ideal. For the radical, if $\mathfrak{a}, \mathfrak{b}$ are solvable ideals, then $\mathfrak{a} + \mathfrak{b}$ is an ideal, and it is isomorphic as a Lie algebra to $(\mathfrak{a} \oplus \mathfrak{b})/(\mathfrak{a} \cap \mathfrak{b})$, which is solvable by Lemma 5.1.6.

Thus, any Lie algebra $\mathfrak{g}$ admits a canonical filtration

$$0 \to R(\mathfrak{g}) \to \mathfrak{g} \to \mathfrak{g}_{ss} \to 0, \tag{5.1.7.1}$$

where $R(\mathfrak{g})$ is the radical of $\mathfrak{g}$, and $\mathfrak{g}_{ss}$ is semisimple. Indeed, the preimage of any solvable ideal in $\mathfrak{g}_{ss}$ is a solvable ideal in $\mathfrak{g}$, and therefore has to equal $R(\mathfrak{g})$.

Note that the definition of nilradical given here does not coincide with Bourbaki's, who wants to avoid calling an abelian Lie algebra its own nilradical, but is quite standard in other references.

Example 5.1.8. In $\mathfrak{gl}_{2n}$, let $\mathfrak{g}$ be the subalgebra of matrices whose lower left $n \times n$-block is zero. The radical of $\mathfrak{g}$ consists of matrices of the form

$$\begin{pmatrix} aI & * \\ 0 & bI \end{pmatrix}$$

while its nilpotent radical consists of matrices of the form

$$\begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$$

(all blocks $n \times n$).

5.1.9. Engel's theorem.

Theorem 5.1.10 (Engel's theorem)**.** *If V is a nonzero finite-dimensional vector space, and $\mathfrak{g} \subset \mathfrak{gl}(V)$ is a Lie subalgebra consisting of nilpotent operators, then there is a nonzero $v \in V$ with $Xv = 0$ for all $X \in \mathfrak{g}$.*

A finite-dimensional Lie algebra $\mathfrak{g}$ is nilpotent if and only if $ad(X)$ is a nilpotent operator, for every $X \in \mathfrak{g}$.

Proof. We proceed by induction on the dimension of $\mathfrak{g}$, the case of dimension 1 being trivial.

In higher dimension, let us consider, besides the given representation V, the adjoint representation of $\mathfrak{g}$ on itself, as well. We claim that $\mathrm{ad}(X) \in \mathrm{End}(\mathfrak{g})$ is a nilpotent operator, for every X. Indeed, $\mathrm{ad}(X)$ is the restriction to $\mathfrak{g}$ of the operator on $\mathrm{End}(V)$:

$$\mathrm{ad}(X) = L_X - R_X,$$

where $L_X(Y) = XY$ and $R_X(Y) = YX \in \mathrm{End}(V)$. Left and right multiplication commute, and there is, by assumption, an n such that $X^n = 0$, so by the binomial formula:

$$\mathrm{ad}(X)^{2n} = (L_X - R_X)^{2n} = \sum_{k=0}^{2n} \binom{2n}{k} L_X^k R_X^{2n-k} = 0.$$

Now, take any nontrivial proper subalgebra $\mathfrak{h} \subset \mathfrak{g}$. Since $\mathfrak{h}$ is $\mathrm{ad}(\mathfrak{h})$-stable, we get an action of $\mathfrak{h}$ on the vector space $\mathfrak{g}/\mathfrak{h}$, i.e., a morphism of Lie algebras

$$\mathfrak{h} \to \mathfrak{gl}(\mathfrak{g}/\mathfrak{h}).$$

Let $\bar{\mathfrak{h}}$ denote its image. Since $\mathrm{ad}(X)$ is nilpotent for every $X \in \mathfrak{h}$, the same will hold for $\bar{\mathfrak{h}}$. By the induction hypothesis, there is a nontrivial subspace of $\mathfrak{g}/\mathfrak{h}$ which is killed by $\mathrm{ad}(h)$ or, equivalently, *the normalizer of* $\mathfrak{h}$ *in* $\mathfrak{g}$ *is strictly larger than* $\mathfrak{h}$. By successively replacing $\mathfrak{h}$ by its normalizer, we arrive at a proper Lie subalgebra $\mathfrak{h}$ whose normalizer is $\mathfrak{g}$, i.e., $\mathfrak{h}$ is an ideal in $\mathfrak{g}$.

By the induction hypothesis, the kernel V_0 of $\mathfrak{h}$ acting on V is nontrivial. But this kernel is acted by the quotient Lie algebra $\mathfrak{g}/\mathfrak{h}$, and by the induction hypothesis again, it contains a nonzero vector killed by $\mathfrak{g}$.

For the second statement, take $V = \mathfrak{g}$. If $\mathrm{ad}(X)$ is nilpotent for every $X \in \mathfrak{g}$, the first statement implies that the center $Z(\mathfrak{g})$ is nontrivial. Replacing $\mathfrak{g}$ by $\mathfrak{g}/Z(\mathfrak{g})$, we get a sequence of ideals

$$\mathfrak{g} = \mathfrak{g}^0 \supset \mathfrak{g}^1 \cdots \supset \mathfrak{g}^n = 0$$

such that $[\mathfrak{g}, \mathfrak{g}^i] \subset [\mathfrak{g}^{i+1}]$. By Lemma 5.1.6, $\mathfrak{g}$ is nilpotent. The other direction is immediate from the definitions: If $\mathfrak{g}$ is nilpotent, then there is an n such that $\mathrm{ad}(X)^n = 0$ for every $X \in \mathfrak{g}$.

$\square$

5.1.11. Lie's theorem.

Theorem 5.1.12. *If V is a finite-dimensional vector space over an algebraically closed field k in characteristic zero, any solvable Lie subalgebra $\mathfrak{g}$ of $\mathfrak{gl}(V)$ has a nonzero eigenvector, and hence a full flag*

$$0 = V_0 \subset V_1 \subset \cdots \subset V_n = V,$$

where V_i has dimension i and is stable under $\mathfrak{g}$.

Proof. As in the proof of Engel's theorem 5.1.10, we argue by induction on the dimension of $\mathfrak{g}$, with the case $\dim \mathfrak{g} = 0$ being trivial. Let, now, $\mathfrak{h} \subset \mathfrak{g}$ be an ideal of codimension one (which exists because $\mathfrak{g}$ is solvable); by induction, $\mathfrak{h}$ has a non-trivial eigenspace V_λ for some linear functional $\lambda : \mathfrak{h} \to k$. Clearly, λ factors through the abelianization $\mathfrak{h}^{\mathrm{ab}} = \mathfrak{h}/[\mathfrak{h}, \mathfrak{h}]$.

Now, we claim that V_λ is $\mathfrak{g}$-stable, i.e., if $x \in \mathfrak{g}$ and $y \in \mathfrak{h}$ then $yxv = \lambda(y)xv$ for all $v \in V_\lambda$. We have $yxv = xyv - [x, y]v = \lambda(y)xv - [x, y]v$, so we need to prove that $[x, y]v = 0$. Consider the flag $0 = W_0 \subset W_1 \subset \dots$ where W_i is spanned by $v, xv, \dots, x^i v$. It is easy to see that y stabilizes this flag, and acts on W_i/W_{i-1} by $\lambda(y)$. Therefore, if $W \subset V$ denotes the maximal subspace of this flag, and $\dim W = n$, the trace of y acting on W is $n\lambda(y)$. This holds for every element of $\mathfrak{h}$, hence also for the element $[x, y]$. Since this is a commutator of two operators, we get

$$0 = \mathrm{tr}([x, y]) = n\lambda([x, y]),$$

and since we are in characteristic zero, $\lambda([x, y]) = 0$.

Thus, V_λ is $\mathfrak{g}$-stable, and because the field is algebraically closed, we can find an eigenvector of $x \in \mathfrak{g} - \mathfrak{h}$ on V_λ, proving the existence of an eigenvector.

The existence of the flag now follows by induction, starting with $V_0 = 0$ and considering the representation of $\mathfrak{g}$ on the quotient spaces V/V_i. □

Remark 5.1.13. The assumption on the characteristic is really necessary; for example, the Lie algebra $\mathfrak{sl}_2$ is solvable in characteristic 2, but its standard representation does not have an eigenspace.

5.1.14. Cartan subalgebras.

Definition 5.1.15. A *Cartan subalgebra* of a Lie algebra is a nilpotent, self-normalizing subalgebra. The *rank* of a Lie algebra is the dimension of a Cartan subalgebra.

We will construct Cartan subalgebras as nilspaces (generalized eigenspaces of zero under the adjoint representation) of *s-regular* elements. Then we will show that they are all conjugate to each other; in particular, the rank is uniquely defined.

Definition 5.1.16. A *regular element* of a Lie algebra $\mathfrak{g}$ is an element whose centralizer is of minimal dimension. An *s-regular element* $X \in \mathfrak{g}$ is an element whose generalized centralizer (i.e., 0-generalized eigenspace under ad) is of minimal dimension. A *regular semisimple* element is a regular element which is also semisimple; equivalently, an s-regular element which is also semisimple.

This terminology "s-regular" is not standard: in many books on Lie algebras, the word "regular" is used for "s-regular". However, the standard use of "regular" nowadays, at least in the case of semisimple Lie algebras, is to refer to elements with minimal *zero eigenspace* (i.e., centralizer), instead of generalized eigenspace. This includes non-semisimple elements, while s-regular, in the case of semisimple Lie algebras in characteristic zero, is equivalent to *regular semisimple*, see Proposition 5.3.7.

Lemma 5.1.17. *s-regular elements form a nonempty Zariski open subset of* $\mathfrak{g}$.

Proof. The dimension of the zero generalized eigenspace is the highest power of t which divides the characteristic polynomial of $\mathrm{ad}(X)$, and therefore having a minimal such dimension is a Zariski open condition. □

Proposition 5.1.18. *The generalized nilspace (under the adjoint representation) of an s-regular element is a Cartan subalgebra.*

Proof. Let X be the s-regular element and $\mathfrak{h}$ its generalized nilspace. We will prove that $\mathfrak{h}$ is nilpotent; equivalently, by Engel's theorem, that the restriction of $\mathrm{ad}(Y)$ to $\mathfrak{h}$, for any $Y \in \mathfrak{h}$, is nilpotent. Let $U \subset \mathfrak{h}$ be the subset of elements which fail to satisfy this; it is a Zariski open subset (again by considerations of the characteristic polynomial). Let $V \subset \mathfrak{h}$ be the subset of elements which act invertibly on $\mathfrak{g}/\mathfrak{h}$. It is again a Zariski open subset, and non-empty since $X \in V$. If $U \neq \emptyset$ then $U \cap V \neq \emptyset$, i.e. there exists an element $Y \in \mathfrak{h}$ such that the dimension of the zero generalized eigenspace for Y is less than the dimension of $\mathfrak{h}$, a contradiction by the s-regularity of X. Thus, $\mathfrak{h}$ is nilpotent.

If Z normalizes $\mathfrak{h}$ then $[Z, X] \in \mathfrak{h}$ which implies that Z is in the generalized nilspace of X, i.e. in $\mathfrak{h}$. □

Hence, every Lie algebra has Cartan subalgebras. We will eventually prove that any two Cartan subalgebras are conjugate (over the algebraic closure) by the group of inner automorphisms of $\mathfrak{g}$ and, in particular, equal to nilspaces of s-regular elements.

5.1.19. Derivations and semidirect products.

Definition 5.1.20. A *derivation* of a Lie algebra $\mathfrak{g}$ is a linear map $D : \mathfrak{g} \to \mathfrak{g}$ satisfying $D([X,Y]) = [X, D(Y)] + [D(X), Y]$.

Let

$$0 \to \mathfrak{a} \to \mathfrak{g} \to \mathfrak{h} \to 0$$

be a split short exact sequence of Lie algebras, i.e., $\mathfrak{h}$ embeds in $\mathfrak{g}$ as a Lie subalgebra. Then, $\mathfrak{h}$ acts on $\mathfrak{a}$ by derivations, i.e., through a morphism $\mathfrak{h} \to \mathrm{Der}(\mathfrak{a})$. Vice versa, given such a morphism, it defines a split extension of $\mathfrak{h}$ by $\mathfrak{a}$ in Lie algebras. This is the *semidirect product* of $\mathfrak{a}$ and $\mathfrak{h}$.

Remarks 5.1.21.
(1) This is a very natural extension of the definition of derivation for an associative algebra, since such a derivation induces a derivation as above on the associated Lie algebra. Vice versa, a derivation of a Lie algebra induces a derivation of its universal enveloping algebra.
(2) Derivations form a Lie subalgebra of $\mathrm{End}(\mathfrak{g})$.
(3) The adjoint representation $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ has image in $\mathrm{Der}(\mathfrak{g})$.

Definition 5.1.22. The derivations in the image of the adjoint map $\mathrm{ad} : \mathfrak{g} \to \mathrm{Der}(\mathfrak{g})$ are called *inner derivations*.

5.1.23. Jordan decomposition of endomorphisms. In this subsection, we prove that every endomorphism x of a finite-dimensional vector space V has a unique decomposition $x = x_s + x_n$, where x_s is semisimple, x_n is nilpotent, and x_s, x_n commute. We will see later (5.2.10) that such a decomposition also exists, and is unique, for *semisimple* Lie algebras, at least in characteristic zero, when the "semisimple" and "nilpotent" parts of an element are defined in terms of the adjoint representation.

Proposition 5.1.24. *Let V be a finite-dimensional vector space over k, and $x \in$ End(V). If k is algebraically closed, there is a unique pair of commuting elements (x_s, x_n) with x_s semisimple, x_n nilpotent, and $x = x_s + x_n$. Moreover, x_s and x_n belong to the subalgebra of operators generated by x.*

For general k, the elements x_s and x_n are defined over a purely inseparable algebraic extension of k.

Proof. First of all, the statement over an algebraically closed field implies the general statement: The uniqueness of x_s, x_n implies that they are fixed under the

Galois group of an algebraic closure of k, hence defined over a purely inseparable subextension.

Hence, we may assume that k is algebraically closed, so V decomposes as a direct sum of generalized eigenspaces V_{λ_i} for x. If such a decomposition exists, since the elements x_s, x_n commute with each other, hence with x, they must preserve the generalized eigenspaces V_{λ_i}. Thus, it is enough to prove existence and uniqueness when V itself is a generalized eigenspace, with eigenvalue λ. But then, we can take $x_s = \lambda I$, and $x_n = x - \lambda_I$. For any other choice $x = x'_s + x'_n$, the nilpotent element $x_n = x - \lambda I$ would have a Jordan decomposition $(x'_s - \lambda_I) + x'_n$, and since commuting nilpotent or semisimple elements have nilpotent, resp. semisimple sum, we would get an equality between the semisimple element $x'_s - \lambda I$ and the nilpotent element $x_n - x'_n$, which means that both are zero.

The element λI is (trivially) in the subalgebra generated by any operator, hence both x_s, x_n are in the subalgebra generated by x, when V is a generalized eigenspace. In the general case, if the characteristic polynomial of x is $\prod_i (T - \lambda_i)^{m_i}$, by the Chinese remainder theorem there are is a polynomial $p \in k[T]$ with $p \equiv \lambda_i \mod (T - \lambda_i)^{m_i}$, and then $p(x) = x_s$. □

The following tiny improvement of the proposition above will be useful in what follows:

Lemma 5.1.25. *In the setting of 5.1.24, x_s can be written as a polynomial of x with zero constant coefficient. For any field automorphism ϕ of k, if $\phi(x_s)$ denotes the semisimple element which acts on the λ-eigenspace of x_s by $\phi(\lambda)$, then $\phi(x_s)$ can also be written as such a polynomial of x. Finally, for any pair of subspaces $W_1 \subset W_2 \subset V$ with $xW_2 \subset W_1$, we also have $yW_2 \subset W_1$, where y is any of $x_s, x_n, \phi(x_s)$.*

Proof. In the proof of Proposition 5.1.24, we could have taken the polynomial p to satisfy the additional congruence $p = 0 \mod T$, if 0 is not among the eigenvalues. (If it is, this condition is among the ones imposed.) This proves the first claim.

For the second, we can similarly find a polynomial q with $q \equiv \phi(\lambda_i) \mod (T - \lambda_i)^{m_i}$ and zero constant coefficient.

For the last claim, if $xW_2 \subset W_1$ then the same is true when x is replaced by $p(x)$, for every polynomial with zero constant coefficient. □

The Jordan decomposition is preserved under tensor operations on the category of representations. (This will be generalized later, Theorem 5.2.15, for arbitrary morphisms of semisimple Lie algebras.)

Notice that for two representations V, W of a Lie algebra $\mathfrak{g}$, the tensor product $V \otimes W$ is a representation under

$$x \cdot (v \otimes w) = (xv) \otimes w + v \otimes (xw).$$

The dual representation on V^* is defined so that the defining pairing

$$V \otimes V^* \to k$$

is invariant, i.e.,

$$\langle xv, v^* \rangle + \langle v, xv^* \rangle = 0. \tag{5.1.25.1}$$

Lemma 5.1.26. *Let V be a finite-dimensional vector space, and $W = V^{\otimes^a} \otimes (V^*)^{\otimes^b}$, for some a, b. For $x \in End(V)$, denote by x' the corresponding endomorphism of W.*

If $x = x_s + x_n$ is the Jordan decomposition of x, then $(x_s)' + (x_n)'$ is the Jordan decomposition of x'.

Proof. It is clear that $x' = (x_s)' + (x_n)'$, with $(x_s)'$ and $(x_n)'$ commuting. It is also clear that $(x_s)'$ is semisimple. To see that $(x_n)'$ is nilpotent, it is enough to show that, if $y \in \mathrm{End}(V_1)$, $z \in \mathrm{End}(V_2)$ are nilpotent endomorphisms of two vector spaces, then the endomorphm $v_1 \otimes v_2 \mapsto (yv_1) \otimes v_2 + v_1 \otimes (zv_2)$ of $V_1 \otimes V_2$ is nilpotent, which is clear by raising it to a sufficiently high power. □

5.1.27. Cartan's criterion for solvability.

Theorem 5.1.28 (Cartan's criterion)**.** *Let $\mathfrak{g} \subset End(V)$ be a Lie subalgebra, where V is a finite-dimensional vector space over a field k in characteristic zero. The Lie algebra $\mathfrak{g}$ is solvable if and only if $tr(xy) = 0$ for all $x \in \mathfrak{g}$, $y \in [\mathfrak{g}, \mathfrak{g}]$.*

Proof. We may assume that the field is algebraically closed, and then by Lie's theorem 5.1.12, if $\mathfrak{g}$ is solvable stabilizes a flag $V_0 \subset V_1 \subset \cdots \subset V_n = V$ with $\dim V_i = i$. Then, every $y \in [\mathfrak{g}, \mathfrak{g}]$ maps $V_i \to V_{i-1}$, hence so does any product xy with $x \in \mathfrak{g}$, so $\mathrm{tr}(xy) = 0$.

Vice versa, assume that $\mathrm{tr}(xy) = 0$ for all $x \in \mathfrak{g}$, $y \in [\mathfrak{g}, \mathfrak{g}]$. To prove that $\mathfrak{g}$ is solvable, it is enough to prove that $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent. By Engel's theorem 5.1.10, this is equivalent to showing that any $y \in [\mathfrak{g}, \mathfrak{g}]$ is nilpotent.

The rest of the proof is written under the assumption that $k = \mathbb{C}$, so that complex conjugate of a semisimple endomorphism makes sense, as in Lemma 5.1.25. For a general field in characteristic zero, complex conjugation should be replaced by other field automorphisms.

Use the Jordan decomposition $y = y_s + y_n$, and observe that $y_s = 0$ iff $\mathrm{tr}(y \cdot \overline{y_s}) = 0$, since the generalized eigenvalues of $y \cdot \overline{y_s}$ are the absolute values of the squares of those of y_s. Now, the element $\overline{y_s}$ does not necessarily belong to $\mathfrak{g}$, so we argue as follows: writing y as a linear combination of commutators in $\mathfrak{g}$:

$$y = \sum_i [x_i, z_i],$$

we easily see that

$$\mathrm{tr}(y \cdot \overline{y_s}) = \sum_i \mathrm{tr} z_i [\overline{y_s}, x_i],$$

and it is enough to show that, even though $\overline{y_s}$ may not be in $\mathfrak{g}$, the operator $\mathrm{ad}(\overline{y_s})$ maps $\mathfrak{g} \to [\mathfrak{g}, \mathfrak{g}]$. By Lemma 5.1.26, and using the fact that for the Lie algebra $\mathfrak{gl}(V)$, the adjoint representation coincides with $V \otimes V^*$, we have $\mathrm{ad}(\overline{y_s}) = \overline{(\mathrm{ad}(y))_s}$. Since $\mathrm{ad}(y)$ maps $\mathfrak{g}$ into $[\mathfrak{g}, \mathfrak{g}]$, by Lemma 5.1.25 the same is true for $\overline{(\mathrm{ad}(y))_s}$.

□

Definition 5.1.29. The *Killing form* on a Lie algebra $\mathfrak{g}$ is the symmetric bilinear form

$$B : S^2\mathfrak{g} \to k$$

given by

$$B(x, y) = \mathrm{tr}(\mathrm{ad}(x)\mathrm{ad}(y)).$$

Lemma 5.1.30. *The Killing form is invariant under the adjoint representation, i.e., for all $x, y, z \in \mathfrak{g}$,*

$$B(ad(x)(y), z) + B(y, ad(x)(z)) = 0.$$

Proof. Easy consequence of the Jacobi identity. □

Theorem 5.1.31. *If the Killing form of a Lie algebra is nondegenerate, the Lie algebra is semisimple. The converse holds in characteristic zero.*

Proof. If $\mathfrak{g}$ has a non-trivial solvable ideal, then it has a non-trivial abelian ideal $\mathfrak{a}$, and then the adjoint action of any $x \in \mathfrak{a}$ maps $\mathfrak{g} \to \mathfrak{a}$ and $\mathfrak{a} \to 0$. Moreover, any $y \in \mathfrak{g}$ preserves a, so $\mathrm{tr}(x)\mathrm{tr}(y)$ maps $\mathfrak{g} \to \mathfrak{a}$ and $\mathfrak{a} \to 0$, and therefore has trace zero. Thus, x is in the radical of the Killing form.

Vice versa, in characteristic zero, if $\mathfrak{h} \subset \mathfrak{g}$ is the radical of the Killing form $B_{\mathfrak{g}}$, it is easy to see from its invariance that $\mathfrak{h}$ is an ideal of $\mathfrak{g}$. For every ideal $\mathfrak{h} \subset \mathfrak{g}$ the Killing form $B_{\mathfrak{g}}$ of $\mathfrak{g}$, restricted to $\mathfrak{h}$, coincides with the Killing form $B_{\mathfrak{h}}$ of $\mathfrak{h}$: indeed, $\mathrm{ad}(x)$ maps $\mathfrak{g} \to \mathfrak{h}$ for $x \in \mathfrak{h}$, so the quotient space $\mathfrak{g}/\mathfrak{h}$ does not contribute to the trace of any product of such endomorphisms. Hence, by Cartan's criterion, Theorem 5.1.28, $\mathfrak{h}$ is solvable, which implies that $\mathfrak{g}$ is not semisimple. □

Remark 5.1.32. The converse fails in positive characteristic, e.g., the Lie algebra $\mathfrak{sl}_p$ is semisimple if $p \neq 2$ is the characteristic of the field, but its Killing form vanishes.

5.2. Semisimple Lie algebras

In this section, all Lie algebras and vector spaces are finite-dimensional, unless otherwise noted. We will mostly be working in characteristic zero, except for some counterexamples that we give in positive characteristic.

Proposition 5.2.1. *Every finite-dimensional, semisimple Lie algebra $\mathfrak{g}$ in characteristic zero is a direct sum of simple Lie algebras in a unique way:*

$$\mathfrak{g} = \bigoplus_i \mathfrak{g}_i.$$

Its derived Lie algebra $[\mathfrak{g}, \mathfrak{g}]$ is equal to $\mathfrak{g}$, and its ideals are precisely the subsums of the simple summands $\mathfrak{g}_i$.

Proof. Let B be the Killing form. Since it is invariant and non-degenerate (by Theorem 5.1.31, the orthogonal complement of any ideal $\mathfrak{h} \subset \mathfrak{g}$ is also an ideal, and $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^{\perp}$. By induction on dimension, $\mathfrak{g}$ is a direct sum of simple Lie algebras.

On every simple summand $\mathfrak{h}$, $[\mathfrak{h}, \mathfrak{h}]$ is an ideal, and since $\mathfrak{h}$ is simple, we must have $[\mathfrak{h}, \mathfrak{h}] = \mathfrak{h}$, hence the same is true for (the direct sum of simple summands) $\mathfrak{g}$.

Fixing a direct sum decomposition $\mathfrak{g} = \bigoplus_i \mathfrak{g}_i$, the image of the projection of any ideal I to the summand $\mathfrak{g}_i$ is an ideal, hence the projection is onto or zero. If it is onto, we have $[\mathfrak{g}_i, I] = \mathfrak{g}_i$ by what was just proven, therefore $\mathfrak{g}_i \subset I$. This shows that ideals are precisely the direct summands $\mathfrak{g}_i$, which implies the uniqueness of the decomposition. □

Remark 5.2.2. Over a field k in positive characteristic p, this theorem does not need to hold. For example (see [**Rum**, §2.4]), there is a nonsplit extension

$$0 \to \mathfrak{h} = \mathfrak{sl}_2(k[z]/z^p) \to \mathfrak{g} \to k \to 0$$

where, for a vector space identification $\mathfrak{g} = \mathfrak{h} \oplus k$, the summand k acts by a certain derivation:

$$[(X, x), (Y, y)] = ([X, Y] - x\partial Y + y\partial X),$$

where, for $X = \begin{pmatrix} f(z) & g(z) \\ h(z) & -f(z) \end{pmatrix}$, we have $\partial X = \begin{pmatrix} f'(z) & g'(z) \\ h'(z) & -f'(z) \end{pmatrix}$. (This is the *semidirect product* of $\mathfrak{h}$ and $k \cdot \partial$, see 5.1.19.)

The derived Lie algebra $[\mathfrak{g}, \mathfrak{g}]$ is equal to $\mathfrak{h}$, which is also the unique minimal ideal.

5.2.3. The Casimir element. The universal enveloping algebra $U(\mathfrak{g})$ typically has a large center, even if $\mathfrak{g}$ does not (e.g., is semisimple). This is a very important reason for invoking the enveloping algebra. The structure of the center, over an algebraically closed field in characteristic zero, is described by the *Harish–Chandra isomorphism*, to be discussed later. For now, we focus on producing some elements in the center.

Let (π, V) be a finite-dimensional representation of a Lie algebra $\mathfrak{g}$, and assume that the trace pairing

$$(X, Y)_\pi = \operatorname{tr}(\pi(X)\pi(Y))$$

is nondegenerate. (In particular, the representation is faithful.)

It is immediate to see that the trace pairing is symmetric and invariant, hence defines a $\mathfrak{g}$-equivariant isomorphism

$$T^2\mathfrak{g} = \mathfrak{g} \otimes \mathfrak{g} \simeq \mathfrak{g}^* \otimes \mathfrak{g}.$$

Proposition 5.2.4. *If (π, V) is a finite-dimensional representation of $\mathfrak{g}$ with a nondegenerate trace pairing, used to identify $T^2\mathfrak{g}$ with End$(\mathfrak{g})$, the element $C_\pi \in U(\mathfrak{g})$ which is the image of the identity operator $I \in$ End$(\mathfrak{g})$ under the quotient $T\mathfrak{g} \to U(\mathfrak{g})$ lies in the center of $U(\mathfrak{g})$.*

If (π, V) is irreducible and k is algebraically closed, C_π acts by a scalar on V; that scalar is equal to dim$(\mathfrak{g})/$dim(V), if the denominator is prime to the characteristic of k.

Explicitly, if $(X_i)_i$ is a basis for $\mathfrak{g}$, and $(Y_i)_i$ is the dual basis with respect to the trace pairing of π, we have

$$C_\pi = \sum_i X_i Y_i.$$

Its definition, which does not make use of the basis, makes it clear that this element is independent of the basis.

Proof. Because of the invariance of the trace form, the adjoint representation of $\mathfrak{g}$ on the second graded piece $T^2\mathfrak{g}$ of the tensor algebra coincides with the representation of $\mathfrak{g}$ on $\mathfrak{g}^* \otimes \mathfrak{g} = \operatorname{End}(\mathfrak{g})$. Since the element $I \in \operatorname{End}(\mathfrak{g})$ is invariant, i.e., $\operatorname{ad}(X)(I) = 0$ for every $X \in \mathfrak{g}$, we have $\operatorname{ad}(X)(C_\pi) = 0$ for every $X \in \mathfrak{g}$, hence C_π is in the center of $U(\mathfrak{g})$.

If (π, V) is irreducible, since C_π is central, any eigenspace of C_π is fixed under $\mathfrak{g}$; thus, if the representation is irreducible and k is algebraically closed, C_π must act by a scalar. If the characteristic of k does not divide $\dim(V)$, that scalar can be computed as $\operatorname{tr}(\pi(C_\pi))/\dim(V)$, and using a dual basis $(X_i)_i$, $(Y_i)_i$, we have

$$\operatorname{tr}(\pi(C_\pi)) = \sum_i \operatorname{tr}(\pi(X_i)\pi(Y_i)) = \dim(\mathfrak{g}).$$

(Notice that the representation is automatically faithful, since the trace form is nondegenerate.) □

The following is a useful observation:

Lemma 5.2.5. *Let $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$ be a faithful, finite-dimensional representation of a semisimple Lie algebra in characteristic zero. The trace pairing $(\ ,\ )_\pi$ is nondegenerate, and different simple summands of $\mathfrak{g}$ (Proposition 5.2.1) are orthogonal with respect to it.*

Proof. By Cartan's criterion 5.1.28, the trace pairing is nonzero on each simple summand of $\mathfrak{g}$; since the radical of an invariant symmetric form is an ideal, it is nondegenerate on each simple summand. The form is invariant, hence the orthogonal complement of each simple summand is $\mathfrak{g}$-stable, and by induction it decomposes into the direct sum of simple ideals in $\mathfrak{g}$ (which is unique, by Proposition 5.2.1). □

Definition 5.2.6. Let $\mathfrak{g}$ be a semisimple Lie algebra in characteristic zero, and use the Killing form, which is nondegenerate by Theorem 5.1.31, to identify $T^2\mathfrak{g} = \mathfrak{g}^* \otimes \mathfrak{g} = \mathrm{End}(\mathfrak{g})$. The element C in the center of $U(\mathfrak{g})$ (by Proposition 5.2.4 applied to the adjoint representation), which is the image of the identity operator $I \in \mathrm{End}(\mathfrak{g}) = T^2\mathfrak{g}$ in $U(\mathfrak{g})$, is called the *Casimir element* of $U(\mathfrak{g})$.

For example, when $\mathfrak{g} = \mathfrak{sl}_2$ with generators (h, e, f) and bracket $[h, e] = 2e$, $[h, f] = -2f$, $[e, f] = h$, the Casimir element is

$$C = \frac{1}{8}h^2 + \frac{1}{4}ef + \frac{1}{4}fe = \frac{1}{8}h^2 - \frac{1}{4}h + \frac{1}{2}ef = \frac{1}{8}h^2 + \frac{1}{4}h + \frac{1}{2}fe.$$

5.2.7. Lie algebra cohomology and complete reducibility. We will examine the question of whether a short exact sequence of $\mathfrak{g}$-representations

$$0 \to A \to B \to C \to 0 \tag{5.2.7.1}$$

admits a splitting: $B \simeq A \oplus C$.

The answer is given in terms of Lie algebra cohomology, which describes isomorphism classes of extensions B as in (5.2.7.1) in terms of a cohomology group.

Notice that any exact sequence of k-vector spaces splits (as vector spaces). That is, there is an element of $\mathrm{Hom}_k(C, B)$ which lifts the identity element in $\mathrm{Hom}_k(C, C)$. We would like to know that there is a $\mathfrak{g}$-invariant such element. Thus, it suffices to show that if we apply the functor of "$\mathfrak{g}$-invariants" to the exact sequence:

$$0 \to \mathrm{Hom}_k(C, A) \to \mathrm{Hom}_k(C, B) \to \mathrm{Hom}_k(C, C) \to 0,$$

it remains exact. (Notice that, in the context of Lie algebras, $\mathfrak{g}$-invariants—equivalently, the trivial representation of $\mathfrak{g}$—simply means that $\mathfrak{g}$ acts by zero.)

This is a problem is cohomology. The functor of $\mathfrak{g}$-invariants is left-exact, and it admits right derived functors $H^n(\mathfrak{g}, \bullet)$ which, in particular, turn any short exact sequence of $\mathfrak{g}$-modules $0 \to U \to V \to W \to 0$ (think of the above Hom spaces here) to a long exact sequence:

$$0 \to U^{\mathfrak{g}} \to V^{\mathfrak{g}} \to W^{\mathfrak{g}} \to H^1(\mathfrak{g}, U) \to H^1(\mathfrak{g}, V) \to H^1(\mathfrak{g}, W) \to \dots.$$

The groups $H^n(\mathfrak{g}, V)$ can also be thought of as $\mathrm{Ext}^n_{\mathfrak{g}}(k, V)$, which are the derived functors of $\mathrm{Hom}_{\mathfrak{g}}(\bullet, \bullet)$ in either argument. The group $\mathrm{Ext}^1(C, A)$, in particular, describes isomorphism classes of extensions (5.2.7.1).

Theorem 5.2.8. *If $\mathfrak{g}$ is a semisimple Lie algebra in characteristic zero, for any finite-dimensional $\mathfrak{g}$-module V we have $H^1(\mathfrak{g}, V) = 0$.*

Every finite-dimensional representation of $\mathfrak{g}$ is semisimple.

The statement on semisimplicity (complete reducibility) is due to Weyl, and referred to Weyl's theorem.

Proof. First, we reduce to simple $\mathfrak{g}$-modules by induction. Suppose that we have a short exact sequence:

$$0 \to U \to V \to W \to 0,$$

and that the first cohomology groups of U and W are trivial, then the long exact sequence shows that $H^1(\mathfrak{g}, V) = 0$, as well.

Now assume that V is irreducible. By the interpretation of $H^1(\mathfrak{g}, V)$ in terms of isomorphism classes of extensions

$$0 \to V \to W \to k \to 0,$$

it is enough to show that any such extension splits (as $\mathfrak{g}$-modules).

We will work separately on the cases where V is the trivial representation, and V is nontrivial.

If $V \neq k$, then we first reduce to the case where $\mathfrak{g}$ acts faithfully on V: Let $\mathfrak{h}$ be the kernel of the map $\mathfrak{g} \to \mathfrak{gl}(V)$ (it is an ideal of $\mathfrak{g}$). We claim that $\mathfrak{h}$ also acts trivially on W. Indeed, $\mathfrak{g}$ maps $W \to V$, and $\mathfrak{h}$ maps $V \to 0$, so $[\mathfrak{h}, \mathfrak{h}]$ acts trivially on W. But $\mathfrak{g}$ is semisimple in characteristic zero, and by Proposition 5.2.1, it is a sum of simple Lie algebras, hence $\mathfrak{h}$ is a subsum of those, hence semisimple. Again by the same proposition, $[\mathfrak{h}, \mathfrak{h}] = \mathfrak{h}$. So, the representation W factors through $\mathfrak{g}/\mathfrak{h}$, and we may replace $\mathfrak{g}$ by that to assume that $\mathfrak{g}$ acts faithfully.

Consider, then, the trace pairing $(X, Y)_\pi \mapsto \operatorname{tr}(\pi(X)\pi(Y))$, where π is the representation of $\mathfrak{g}$ on V. By Lemma 5.2.5, it is nondegenerate, hence the central element C_π of Proposition 5.2.4 is defined. By the same proposition, C_π acts on V by a nonzero scalar; on the other hand, it acts on k by zero. Thus, the short exact sequence can be split $W = V \oplus k$, according to the eigenspaces of C_π.

If $V = k$, that means that the image of $\mathfrak{g}$ in $\operatorname{End}(W)$ consists of nilpotent operators, hence is a nilpotent Lie algebra by Engel's theorem 5.1.10. Being a direct sum of simple Lie algebras by Proposition 5.2.1, $\mathfrak{g}$ has no nontrivial nilpotent quotients, hence W is the trivial representation. This completes the proof that $H^1(\mathfrak{g}, V) = 0$ for any finite-dimensional representation V.

Finally, for a short exact sequence of $\mathfrak{g}$-modules $0 \to A \to B \to C \to 0$, applying the vanishing of cohomology to the modules $\operatorname{Hom}_k(C, \bullet)$, we get a short exact sequence

$$0 \to \operatorname{Hom}_{\mathfrak{g}}(C, A) \to \operatorname{Hom}_{\mathfrak{g}}(C, B) \to \operatorname{Hom}_{\mathfrak{g}}(C, C) \to 0,$$

hence the identity element in $\operatorname{Hom}_{\mathfrak{g}}(C, C)$ can be lifted to a $\mathfrak{g}$-morphism $C \to B$. This proves complete reducibility. □

Remarks 5.2.9.
(1) Complete reducibility fails for infinite-dimensional representations; in fact, in our study of the category O, we will construct the finite-dimensional representations as quotients of infinite-dimensional, non-semisimple representations.
(2) Complete reducibility fails in positive characteristic p. For example, the Lie algebra $\mathfrak{g} = \mathfrak{sl}_2$ is semisimple if $p \neq 2$, but the p-th symmetric power of

its standard representation is not semisimple. Indeed, the space of polynomials of degree p in two variables (x, y) contains the invariant subspace generated by x^p and y^p (with trivial action of $\mathfrak{g}$), but the $\mathfrak{g}$-span of any other nonzero vector $v = \sum_{i=0}^{p} a_i x^{p-i} y^i$ meets that subspace, because if m is the maximal i with $i \neq p$ such that $a_i \neq 0$, and $e = x\frac{\partial}{\partial y} \in \mathfrak{g}$, we have $e^m v = m! a_m x^m$.

5.2.10. Derivations and the Jordan decomposition.

Proposition 5.2.11. *Every derivation of a semisimple Lie algebra in characteristic zero is inner (Definition 5.1.22).*

Proof. The formula $[D, \mathrm{ad}(X)] = \mathrm{ad}(DX)$ shows that the image of ad is an ideal in $\mathrm{Der}(\mathfrak{g})$. Since the image is a semisimple Lie algebra, there is a complementary ideal I (namely, its orthogonal complement under the Killing form on $\mathrm{Der}(\mathfrak{g})$). But if $D \in I$, and I is an ideal, the same formula shows that $\mathrm{ad}(DX) \in I \cap \mathrm{ad}(\mathfrak{g}) = 0$, which since ad is injective means that $DX = 0$, i.e. $D = 0$. $\square$

Remark 5.2.12. We already saw in Remark 5.2.2 that Proposition 5.2.11 fails in positive characteristic.

Proposition 5.2.13. *If $D \in \mathrm{Der}(\mathfrak{g}) \subset \mathrm{End}(\mathfrak{g})$ then $D_s, D_n \in \mathrm{Der}(\mathfrak{g})$.*

Proof. We may assume that the field is algebraically closed. If X is in the generalized λ-eigenspace and Y is in the generalized μ-eigenspace for D, then it can be shown by induction that:

$$(D - (\lambda + \mu))^n([X, Y]) = \sum_{r=0}^{n} \binom{n}{r} [(D - \lambda)^r(X), (D - \mu)^{n-r}(Y)],$$

hence $[X, Y]$ is in the generalized $\mu + \lambda$-eigenspace. This shows that D_s is a derivation, and then $D_n = D - D_s$ is a derivation. $\square$

Definition 5.2.14. Let $\mathfrak{g}$ be a Lie algebra. An element $X \in \mathfrak{g}$ is called *semisimple* if $\mathrm{ad}(X)$ is a semisimple operator, and *nilpotent* if $\mathrm{ad}(X)$ is nilpotent.

Theorem 5.2.15 (Jordan–Chevalley decomposition)**.** *Let $\mathfrak{g}$ be a semisimple Lie algebra in characteristic zero. Every $X \in \mathfrak{g}$ admits a unique decomposition $X = X_s + X_n$, with X_s semisimple, X_n nilpotent, and $[X_s, X_n] = 0$.*

Moreover, for any finite-dimensional representation $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$, and any $X \in \mathfrak{g}$, we have $\rho(X_s) = \rho(X)_s$, $\rho(X_n) = \rho(X)_n$, where $\rho(X) = \rho(X)_s + \rho(X)_n$ is the Jordan decomposition of $\rho(X)$.

For any morphism $\pi : \mathfrak{g}_1 \to \mathfrak{g}_2$ of semisimple Lie algebras in characteristic zero, and any $X \in \mathfrak{g}_1$, we have $\pi(X_s) = \pi(X)_s$, $\pi(X_n) = \pi(X)_n$.

Proof. By the Propositions 5.2.13, $\mathrm{ad}(X)_s$ and $\mathrm{ad}(X)_n$ are derivations in $\mathrm{End}(\mathfrak{g})$. By Proposition 5.2.11, they belong to the image of ad. Since the adjoint representation is faithful, this proves the existence and uniqueness of X_s and X_n.

By complete reducibility of $\mathrm{End}(V)$ under the adjoint $\mathfrak{g}$-action, we have:

$$\mathrm{End}(V) = \rho(\mathfrak{g}) \oplus \mathfrak{m},$$

where $\mathfrak{m}$ is an $\mathrm{ad}(\rho(\mathfrak{g}))$-invariant subspace. (Notice that in Proposition 5.2.11 we were able to obtain a similar decomposition in $\mathrm{Der}(\mathfrak{g})$ by using the Killing form, so we did not need to know reducibility.)

Since $\rho(X)_s, \rho(X)_n$ are polynomials in $\rho(X)$, their adjoint action preserves both $\rho(\mathfrak{g})$ and $\mathfrak{m}$. Let $\rho(X)_n = \rho(a)+b$ with $a \in \mathfrak{g}, b \in \mathfrak{m}$. Since $[\rho(\mathfrak{g}), b] = 0$, $b \in \mathrm{End}(V)$ is a $\mathfrak{g}$-endomorphism. If $V = \oplus V_i$ is a decomposition into irreducibles, b acts by a scalar on each one of them, by Schur's lemma. On the other hand, we know that $\rho(X)_n$ is nilpotent, $\rho(a)$ and b commute, and $\mathrm{tr}_{V_i}(\rho(a)) = 0$ because a (like every element of $\mathfrak{g}$) is a sum of commutators. Therefore, $\mathrm{tr}_{V_i}(b) = 0$, hence b acts by zero on all V_i, i.e. $b = 0$.

Now, $\rho(X)_n = \rho(a)$ acts nilpotently on V, hence it acts nilpotently on $\mathrm{End}(V)$ under the adjoint representation. By the decomposition $\mathrm{End}(V) = \mathfrak{g} \oplus \mathfrak{m}$ it follows that it acts nilpotently on $\mathfrak{g}$. By the uniqueness of the Jordan decomposition we can now infer that $\rho(X)_n = \rho(X_n)$.

Finally, for any morphism $\pi : \mathfrak{g}_1 \to \mathfrak{g}_2$, setting $\rho = \mathrm{ad} \circ \pi$ and applying the previous statement, we obtain the last statement. □

5.3. Root systems and the structure of semisimple Lie algebras

In this section, the underlying field is of characteristic zero, and we keep assuming that all vector spaces are finite dimensional.

5.3.1. Representations of $\mathfrak{sl}_2$. The Lie algebra of $\mathfrak{sl}_2$ is generated over the underlying field by three elements H, E, F with bracket relations:

$$[H,E] = 2E,$$
$$[H,F] = -2F,$$
$$[E,F] = H.$$

Let $\Delta = 8C = 4FE + (H+2)H$ in the center of $U(\mathfrak{g})$. (It turns out—see the Harish-Chandra isomorphism, Theorem 6.5.7— that $Z(U\mathfrak{g})$ is a polynomial ring generated by this element.)

Given a representation V of $\mathfrak{sl}_2$ (recall that all spaces are assumed finite-dimensional in this chapter), and $\lambda \in k$, let V_λ denote the λ-eigenspace of H. We don't know yet that H acts semisimply, so a priori V is not the direct sum of the V_λ's.

Lemma 5.3.2. *$E \cdot V_\lambda \subset V_{\lambda+2}$; $F \cdot V_\lambda \subset V_{\lambda-2}$.*

There is a non-zero vector $v \in V$ which is an eigenvector for H and such that $Ev = 0$.

Proof. The first statement is clear from the bracket relations. From the finite-dimensionality of V, there must be a nonzero H-eigenvector annihilated by E. □

Definition 5.3.3. A *highest weight vector of an $\mathfrak{sl}_2$-module* is a nonzero vector annihilated by E. A *lowest weight vector of an $\mathfrak{sl}_2$-module* is a nonzero vector annihilated by F.

Proposition 5.3.4. *Fix a heighest weight vector $v \in V_\lambda$, and let V' be the span of $\{F^i v\}_{i\in\mathbb{N}}$. Then V' is $\mathfrak{sl}_2$-stable, irreducible, and Δ acts by $\lambda(\lambda+2)$. The highest weight λ is a non-negative integer, and V' is the sum of one-dimensional weight spaces V'_μ for $\mu = \lambda, \lambda-2, \lambda-4, \dots, -\lambda$.*

Proof. It is clearly stable under F and H. We easily compute:

$$EF^n v_\lambda = n(\lambda - (n-1))F^{n-1}v_\lambda.$$

Hence, the space is E-stable.

Moreover, since it is finite-dimensional, we must have $n(\lambda - (n-1)) = 0$ for some $n \geq 1$, hence λ is a non-negative integer. In that case, $n = \lambda + 1$, and $F^n v_\lambda$ must be zero (because it is a highest weight vector of weight $-\lambda - 2$ and, by the same argument, it cannot generate a finite-dimensional representation). On the other hand, for $n < \lambda + 1$ $EF^n v_\lambda \neq 0$, hence $F^n v_\lambda \neq 0$. The statement about the weight spaces of V' follows.

We have: $\Delta v = 4FEv + (H+2)Hv = 0 + \lambda(\lambda+2)v$. Since Δ commutes with the action of $\mathfrak{sl}_2$ and is generated by v, all elements of V' have the same Δ-eigenvalue.

On the other hand, V' has at most one eigenvector for each H-eigenvalue. If V' was reducible, there would be some highest weight vector with eigenvalue $\neq$ ☐

Theorem 5.3.5. *For every nonnegative integer n there is a unique, up to isomorphism, irreducible finite-dimensional representation V_n of $\mathfrak{sl}_2$ of heighest weight n (in characteristic zero). It has dimension $n+1$, and eigenvalue $n(n+2)$ under the operator Δ.*

All finite-dimensional representations of $\mathfrak{sl}_2$ are H-semisimple, and direct sums of the modules V_n.

Proof. If V denotes the standard, 2-dimensional representation, then it is easy to see that $S^n V$ has a unique highest weight vector with weight $n+1$, hence is irreducible. Uniqueness follows from the explicit description of the action of E, F and H above, and the rest follow from complete reducibility (Theorem 5.2.8) and Proposition 5.3.4. ☐

This existence statement will require a lot more work in the general case.

5.3.6. Cartan subalgebras of semisimple Lie algebras.

Proposition 5.3.7. *Assume that $\mathfrak{g}$ is a semisimple Lie algebra in characteristic zero, and let $\mathfrak{h}$ be the generalized nilspace of an s-regular element (hence[1], by Proposition 5.1.18, a Cartan subalgebra). Then:*

(1) *$\mathfrak{h}$ is a maximal abelian subalgebra.*
(2) *The centralizer of $\mathfrak{h}$ is $\mathfrak{h}$.*
(3) *Every element of $\mathfrak{h}$ is semisimple. Every s-regular element of $\mathfrak{g}$ is semisimple.*
(4) *The restriction of the Killing form (or any non-degenerate invariant symmetric bilinear form) of $\mathfrak{g}$ to $\mathfrak{h}$ is non-degenerate.*

Proof. If we prove (**??**), the rest of the statements will follow; let us see how:

Cartan's criterion says that a Lie subalgebra $\mathfrak{a}$ of $\mathrm{End}(V)$ is solvable if and only if $\mathrm{tr}(XY) = 0$ for every $X \in \mathfrak{a}, Y \in [\mathfrak{a}, \mathfrak{a}]$. Applying this to $\mathrm{ad}(\mathfrak{h}) \subset \mathrm{End}(\mathfrak{g})$ (which is nilpotent, hence solvable), we get that $B(X,Y) = 0$ for all $X \in \mathfrak{h}, Y \in [\mathfrak{h}, \mathfrak{h}]$ (where B is the Killing form for $\mathfrak{g}$). Therefore, the radical of the restriction of B to $\mathfrak{h}$ contains the commutator, which means that $[\mathfrak{h}, \mathfrak{h}] = 0$. Thus, $\mathfrak{h}$ is abelian.

The centralizer is contained in the normalizer, which is $\mathfrak{h}$, but since $\mathfrak{h}$ is abelian it coincides with it. Thus, $\mathfrak{h}$ is maximal abelian.

Finally, let $X \in \mathfrak{h}$ and let $X = X_s + X_n$ be its Jordan decomposition. Since X_s, X_n commute with the centralizer of X, which contains $\mathfrak{h}$, it follows that X_s, X_n are in the centralizer of $\mathfrak{h}$, which is $\mathfrak{h}$. Thus, if $Y \in \mathfrak{h}$, $\mathrm{ad}(Y)\mathrm{ad}(X_n)$ is nilpotent,

[1]Eventually, since they are conjugate, all Cartan subalgebras are of this form

which implies that $\mathrm{ad}(X_n)$ is orthogonal to $\mathfrak{h}$ under the Killing form. By non-degeneracy of the Killing form on $\mathfrak{h}$, $X_n = 0$. Every s-regular element of $\mathfrak{g}$ is contained in its generalized nilspace $\mathfrak{h}$, hence is semisimple.

We come to the proof of (**??**): if X is an s-regular element such that $\mathfrak{h}$ is the generalized nilspace of X, let $\mathfrak{g} = \bigoplus_\lambda \mathfrak{g}_\lambda$ be a decomposition of $\mathfrak{g}$ into generalized $\mathrm{ad}(X)$-eigenspaces. As we saw in the proof of Proposition 5.2.13, $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subset \mathfrak{g}_{\lambda+\mu}$, which implies that $\mathfrak{g}_\lambda \perp \mathfrak{g}_\mu$ (under the Killing form), unless $\lambda + \mu = 0$. Therefore, the decomposition:

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus (\mathfrak{g}_\lambda \oplus \mathfrak{g}_{-\lambda})$$

is orthogonal, and since B is nondegenerate, it has to be non-degenerate on each of the summands, in particular on $\mathfrak{h} = \mathfrak{g}_0$. $\square$

5.3.8. The root system of a semisimple Lie algebra. Let $\mathfrak{g}$ be a semisimple Lie algebra over an algebraically closed field k in characteristic zero. Fix $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra. All constructions that follow depend, a priori, on $\mathfrak{h}$. In Section 5.5 we will see that all Cartan subalgebras are conjugate, and this will establish independence of the root system of $\mathfrak{g}$ from the choice of $\mathfrak{h}$. Recall that, by Proposition 5.3.7, $\mathfrak{h}$ is abelian. It

By Proposition 5.3.7, the restriction of the adjoint representation to a Cartan subalgebra $\mathfrak{h}$ reads:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_\alpha \mathfrak{g}_\alpha, \tag{5.3.8.1}$$

where the $\mathfrak{g}_\alpha$'s are eigenspaces with *nonzero* eigencharacter $\alpha \in \mathfrak{h}^*$.

Definition 5.3.9. The set Φ of nonzero elements $\alpha \in \mathfrak{h}^*$ such that $\mathfrak{g}_\alpha \neq 0$ in the decomposition (5.3.8.1) is called the set of *roots* of $\mathfrak{g}$.

Theorem 5.3.10. *The following hold for the set of roots Φ of a semisimple Lie algebra $\mathfrak{g}$:*

1. *Φ spans $\mathfrak{h}^*$.*
2. *If $\alpha \in \Phi$, then $-\alpha \in \Phi$.*
3. *If $\alpha \in \Phi$, let $t_\alpha \in \mathfrak{h}$ be the image of α under the isomorphism: $\mathfrak{h}^* \to \mathfrak{h}$ defined by a non-degenerate invariant symmetric form $(\ ,\ )$ on $\mathfrak{g}$. Then, for all $X \in \mathfrak{g}_\alpha, Y \in \mathfrak{g}_{-\alpha}$ we have: $[X,Y] = (X,Y)t_\alpha$.*
4. *$(t_\alpha, t_\alpha) \neq 0$.*
5. *The sum $\mathfrak{h}_\alpha + \mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha}$ is a subalgebra isomorphic to $\mathfrak{sl}_2$; denote it by $\mathfrak{sl}_{2,\alpha}$.*
6. *If $\alpha, c\alpha \in \Phi$ then $c = \pm 1$.*
7. *For any $\alpha, \beta \in \Phi$ and non-proportional, the $\mathfrak{sl}_{2,\alpha}$-stable subspace $\mathfrak{g}_{\beta+\mathbb{Z}\alpha}$ is an irreducible representation of $\mathfrak{sl}_{2,\alpha}$.*
8. *For any $\alpha \in \Phi$, let w_α be the linear transformation $\lambda \mapsto \lambda - \langle \lambda, h_\alpha \rangle$ on $\mathfrak{h}^*$. Then, w_α fixes Φ.*

Remark 5.3.11. For the last statement of Theorem 5.3.10, we could have used any positive definite invariant inner product; indeed, on the simple summands of $\mathfrak{g}$ (see Proposition 5.2.1) any two of them are equal up to a scalar, and the different summands are orthogonal, so the reflections do not depend on the choice of invariant inner product.

Proof. (1) Any $x \in \mathfrak{h}$ in the kernel of Φ is central in $\mathfrak{g}$, hence zero since $\mathfrak{g}$ is semisimple.

(2) Choose a non-degenerate invariant symmetric form $(\, , \,)$. For α, β in Φ (or zero), $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_\beta$ and any $z \in \mathfrak{h}$, by invariance we have
$$\alpha(z)(x, y) = ([z, x], y) = -(x, [z, y]) = -\beta(z)(x, y).$$
Thus, $(x, y) = 0$ unless $\alpha + \beta = 0$. Since the form is nondegenerate, for every x there must be a y with $(x, y) \neq 0$, thus if $\alpha \in \Phi$, so is $-\alpha$.

(3) Notice first that $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$, so $[x, y]$ must belong to $\mathfrak{h}$. For all $z \in \mathfrak{h}$, $x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_{-\alpha}$ we have: $(z, [x, y]) = ([z, x], y) = \alpha(z)(x, y) = (z, (x, y)t_\alpha)$. Since the form is nondegenerate on $\mathfrak{h}$, by Proposition 5.3.7, we deduce that $[x, y] = (x, y)t_\alpha$.

(4) Notice that the pairing $(\, , \,)$ between $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ is nonzero because otherwise it would be degenerate on $\mathfrak{g}$. It follows that $\mathfrak{h}_\alpha := [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ is one-dimensional, spanned by the element t_α. If $(t_\alpha, t_\alpha) = 0$ then $(t_\alpha, [x, y]) = 0$ (for x, y as before), which implies that $[t_\alpha, x] = [t_\alpha, y] = 0$, and the elements x, y, t_α span a solvable Lie algebra. Thus, its image under the adjoint representation stabilizes a full flag, and since t_α is in its commutator, $\mathrm{ad}(t_\alpha)$ is nilpotent. But by Proposition 5.3.7, $\mathrm{ad}(t_\alpha)$ is also semisimple, so it has to be zero, which means that t_α is a central element in $\mathfrak{g}$, a contradiction.

(5) Let $h_\alpha \in \mathfrak{h}_\alpha$ be the element characterized by $\alpha(h_\alpha) = 2$, that is, $h_\alpha = \frac{2}{(t_\alpha, t_\alpha)} t_\alpha$. Choose any nonzero element $x = e \in \mathfrak{g}_\alpha$, and then choose $y = f \in \mathfrak{g}_{-\alpha}$ with $(e, f) = \frac{2}{(t_\alpha, t_\alpha)}$. Then, (h, e, f) is an $\mathfrak{sl}_2$-triple, and we denote its span by $\mathfrak{sl}_{2,\alpha}$. The subspace
$$\mathfrak{m} = \mathfrak{h}_\alpha \oplus \bigoplus_{n \in \mathbb{Z}, n \neq 0} \mathfrak{g}_{n\alpha}$$
decomposes, as an $\mathfrak{sl}_{2,\alpha}$-module, into finite-dimensional irreducible representations with even weights (for h_α). Since the zero weight space is one-dimensional, this implies (from the classification of irreducible, finite-dimensional representations of $\mathfrak{sl}_2$) that it is irreducible. It contains the adjoint representation of $\mathfrak{sl}_2$, therefore $\mathfrak{m} = \mathfrak{sl}_{2,\alpha}$.

(6) The previous point proves that there are no integral multiples of α in Φ, other than $\pm\alpha$. Similarly, again by the classification of irreducible, finite-dimensional representations of $\mathfrak{sl}_2$, the only other multiples are half-integral, and if one of them appears, then $\frac{\alpha}{2}$ must appear; indeed, the weights for h_α have to be integers, and if an odd integer appears, then so must the weight 1. But, repeating this argument with $\frac{\alpha}{2}$ in place of α, we get that $2\frac{\alpha}{2}$ cannot appear, a contradiction. Hence, $\frac{\alpha}{2} \notin \Phi$.

(7) For any $\beta \in \Phi$ which is non-proportional to α, consider the subspace
$$\mathfrak{n} = \mathfrak{g}_{\beta + \mathbb{Z}\alpha}.$$
It is a $\mathfrak{sl}_{2,\alpha}$-stable subspace with weights (for h_α) of the same parity, and each nonzero-weight space is one-dimensional, therefore it is an irreducible representation of $\mathfrak{sl}_{2,\alpha}$.

(8) By the classification of irreducible representation of $\mathfrak{sl}_2$, the subspace $\mathfrak{n}$ must include a range of weights differing by 2; therefore,
$$\Phi \cap (\beta + \mathbb{Z}\alpha) = \{\beta + n\alpha\}_{n=-r}^{q}$$

for some nonnegative integers r, q. This has $r+q+1$ weights, therefore the lowest weight must be $-(r+q)$, the heighest weight must be $(r+q)$. Therefore,

$$\langle \beta, h_\alpha \rangle = r - q, \tag{5.3.11.1}$$

and we get

$$w_\alpha(\beta) = \beta - \langle \beta, h_\alpha \rangle \, \alpha = \beta - (r-q)\alpha \in \Phi.$$

□

Definition 5.3.12. A (crystallographic) *root system* is a triple (E, Φ, W), where

- E be a finite-dimensional real vector space;
- Φ is a finite subset of E not containing zero;
- $W \subset \mathrm{GL}(E)$ is a (necessarily finite) group of automorphisms preserving Φ, generated by elements w_α, $\alpha \in \Phi$, which fix a hyperplane and send the root α to $-\alpha$.

The group W is called the *Weyl group* of the root system.

The root system is said to be *reduced* if $\Phi \cap \mathbb{R}\alpha = \{\pm\alpha\}$.

Definition 5.3.13. The *root system of a semisimple Lie algebra* $\mathfrak{g}$, with respect to a Cartan subalgebra $\mathfrak{h}$, is the triple (E, Φ, W), where $\Phi \subset \mathfrak{h}^*$, E is the $\mathbb{R}$-span of Φ, and W is the group generated by the reflections w_α, $\alpha \in \Phi$.

Remark 5.3.14. In many references, a root system is defined with W replaced by an inner product on E, such that the elements w_α are orthogonal reflections on the hyperplanes perpendicular to the roots. Clearly, such an inner product determines the Weyl group W; vice versa, any finite-dimensional representation of a finite group is unitarizable, so for any root system in the sense of Definition 5.3.12, there is an inner product with respect to which the w_α's are orthogonal reflections. Moreover, up to the obvious freedom of rescaling the inner product on each *irreducible* summand of the root system (i.e., a summand that cannot be further decomposed as a direct sum of two root systems), the inner product is unique.

The following proposition shows a way to produce such an inner product for a given Lie algebra.

Lemma 5.3.15. *Let $(\ ,\ )$ be the Killing form, and use it to identify $\mathfrak{h}^* = \mathfrak{h}$. Let E be the $\mathbb{R}$-span of the roots. Then, the Killing form is positive definite on the span of roots.*

Proof. Let $t_\lambda \in \mathfrak{h}$ be the element that corresponds to $\lambda \in \mathfrak{h}^*$ under the identification. We compute:

$$(\lambda, \lambda) = (t_\lambda, t_\lambda) = \mathrm{tr}((\mathrm{ad}(t_\lambda))^2) = \sum_{\alpha \in \Phi} \alpha(t_\lambda)^2.$$

If we can show that $(\alpha, \beta) = \alpha(t_\beta) \in \mathbb{R}$ for every root β, that would imply positivity (since we already know, from Theorem 5.3.10, that $\alpha(t_\alpha) = (t_\alpha, t_\alpha) \neq 0$.

From (5.3.11.1), we have that $(\alpha, \beta) = \alpha(t_\beta) = \frac{\alpha(h_\beta)}{2}(\beta, \beta) \in \mathbb{Q} \cdot (\beta, \beta)$, and therefore

$$(\beta, \beta) = \sum_{\alpha \in \Phi} (\alpha, \beta)^2 \in \mathbb{Q} \cdot (\beta, \beta)^2,$$

therefore $(\beta, \beta) \in \mathbb{Q}$ and $(\alpha, \beta) \in \mathbb{Q}$. □

Definition 5.3.16. Given a Cartan algebra $\mathfrak{h} \subset \mathfrak{g}$, the element $h_\alpha \in \mathfrak{h}$ of any $\mathfrak{sl}_2$-triple (h_α, e, f) with $e \in \mathfrak{g}_\alpha$, $f \in \mathfrak{g}_{-\alpha}$ is called the *coroot* associated to the root α, and denoted by $\check{\alpha}$.

Notice that $\mathfrak{h}_\alpha$ does not depend on the choice of e and f. We will denote the set of coroots by $\check{\Phi}$. Hence, the simple reflection w_α on $\mathfrak{h}^*$ associated to the root α can be written:

$$w_\alpha(x) = x - \langle x, \check{\alpha} \rangle \alpha. \tag{5.3.16.1}$$

Lemma 5.3.17. *If (E, Φ, W) is a root system, then $(E^*, \check{\Phi}, W)$ is also a root system, where W acts on E^* by the dual representation to E.*

Proof. Obvious. □

Definition 5.3.18. If (E, Φ, W) is a root system, the *dual root system* is the triple $(E^*, \check{\Phi}, W)$ of Lemma 5.3.17.

We will need the classification of root systems of rank two, and a corollary of that.

Proposition 5.3.19. *The only root systems of rank two are the root systems $A_1 \times A_1$, A_2, B_2 and G_2. If α, β are two non-proportional roots in a root system Φ with $\langle \alpha, \check{\beta} \rangle < 0$, then $\alpha + \beta \in \Phi$.*

Proof. [Definitions and proofs for the first statement to be added later—easy to look up!]

If α, β are non-proportional roots in a root system, the intersection of the set of roots with their linear span, together with the subgroup of the Weyl group generated by the reflections w_α, w_β, form a root system of rank two. The classification shows that if $\langle \alpha, \check{\beta} \rangle < 0$, then $\alpha + \beta$ is also a root.

□

5.4. Parabolic subalgebras

Definition 5.4.1. A *based root system* is a quadruple $(E, \Phi \supset \Phi^+, W)$ consisting of a root system, and a subset Φ^+ which consisting of the elements of Φ on one side of a hyperplane not meeting Φ (that is, $\Phi^+ = \{\alpha \in \Phi | t(\alpha) > 0\}$ for some linear functional $t \in E^*$ such that $\ker(t)$ does not contain any roots).

The *simple roots* of a based root system are those elements of Φ^+ which cannot be written (non-trivially) as a nonnegative integral linear combination of other elements of Φ^+. The set Δ of simple roots is called a *basis* of the root system.

The name is due to the following:

Proposition 5.4.2. *Given a based root system $(E, \Phi \supset \Phi^+, W)$ with set of simple roots Δ, the* root lattice *$R = \langle \Phi \rangle_{\mathbb{Z}}$ (the subgroup of E generated by Φ) is freely generated by Δ; in particular, the elements of Δ are linearly independent.*

Proof. Choose an inner product $(\ ,\ )$ with respect to which the elements of W are orthogonal.

By Proposition 5.3.19, we have $(\alpha, \beta) \leq 0$ for all $\alpha, \beta \in \Delta$, because otherwise $\alpha - \beta$ or $\beta - \alpha$ would belong to Φ^+, contradicting the fact that they are indecomposable.

Consider a nontrivial zero linear combination

$$\sum_{\alpha\in\Delta} c_\alpha \alpha = 0.$$

Since all elements of Δ belong to the same open half-plane, some of the coefficients are positive and some are negative, so we can write

$$\lambda := \sum_{\alpha\in\Delta, c_\alpha>0} c_\alpha \alpha = \sum_{\alpha\in\Delta, c_\alpha<0} (-c_\alpha)\alpha,$$

with none of the two sums being empty. Then,

$$(\lambda,\lambda) = \left(\sum_{\alpha\in\Delta, c_\alpha>0} c_\alpha \alpha, \sum_{\alpha\in\Delta, c_\alpha<0} (-c_\alpha)\alpha \right) \leq 0,$$

contradicting the positivity of the inner product.

Thus, the elements of Δ are linearly independent, and since by definition every other element of Φ^+ belongs to their $\mathbb{Z}$-span, they are a free basis for the root lattice. □

Definition 5.4.3. A *Borel subalgebra* of a Lie algebra is a maximal solvable subalgebra. A *parabolic subalgebra* is a subalgebra containing a Borel subalgebra.

Obviously, every Cartan subalgebra is contained in a Borel subalgebra.

Lemma 5.4.4. *A Borel subalgebra $\mathfrak{b}$ of a Lie algebra $\mathfrak{g}$ is self-normalizing.*

Proof. If $x \in \mathfrak{g}$ normalizes $\mathfrak{b}$, then the subalgebra $\mathfrak{b}'$ generated by $\mathfrak{b}$ and x is solvable. By the maximality of $\mathfrak{b}$, $\mathfrak{b}' = \mathfrak{b}$. □

The following will be useful for the results that follow:

Lemma 5.4.5. *Any Lie subalgebra of $\mathfrak{g}$ containing the Cartan subalgebra $\mathfrak{h}$ is of the form*

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in P} \mathfrak{g}_\alpha,$$

for some closed *subset $P \subset \Phi$, in the sense that if $\alpha,\beta \in P$ and $\alpha+\beta \in \Phi$, then $\alpha+\beta \in P$.*

Proof. Any Lie subalgebra containing $\mathfrak{h}$ is an $\mathfrak{h}$-submodule, therefore a sum of $\mathfrak{h}$ with some of the (one-dimensional) root spaces.

The fact that the subset P of roots appearing is closed follows from Theorem 5.3.10: for α, β non-proportional, the subspace $\mathfrak{g}_{\beta+\mathbb{Z}\alpha}$ is an irreducible $\mathfrak{sl}_{2,\alpha}$-representation; therefore, a nonzero element $e \in \mathfrak{g}_\alpha$ does not kill $\mathfrak{g}_\beta$, unless this is the highest weight space, that is, unless $\alpha+\beta \notin \Phi$. □

Proposition 5.4.6. *Given a Cartan subalgebra $\mathfrak{h}$ in a semisimple Lie algebra $\mathfrak{g}$, the set of Borel subalgebras containing $\mathfrak{h}$ is in bijection with the set of bases on the root system (E,Φ,W) associated to $\mathfrak{h}$, where a choice $\Phi^+ \subset \Phi$ of positive roots corresponds to the Borel subalgebra*

$$\mathfrak{b} = \mathfrak{h} + \bigoplus_{\alpha\in\Phi^+} \mathfrak{g}_\alpha.$$

The parabolic subalgebras containing this Borel subalgebra are determined by subsets of the set Δ of simple roots, with $I \subset \Delta$ corresponding to the parabolic subalgebra

$$\mathfrak{p}_I = \mathfrak{b} + \bigoplus_{\alpha \in \Phi_I^+} \mathfrak{g}_{-\alpha},$$

where Φ_I^+ is the set of elements of Φ^+ in the span of I.

Proof. Any Lie subalgebra containing $\mathfrak{h}$ must be a sum of root spaces, and if it is solvable it cannot contain $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ at the same time, because then it would contain a copy of $\mathfrak{sl}_2$. Thus, if P is the set of roots whose root spaces are contained in $\mathfrak{b}$, we have $P \cap (-P) = \emptyset$. We will prove that this implies that P lies in a half-space.

We claim that no nontrivial sum $\alpha_1 + \cdots + \alpha_n$ of elements of P is zero; indeed, if this is the case, then $(\alpha_1, \alpha_j) < 0$ for some j (for a chosen invariant inner product), therefore $\alpha_1 + \alpha_j \in \Phi$ by Proposition 5.3.19, and therefore $\alpha_1 + \alpha_j \in P$, by Lemma 5.4.5. This reduces the claim to a sum of $n-1$ elements, and the claim follows by induction.

Now we claim that there exists an $\beta \in P$ with $(\alpha, \beta) \geq 0$ for all $\alpha \in P$. If not, we would be able to choose an infinite sequence $\alpha_\bullet : \mathbb{N} \to P$ with $\beta_n = \alpha_1 + \alpha_2 + \cdots + \alpha_n \in P$ for all n; indeed, having chosen the first n elements of this sequence, by assumption (α, β_n) is not ≥ 0 for all $\alpha \in P$, but then choosing α_{n+1} with $(\alpha_{n+1}, \beta_n) < 0$, again by Proposition 5.3.19 and Lemma 5.4.5, we would get that $\beta_n + \alpha_{n+1} \in P$. Now, the finiteness of P implies that $\beta_i = \beta_j$ for some $i < j$, implying that $\alpha_{i+1} + \cdots + \alpha_j = 0$, a contradiction.

This proves that P lies in a half-space (open, since no opposite elements of Φ are in P), and it is clear that a Lie algebra of the form $\mathfrak{h} + \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$ is solvable, therefore a maximal solvable Lie algebra containing $\mathfrak{h}$ has to be of this form.

For a parabolic subalgebra $\mathfrak{p}$ properly containing $\mathfrak{b}$, if now P denotes the set of roots in the weight decomposition of $\mathfrak{p}$, if $-\beta \in P$ for some positive root β, and if $\beta = \alpha_1 + \cdots + \alpha_n$ is its decomposition as a sum of elements of Δ, we will prove by induction on n that all $-\alpha_i \in P$. We have $(\beta, \beta) > 0$, therefore $(-\beta, \alpha_i) < 0$ for some i, hence $\gamma := -\beta + \alpha_i \in P$ (or is zero), by the same argument as before. If $\gamma \neq 0$, then $-\gamma \in P$, and again by Lemma 5.4.5, $-\alpha_i = -\beta + (-\gamma) \in P$. By the induction hypothesis, for all elements $\alpha \in \Delta$ in the decomposition of $-\gamma$, $-\alpha \in P$, and the claim is proven.

Vice versa, if $I = \Delta \cap (-P)$, and $\beta \in \Phi^+$ is in the linear span of I, then β can be written as a sum $\alpha_1 + \cdots + \alpha_n$ of elements of I, and we will prove by induction on n that $-\beta \in P$. Again, $(-\beta, \alpha_i) < 0$ for some i, so $-\beta + \alpha_1$, if nonzero, is in P, by the induction hypothesis. Since $-\alpha_1 \in P$, by Lemma 5.4.5, $-\beta \in P$, as well. ☐

5.5. Conjugacy of Borel and Cartan subalgebras

In this section, $\mathfrak{g}$ is a finite-dimensional Lie algebra over an algebraically closed field k of characteristic zero.

Definition 5.5.1. Given a Lie algebra $\mathfrak{g}$, let $N(\mathfrak{g}) = \{X \in \mathfrak{g} | \exists Y \in \mathfrak{g}, a \neq 0 \text{ with } [X, Y] = aX\}$, i.e., all elements which belong to a nonzero eigenspace under the adjoint action of some other element. The group $\mathcal{E}(\mathfrak{g})$ is the subgroup of all automorphisms of $\mathfrak{g}$ generated by the automorphisms $\exp(\mathrm{ad}(X))$, $X \in N(\mathfrak{g})$.

The following lemma shows that the automorphisms of Definition 5.5.1 are well-defined:

Lemma 5.5.2. *The elements of the set $N(\mathfrak{g})$ of Definition 5.5.1 are nilpotent.*

Proof. The relation $[X, Y] = aX$ means that X takes the λ-generalized eigenspace of Y to the $(\lambda - a)$-generalized eigenspace. Since we are in characteristic zero, all $\lambda - na$, $n \in \mathbb{Z}$, are distinct, and by finite-dimensionality, the $\lambda - na$-eigenspace has to be zero for large n. ☐

Remark 5.5.3. In semisimple Lie algebras, it turns out that $\mathcal{E}(\mathfrak{g})$ is the entire group of *inner automorphisms*, i.e., automorphisms generated by nilpotent elements. On the opposite end, if $\mathfrak{g}$ is nilpotent, then $\mathcal{E}(\mathfrak{g})$ is trivial.

Theorem 5.5.4. *Let k be algebraically closed, in characteristic zero. Any two Borel subalgebras of $\mathfrak{g}$ are conjugate under $\mathcal{E}(\mathfrak{g})$, and any two Cartan subalgebras of $\mathfrak{g}$ are conjugate under $\mathcal{E}(\mathfrak{g})$.*

Proof. Omitted, for now. See [**Hum78**] or [**Ste09**]. The analog of this theorem for algebraic groups appears in Section 7.4. ☐

5.6. Jacobson–Morosov

5.7. Ado's theorem

5.8. Other chapters

CHAPTER 6

Verma modules and the category $\mathcal{O}$.

A good reference for the category $\mathcal{O}$ is the book [**Hum08**] of Humphreys.

In this section, all Lie algebras and representations are over an algebraically closed field in characteristic zero, which for notational simplicity we take to be $\mathbb{C}$.

6.1. Verma modules

We have seen 5.2.8 that finite-dimensional representations of semisimple Lie algebras are completely reducible. We now want to *construct* those irreducible representations (in particular, to show that there is a unique one up to unique isomorphism for each given weight), and to compute their *characters*.

In specific cases one can do that "by hand", constructing first the irreducible representations attached to fundamental weights, and then the rest by taking tensor products of those, and removing copies of the representations already constructed. For instance, for $\mathfrak{sl}_n$ the $n-1$ fundamental representations are the first $n-1$ exterior powers of the standard, n-dimensional represntation.

For a more systematic approach, it is better to move outside the realm of finite-dimensional representations. The category to consider is motivated by the following definition and lemma:

Definition 6.1.1. Let $\mathfrak{g}$ be a semisimple (or reductive) Lie algebra, $\mathfrak{b}$ a Borel subalgebra, and $\mathfrak{h}$ its quotient by its commutator. Let V be a representation of $\mathfrak{g}$. A *heighest weight vector* (for the given choice of Borel subgroup) is an eigenvector for $\mathfrak{b}$, and the eigencharacter $\lambda \in \mathfrak{h}^*$ is called the *weight* of the highest weight vector.

Lemma 6.1.2. *A finite-dimensional representation of a semisimple Lie algebra is generated by its highest weight vectors.*

Proof. Since the representation is semisimple by Theorem 5.2.8, it is enough to show that any irreducible representation V contains a highest weight vector. This follows from Lie's theorem 5.1.12. $\square$

Thus, we will attempt to construct all finite-dimensional representations by constructing the universal objects with highest weight.

More precisely, we consider the category of $\mathfrak{g}$-modules of arbitrary, possibly infinite, dimension (no topology), and for $\lambda \in \mathfrak{h}^*$ (where $\mathfrak{h}$ denotes a universal Cartan, later to be identified with a Cartan subgroup of $\mathfrak{g}$) we let M_λ denote the module defined by the following universal property:

Definition 6.1.3. Let $\mathfrak{g}$ be a semisimple Lie algebra, $\mathfrak{b}$ a Borel subalgebra, and $\mathfrak{h}$ its reductive quotient. Fix $\lambda \in \mathfrak{h}^*$. The *Verma module* of highest weight λ is a $\mathfrak{g}$-module M_λ with the property

$$\mathrm{Hom}_{\mathfrak{g}}(M_\lambda, V) = \mathrm{Hom}_{\mathfrak{b}}(\mathbb{C}_\lambda, V),$$

where $\mathbb{C}_\lambda$ is the one-dimensional module where $\mathfrak{b}$ acts by the character λ.

Lemma 6.1.4. *For every $\lambda \in \mathfrak{h}^*$, the Verma module M_λ exists and can be identified as*

$$M_\lambda = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda.$$

Proof. Simply the universal property of tensor products. ☐

Let us now fix an opposite Borel $\mathfrak{b}^-$, identifying $\mathfrak{h}$ with $\mathfrak{b} \cap \mathfrak{b}^-$. We denote by $\mathfrak{n}$, $\mathfrak{n}^-$ the nilpotent radicals of $\mathfrak{b}$, $\mathfrak{b}^-$. Notice that, by the PBW theorem, as a $\mathfrak{b}^-$-module:

$$M_\lambda = U(\mathfrak{n}^-) \otimes_{\mathbb{C}} \mathbb{C}_\lambda, \tag{6.1.4.1}$$

where $U(\mathfrak{n}^-)$ acts by left multiplication on the first factor, and the $\mathfrak{h}$-action is the tensor product of the adjoint representation and the representation on $\mathbb{C}_\lambda$.

Therefore:

Lemma 6.1.5. (1) *M_λ is $\mathfrak{h}$-locally finite and semisimple. The ($\mathfrak{h}$-)weights of M_λ are of the form $\lambda - \sum_i c_i\alpha_i$, where α_i range over simple positive roots (we will denote their set by Δ) and $c_i \in \mathbb{Z}_{\geq 0}$. The weight spaces are finite-dimensional, and the λ-weight space M_λ^λ is one-dimensional.*
(2) *M_λ is $\mathfrak{n}$-locally finite.*

Proof. The first statement follows immediately from the presentation (**??**), and the second from the first and the fact that the action of $\mathfrak{n}$ raises weights. ☐

Proposition 6.1.6. *M_λ has a unique irreducible quotient L_λ.*

Proof. Any proper submodule is spanned by its $\mathfrak{h}$-eigenspaces (since the $\mathfrak{h}$-action is locally finite), and if it is proper, it cannot meet M_λ^λ, since this generates M_λ. Therefore, the sum of all proper submodules is proper. ☐

6.2. The category $\mathcal{O}$.

Definition 6.2.1. The *category $\mathcal{O}$* is the full subcategory of the category of $\mathfrak{g}$-modules consisting of those objects which are:

- $\mathfrak{h}$-locally finite and semisimple;
- $\mathfrak{n}$-locally finite;
- finitely generated.

By Lemma 6.1.5, Verma modules belong to the category $\mathcal{O}$.

Lemma 6.2.2. *A $\mathfrak{g}$-submodule or a quotient module of a module in $\mathcal{O}$ is in $\mathcal{O}$. The category $\mathcal{O}$ is a Noetherian abelian category.*

Proof. For a $\mathfrak{g}$-submodule $N \subset M$, if M is $\mathfrak{h}$-locally finite and semisimple and $\mathfrak{n}$-locally finite, so is N. Moreover, the universal enveloping algebra $U(\mathfrak{g})$ is Noetherian, by Proposition 4.3.8, hence if M is finitely generated, so is N. The same properties for quotient modules are obvious.

The category being Noetherian means that the union of an increasing chain of submodules has to stabilize. But the union is a submodule, hence finitely generated, therefore stabilizes.

The category of $\mathfrak{g}$-modules is abelian, and it is clear that coproducts (direct sums) of objects in $\mathcal{O}$ are also in $\mathcal{O}$. Since submodules and quotient modules (hence, kernels and cokernels) are also in $\mathcal{O}$, the category is abelian. ☐

We will eventually see that it is also Artinian, i.e. every object is of finite length.

Lemma 6.2.3. *Every object in $\mathcal{O}$ has a filtration whose quotients are surjective images of Verma modules.*

Proof. Let V be in $\mathcal{O}$, and let $W \subset V$ be a finite-dimensional, generating subspace. Without loss of generality, W is $\mathfrak{b}$-stable (for $U(\mathfrak{b})W$ is, in any case, finite-dimensional). By Lie's theorem 5.1.12, it has a filtration with one-dimensional quotients. Therefore, V has a filtration with quotients generated by $\mathfrak{b}$-eigenvectors. Each such representation is the surjective image of a Verma module. □

Definition 6.2.4. The *character* ch_V of an object V in the category $\mathcal{O}$ (or a sub-$\mathfrak{h}$-module) is the formal sum

$$\sum_{\lambda\in\mathfrak{h}^*} \dim V_\lambda \cdot e^\lambda.$$

Lemma 6.2.5. *For every $V \in \mathcal{O}$, the character ch_V belongs to the ring C of formal sums $\sum_{\lambda\in\mathfrak{h}^*} c(\lambda)e^\lambda$, where $c_\bullet : \mathfrak{h}^* \to \mathbb{Z}$ is supported in a finite number of translates of the negative root monoid, and multiplication defined by $e^\lambda \cdot e^\mu = e^{\lambda+\mu}$.*

The character map $ch : \mathcal{O} \to C$ factors through the Grothendieck group $\mathbb{Z}[\mathcal{O}]$ of the category $\mathcal{O}$; moreover, for any finite-dimensional $\mathfrak{g}$-module L, and any $M \in \mathcal{O}$, we have $L \otimes M \in \mathcal{O}$, and $ch_L \cdot ch_M = ch_{L\otimes M}$.

Recall that the *Grothendieck group* of an abelian category $\mathcal{C}$ is the free group on its objects, modulo the relation: $[B] = [A] + [C]$ for every short exact sequence $0 \to A \to B \to C \to 0$. We will eventually see that the Grothendieck group of $\mathcal{O}$ is generated by Verma modules, in fact: it is free on the set of Verma modules.

Proof. It is clear from the definition that the character factors through the Grothendieck group. It follows from Lemma 6.2.3 that the support of the character of any object is contained in the support of the character of a finite direct sum of Verma modules. By Lemma 6.1.5, the support of the characters of those are translates of the negative root monoid. □

6.3. The case of $\mathfrak{sl}_2$, and application to general Lie algebras.

Let $\mathfrak{g} = \mathfrak{sl}_2$. We identify $\mathfrak{h}^* \simeq \mathbb{C}$, by applying the positive root $\check{\alpha}$. Under this, the half-sum of positive roots $\rho = \frac{\alpha}{2}$ corresponds to 1.

Lemma 6.3.1. *For $\mathfrak{g} = \mathfrak{sl}_2$, $M_{\lambda-\rho}$ is irreducible, unless $\lambda \in \mathbb{Z}_{>0}$, in which case there is an exact sequence:*

$$0 \to M_{-\lambda-\rho} \to M_{\lambda-\rho} \to L_{\lambda-\rho} \to 0.$$

Proof. Every submodule must have a highest weight vector, which must be of the form $F^n v_{\lambda-\rho}$. We compute that:

$$EF^n v_{\lambda-\rho} = n(\lambda - n)F^{n-1}v_{\lambda-\rho},$$

therefore for it to be zero, for some $n > 0$, we must have $\lambda \in \mathbb{Z}_{>0}$. □

We return to the case of a general semisimple $\mathfrak{g}$. Then:

Proposition 6.3.2. *If α is a simple root such that $\langle\lambda, \check{\alpha}\rangle \in \mathbb{Z}_{>0}$ then there is an embedding: $M_{w_\alpha\lambda-\rho} \hookrightarrow M_{\lambda-\rho}$. The quotient $V = M_{\lambda-\rho}/M_{w_\alpha\lambda-\rho}$ has the property that it is locally $(\mathfrak{sl}_2)_\alpha$-finite, where $(\mathfrak{sl}_2)_\alpha$ denotes the embedding of $\mathfrak{sl}_2$ into $\mathfrak{g}$ determined by the root α. The character of any subquotient of V is w_α-stable.*

Proof. As in Lemma 6.3.1, we calculate that there is a highest weight vector with weight $w_\alpha\lambda$, hence there is a non-trivial map: $M_{w_\alpha\lambda-\rho} \to M_{\lambda-\rho}$. Since $M_{w_\alpha\lambda-\rho}, M_{\lambda-\rho} \simeq U(\mathfrak{n}^-)$ as $U(\mathfrak{n}^-)$-modules, and $U(\mathfrak{n}^-)$ does not have zero divisors, such a map has to be injective.

With notation $(H_\alpha, E_\alpha, F_\alpha)$ for the $\mathfrak{sl}_2$-triple corresponding to α, we need to show that the quotient is F_α-locally finite. (Finiteness under the other two is automatic for the category $\mathcal{O}$.) If V' is the set of F_α-finite vectors, then $V' \ni v_{\lambda-\rho}$; we claim that V' is $\mathfrak{g}$-stable. Indeed, we have a homomorphism of F_α-modules: $\mathfrak{g} \otimes V' \to V$, where F_α acts on $\mathfrak{g}$ via the adjoint representation. But $\mathfrak{g}$ is F_α-finite and V' is F_α-locally finite, hence their tensor product is locally finite, therefore $\mathfrak{g}V' \subset V'$. Together with $v_{\lambda-\rho} \in V'$, this implies that $V' = V$.

The last assertion follows from the corresponding statement on finite-dimensional $\mathfrak{sl}_2$-modules, see 5.3.1. □

Because of the shift by ρ in the previous proposition, it is convenient to define a modified action of the Weyl group on $\mathfrak{h}^*$.

Definition 6.3.3. The *dot action* of W on $\mathfrak{h}^*$ is defined by

$$w \bullet \lambda = w(\lambda + \rho) - \rho.$$

Hence, replacing $\lambda-\rho$ by λ, the embedding of Proposition 6.3.2 reads: $M_{w_\alpha \bullet \lambda} \hookrightarrow M_\lambda$, when $\langle \lambda, \check{\alpha} \rangle \in \mathbb{Z}_{\geq 0}$.

Proposition 6.3.4. *If V is a finite-dimensional $\mathfrak{g}$-module, its character is W-invariant. Moreover, if V is irreducible, it is equal to the irreducible quotient L_λ, for some weight λ that is* integral *(i.e. $\langle \check{\alpha}, \lambda \rangle \in \mathbb{Z}$ for all roots α) and* dominant *(i.e. $\langle \check{\alpha}, \lambda \rangle \geq 0$ for all positive roots α).*

Vice versa, assume that λ is integral and dominant. Then, the representation:

$$M_\lambda / \left(\sum M_{w_\alpha \bullet \lambda} \right)$$

(sum over simple positive roots) is finite dimensional, and equal to the unique irreducible quotient L_λ of M_λ.

Proof. If V is finite-dimensional, by Weyl's theorem 5.2.8 it is a direct sum of irreducibles, hence a direct sum of L_λ's, for various weights λ. Restricting to $\mathfrak{sl}_{2,\alpha}$, the copy of $\mathfrak{sl}_2$ corresponding to a simple root α, we see that, in order for L_λ to be finite-dimensional, the highest weight λ must satisfy $\langle \lambda, \check{\alpha} \rangle \in \mathbb{Z}_{\geq 0}$. This holds for every α, therefore λ must be integral and dominant.

In this case, by Proposition 6.3.2, the irreducible quotient of M_λ will have a w_α-stable set of weights, for every simple root α, therefore a W-stable set of weights.

On the other hand, all weights are $\leq \lambda$ and differ from λ by an element of the root lattice, so there is a finite set of weights only. Finally, the weight spaces are finite dimensional, so the quotient is finite-dimensional. If the quotient were not irreducible, by complete reducibility it would be a direct sum of irreducibles, contradicting the fact that L_λ is the unique irreducible quotient of M_λ. □

6.4. The Chevalley isomorphism

Let $\mathfrak{g}$ be a reductive Lie algebra, $\mathfrak{h}$ a Cartan subalgebra. The goal of this section is to study the ring $\mathbb{C}[\mathfrak{g}]^{\mathfrak{g}}$ of invariant polynomials on $\mathfrak{g}$ under the adjoint representation. If $\mathfrak{g}$ is the Lie algebra of a connected complex group G, this is equal to $\mathbb{C}[\mathfrak{g}]^G$.

Theorem 6.4.1 (Chevalley isomorphism)**.** *The restriction map under $\mathfrak{h} \hookrightarrow \mathfrak{g}$ gives rise to an isomorphism*

$$res : \mathbb{C}[\mathfrak{g}]^{\mathfrak{g}} \to \mathbb{C}[\mathfrak{h}]^W. \tag{6.4.1.1}$$

Proof. For injectivity, we need to use conjugacy of Cartan subalgebras, Theorem 5.5.4: If G is the group $\mathcal{E}(\mathfrak{g})$ of inner automorphisms of Definition 5.5.1, we have $\mathbb{C}[\mathfrak{g}]^{\mathfrak{g}} = \mathbb{C}[\mathfrak{g}]^G$ and $\mathrm{Ad}(G)(\mathfrak{h})$ is dense in $\mathfrak{g}$, so the restriction map of G-invariant functions to $\mathfrak{h}$ is injective.

For W-invariance, it is enough to show that the image is invariant under the reflection w_α corresponding to any simple root α. This simple root defines an embedding $\mathfrak{m} := \mathfrak{sl}_2 \oplus \alpha^\perp \hookrightarrow \mathfrak{g}$, where $\alpha^\perp \subset \mathfrak{h}$ is the orthogonal complement of α in $\mathfrak{h}$. The Lie algebra $\mathfrak{m}$ contains $\mathfrak{h}$, and we have restriction maps

$$\mathbb{C}[\mathfrak{g}]^{\mathfrak{g}} \to \mathbb{C}[\mathfrak{m}]^{\mathfrak{m}} \to \mathbb{C}[\mathfrak{h}].$$

This reduces us to the case of $\mathfrak{sl}_2 = \langle h, e, f\rangle$.

Now, for a nilpotent element X of a Lie algebra $\mathfrak{g}$, the automorphism $\exp(\mathrm{ad}(X)) = \sum_{n\geq 0} \frac{1}{n!}\mathrm{ad}(X)^n$ of $\mathfrak{g}$ makes sense, since $\mathrm{ad}(X)$ is nilpotent. In the case of $\mathfrak{sl}_2$, with w the nontrivial element of the Weyl group, we notice that the automorphism w of $\mathfrak{h}$ is induced by the automorphism

$$\tilde{w} = \exp(\mathrm{ad}(e))\exp(\mathrm{ad}(-f))\exp(\mathrm{ad}(e))$$

of $\mathfrak{g}$. (This is simply conjugation by the element $\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$ of SL_2.) This proves that the restriction of a $\mathfrak{g}$-invariant polynomial function on $\mathfrak{g}$ to $\mathfrak{h}$ is W-invariant.

We now pass to surjectivity, which is the deepest part of the theorem. Both $\mathbb{C}[\mathfrak{g}]^{\mathfrak{g}}$ and $\mathbb{C}[\mathfrak{h}]^W$ are graded by the degree of a polynomial, which we will denote by an index $\mathbb{C}[\ ]_d$, and the map between them preserves the grading. (We will introduce a filtration on these modules below.) Since W is a finite group, the symmetrization map

$$\mathbb{C}[\mathfrak{h}]_d \ni f \mapsto f_W := \frac{1}{|W|}\sum_{w\in W} w\cdot f \in \mathbb{C}[\mathfrak{h}]_d^W$$

is a W-equivariant projection, and certainly the elements of the form λ^d, $\lambda \in \mathfrak{h}^*$, span $\mathbb{C}[\mathfrak{h}]_d$. We can even restrict to λ integral and dominant.

If λ is a dominant, integral weight, and $(\rho_\lambda, V_\lambda)$ is the irreducible finite-dimensional representation with heighest weight λ, the left hand side contains the trace function

$$f_{\lambda,d}(X) = \mathrm{tr}\rho_\lambda(X)^d.$$

We define a filtration of $\mathbb{C}[\mathfrak{g}]^{\mathfrak{g}}$ by dominant integral weights, where the filtered piece $F^\lambda\mathbb{C}[\mathfrak{g}]^{\mathfrak{g}}$ consists of the span of all $f_{\mu,d}(X)$ with $\mu \leq \lambda$, where, by definition, $\mu \leq \lambda \iff \lambda - \mu \in R^+$, where R^+ is the monoid spanned by positive roots. (This is a standard partial ordering on the weight lattice; weights which do not differ by an element in the root lattice do not have a common upper bound, but we won't worry about this since for this argument we can restrict λ further to be in any lattice of finite index, such as the root lattice.)

Similarly, we define a filtration of $\mathbb{C}[\mathfrak{h}]_d^W$, with $F^\lambda\mathbb{C}[\mathfrak{h}]_d^W$ spanned by the elements $(\mu^d)_W$ with $\mu \leq \lambda$. Since V_λ has a W-invariant set of weights, all $\leq \lambda$, and with $\dim V_\lambda^\lambda = 1$, we get that res maps $F^\lambda\mathbb{C}[\mathfrak{g}]^{\mathfrak{g}} \to F^\lambda\mathbb{C}[\mathfrak{h}]_d^W$, with $\mathrm{res}(f_{\lambda,d}) \equiv |W|(\lambda^d)_W$ in $\mathrm{gr}^\lambda\mathbb{C}[\mathfrak{h}]_d^W$. The element $(\lambda^d)_W$ spans $\mathrm{gr}^\lambda\mathbb{C}[\mathfrak{h}]_d^W$, thus we get by induction (the base case $\lambda = 0$ being trivial) that the map (6.4.1.1) is surjective. $\square$

There is a second part to Chevalley's theorem, which asserts that the algebra of invariant functions is a polynomial algebra.

Theorem 6.4.2. *Let E be a complex vector space, and $W \subset GL(E)$ a finite subgroup of automorphisms, which is generated by pseudoreflections (i.e., elements that fix a hyperplane). Then, the algebra $\mathbb{C}[E]^W$ is a polynomial algebra in $d = \dim(E)$ generators. In particular, for a semisimple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$, the algebra of invariants $\mathbb{C}[\mathfrak{g}]^{\mathfrak{g}}$ is a polynomial algebra over $\mathbb{C}$ in rank$(\mathfrak{g})$ generators.*

Proof. See [**Ste09**], for now. □

Example 6.4.3. For $\mathfrak{g} = \mathfrak{sl}_n$, the coefficients of the characteristic polynomial of an element X:

$$\chi_X(t) = t^n + \operatorname{tr}(-X; \wedge^2\mathbb{C}^n)t^{n-2} + \cdots + \operatorname{tr}(-X; \wedge^{n-1}\mathbb{C}^n)t + \det(-X)$$

generate the ring $\mathbb{C}[\mathfrak{g}]^{\mathfrak{g}}$ freely.

Definition 6.4.4. The *fundamental degrees* of a finite reflection group W acting on a Euclidean space E are the degrees of a set of homogeneous free generators of the polynomial ring $\mathbb{C}[E]^W$. The fundamental degrees of a semisimple Lie algebra are the fundamental degrees of its root system.

For example, for $\mathfrak{g} = \mathfrak{sl}_n$, the fundamental degrees are $2, \ldots, n$. These degrees $(d_i,\ i = 1, \ldots, \dim E)$ are uniquely defined, and have some interesting properties, for example:

$$\prod_i d_i = |W|, \tag{6.4.4.1}$$

$$\sum_i d_i = \frac{|\Phi|}{2} + \dim E. \tag{6.4.4.2}$$

See [**Hum90**, §3].

6.5. The Harish-Chandra homomorphism

Definition 6.5.1. The *Harish-Chandra center* of a Lie algebra $\mathfrak{g}$ is the center $Z(\mathfrak{g})$ of the universal enveloping algebra $U(\mathfrak{g})$.

Notice that, although we write $Z(\mathfrak{g})$, this is not the center of $\mathfrak{g}$ itself (which is trivial for semisimple algebras), but of its universal enveloping algebra.

What goes by the name of "Harish-Chandra homomorphism" is actually an isomorphism, between $Z(\mathfrak{g})$ and the polynomial ring $\mathbb{C}[\mathfrak{h}^*]^{W,\bullet} = U(\mathfrak{h})^{W,\bullet}$, where $\bullet$ denotes the dot action of W, see 6.3.3. To construct it, we consider the action of $Z(\mathfrak{g})$ on a specific $\mathfrak{g}$-module with a commuting $\mathfrak{h}$-action.

Definition 6.5.2. The *universal Verma module* is the $\mathfrak{g}$-module M with the property that

$$\operatorname{Hom}_{\mathfrak{g}}(M, V) = \operatorname{Hom}_{\mathfrak{n}}(\mathbb{C}, V).$$

As with Verma modules, the universal Verma module M exists, and can be identified with

$$M = U(\mathfrak{g}) \otimes_{U(\mathfrak{n})} \mathbb{C} = U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{n}.$$

Writing

$$M = U(\mathfrak{g}) \otimes_{U(\mathfrak{n})} \mathbb{C} = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} (U(\mathfrak{b}) \otimes_{U(\mathfrak{n})} \mathbb{C}) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} U(\mathfrak{h}),$$

we see that M is a $\mathfrak{g} \times \mathfrak{h}$-module. The following lemma will give rise to the Harish-Chandra homomorphism:

Proposition 6.5.3. *For every $X \in Z(\mathfrak{g})$, there is a unique element $\phi(X) \in U(\mathfrak{h})$ such that the action of X on the universal Verma module M coincides with the action of $\phi(X)$. The resulting map*

$$\phi : Z(\mathfrak{g}) \to U(\mathfrak{h})$$

is a ring homomorphism.

Remark 6.5.4. Explicitly, Proposition 6.5.3 says that, under the identification $M = U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{n}$, the image of $Z(\mathfrak{g})$ in the quotient lies in the image of $U(\mathfrak{h})$; in other words, $Z(\mathfrak{g}) \subset U(\mathfrak{h}) + U(\mathfrak{g})\mathfrak{n}$.

Notice that $U(\mathfrak{h}) = S(\mathfrak{h}) = \mathbb{C}[\mathfrak{h}^*]$.

Proof. The action of $Z(\mathfrak{g})$ commutes with that of $U(\mathfrak{g})$, so it suffices to show that the action of X on a generator of the module M coincides with the action of some $\phi(X) \in U(\mathfrak{h})$. Take this generator to be the element $1 := 1 \otimes 1 \in U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} U(\mathfrak{h})$. This element is annihilated by the adjoint action of $\mathfrak{g}$, in particular, under the action of $\mathfrak{h}$ considered as a subalgebra of $\mathfrak{g}$. Using the Poincaré–Birkhoff–Witt theorem, we compute that M, restricted to $\mathfrak{h} \subset \mathfrak{g}$, is isomorphic to

$$U(\mathfrak{n}^-) \otimes U(\mathfrak{h}).$$

Since $U(\mathfrak{n}^-)^{\mathfrak{h}} = \mathbb{C}$, we get that $M^{\mathfrak{h}} = U(\mathfrak{h})$, so $X \cdot 1 = \phi(X) \in U(\mathfrak{h})$.

But $\phi(X)$ is also the image of the element $\phi(X) \in U(\mathfrak{h})$ acting on 1 via the action of $\mathfrak{h}$ that commutes with the action of $\mathfrak{g}$, which we will denote as a right action:

$$\phi(X) = 1 \cdot \phi(X).$$

Since the action of $U(\mathfrak{g})$ commutes with the action of $Z(\mathfrak{g})$, the same holds when we replace 1 by any element $Z \in M$:

$$X \cdot M = M \cdot \phi(X).$$

Therefore $\phi(XY) = 1 \cdot \phi(XY) = XY \cdot 1 = X(1 \cdot \phi(Y)) = (1 \cdot \phi(Y)) \cdot \phi(X) = 1 \cdot \phi(X)\phi(Y)$, and the map ϕ is a homomorphism. □

Definition 6.5.5. The homomorphism $\phi : Z(\mathfrak{g}) \to U(\mathfrak{h})$ of Proposition 6.5.3 is the *Harish-Chandra homomorphism.*

Proposition 6.5.6. *The center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$ acts on each Verma module M_λ by a character $\chi_\lambda : Z(\mathfrak{g}) \to \mathbb{C}$. If λ, μ are integral and conjugate by the dot action of the Weyl group (Definition 6.3.3), then $\chi_\lambda = \chi_\mu$.*

Proof. The center preserves the eigenspaces for the $\mathfrak{h}$-action, and since M_λ^λ is one-dimensional, it acts on it by a scalar. This generates M_λ under the $U(\mathfrak{g})$-action, so the center acts by the same scalar on all of M_λ.

If λ, μ are integral, λ is dominant, and $w \bullet \mu = \lambda$ for some $w \in W$, we saw in Proposition 6.3.2 that there is an embedding $M_\mu \hookrightarrow M_\lambda$, therefore $\chi_\mu = \chi_\lambda$. Since w is arbitrary, the same holds without the assumption that λ be dominant. □

Theorem 6.5.7. *The Harish-Chandra homomorphism is injective, and gives rise to an isomorphism*

$$Z(\mathfrak{g}) \simeq \mathbb{C}[\mathfrak{h}^*]^{W,\bullet},$$

where the exponent on the right means invariants with respect to the dot action 6.3.3.

Of course, the map $\lambda \mapsto \rho + \lambda$ induces, by pullback, an isomorphism $\mathbb{C}[\mathfrak{h}^*]^W \simeq \mathbb{C}[\mathfrak{h}^*]^{W,\bullet}$, but it is good to keep in mind that the most natural map gives rise to invariants with respect to the dot action.

Proof. Any $\lambda \in \mathfrak{h}^*$ defines a morphism of $\mathfrak{b}$-modules $U(\mathfrak{h}) \to \mathbb{C}_\lambda$, which by induction gives rise to a morphism $M \to M_\lambda$. Therefore, the character χ_λ by which $Z(\mathfrak{g})$ acts on M_λ is equal to $\lambda \circ \phi$, where ϕ is the Harish-Chandra homomorphism. For every integral λ, and every $w \in W$, we have, by Proposition 6.5.6, $\lambda \circ \phi = (w \bullet \lambda) \circ \phi$, and since those λ's are Zariski dense in $\mathfrak{h}^*$, the image of the Harish-Chandra homomorphism lies in the invariants for the dot action.

Having constructed the homomorphism $Z(\mathfrak{g}) \to \mathbb{C}[\mathfrak{h}^*]^{W,\bullet}$, there remains to show that it is a bijection. The argument to be used is quite general: once a homomorphism between filtered rings is constructed, to show that it is an isomorphism, it is enough to show this for their associated graded rings.

Notice that the natural filtration of $U(\mathfrak{g})$ is $\mathfrak{g}$-stable, and therefore

$$Z(\mathfrak{g}) = \lim_{\rightarrow} (F^d U(\mathfrak{g}))^{\mathfrak{g}}.$$

Thus, $Z(\mathfrak{g})$ is filtered by $F^d Z(\mathfrak{g}) = (F^d U(\mathfrak{g}))^{\mathfrak{g}}$. The Harish-Chandra homomorphism respects this filtration, and induces a homomorphism of the associated graded:

$$\mathrm{gr}\phi : \mathrm{gr}Z(\mathfrak{g}) \to \mathrm{gr}U(\mathfrak{h})^{W,\bullet}.$$

Also, notice that the shift by ρ in the definition of the dot action of W is not seen at the graded level, so $\mathrm{gr}U(\mathfrak{h})^{W,\bullet}$ is canonically equal to $S(\mathfrak{h})^W$ (invariants for the standard action of W).

In what follows, we will use an invariant bilinear form to identify $\mathfrak{g} \simeq \mathfrak{g}^*$, $\mathfrak{h} = \mathfrak{h}^*$, and apply the Chevalley isomorphism of Theorem 6.4.1. In doing so, we keep in mind that the invariant bilinear form identifies $\mathfrak{n}$ as the orthogonal complement of $\mathfrak{b}$. Therefore, the restriction map $S(\mathfrak{g}) = \mathbb{C}[\mathfrak{g}^*] \to \mathbb{C}[\mathfrak{h}^*] = S(\mathfrak{h})$ takes the ideal $S(\mathfrak{g})\mathfrak{n}$ to zero.

By Remark 6.5.4, any element in $Z(\mathfrak{g})$ belongs to the subspace $U(\mathfrak{h}) \oplus U(\mathfrak{g})\mathfrak{n} \subset U(\mathfrak{g})$. Restricting to the d-th piece of the filtration, we obtain a commutative diagram

[Look at pdf file if diagram does not appear.]

$$\begin{tikzcd}
F^d Z(\mathfrak{g}) \arrow[r, hook] \arrow[d, "HC"] & F^d\left(U(\mathfrak{h}) \oplus U(\mathfrak{g})\mathfrak{n}\right) \arrow[r, hook] \arrow[dl, "\mathrm{proj}"'] \arrow[d, "\mathrm{gr}"] & F^d U(\mathfrak{g}) \arrow[d, "\mathrm{gr}"] \\
F^d U(\mathfrak{h}) \arrow[dr, "\mathrm{gr}"] & S^d(\mathfrak{h}) \oplus S^{d-1}(\mathfrak{g})\mathfrak{n} \arrow[r, hook] \arrow[d, "\mathrm{proj}"] & S^d(\mathfrak{g}) \arrow[dl, "\mathrm{res}"] \\
 & S^d(\mathfrak{h}). &
\end{tikzcd}$$

The composition $F^d Z(\mathfrak{g}) \to S^d(\mathfrak{h})$ is precisely the grading of the Harish-Chandra homomorphism $\mathrm{gr}^d\phi : \mathrm{gr}^d Z(\mathfrak{g}) \to \mathrm{gr}^d U(\mathfrak{h})^{W,\bullet}$, as can be seen by following the arrows on the left.

On the other hand, because of complete reducibility (Theorem 5.2.8), the associated graded of $Z(\mathfrak{g})$ is

$$\mathrm{gr}^d Z(\mathfrak{g}) = \left(\mathrm{gr}^d U(\mathfrak{g})\right)^{\mathfrak{g}} = S^d(\mathfrak{g})^{\mathfrak{g}},$$

where we have used complete reducibility to say that the functor of $\mathfrak{g}$-invariants, applied to the short exact sequence

$$0 \to F^{d-1}U(\mathfrak{g}) \to F^d U(\mathfrak{g}) \to S^d(\mathfrak{g}) \to 0,$$

preserves exactness.

Thus, applying the functor of $\mathfrak{g}$-invariants on the right-most arrows in the diagram, and the Chevalley isomorphism 6.4.1, we obtain that $\mathrm{gr}^d\phi$ is an isomorphism: $\mathrm{gr}^d Z(\mathfrak{g}) \to S(\mathfrak{h})^W$. Thus, the Harish-Chandra homomorphism ϕ is an isomorphism onto $U(\mathfrak{h})^{W,\bullet} = \mathbb{C}[\mathfrak{h}^*]^{W,\bullet}$.

□

6.6. Localization with respect to $Z(\mathfrak{g})$

We return to the study of the category $\mathcal{O}$.

Lemma 6.6.1. *For every object V in $\mathcal{O}$, the action of $Z(\mathfrak{g})$ on V is locally finite.*

Proof. We have seen in Lemma 6.2.3 that every object can be filtered by surjective images of Verma modules. By Proposition 6.5.6, $Z(\mathfrak{g})$ acts by a scalar on Verma modules, hence also on their quotients. Therefore, it acts locally finitely on finite extensions of such objects. □

Set $\mathfrak{h}^*//_\bullet W = \mathrm{Spec}\mathbb{C}[\mathfrak{h}^*]^{W,\bullet}$, so that the maximal ideals of $Z(\mathfrak{g})$ are the complex points of the quotient $\mathfrak{h}^*//_\bullet W$, which are the points of the set-theoretic quotient of $\mathfrak{h}^*$ by the dot action of W.

For $\chi \in \mathfrak{h}^*//_\bullet W$, we let $\mathcal{O}_\chi$ denote the full subcategory consisting of objects of $\mathcal{O}$ which are generalized eigenspaces for $Z(\mathfrak{g})$ with generalized eigencharacter χ.

Theorem 6.6.2. (1) *The category $\mathcal{O}$ is a direct sum of categories $\mathcal{O}_\chi$, with χ varying over the complex points of $\mathfrak{h}^*//_\bullet W$.*

(2) *If λ is such that $\lambda - w\bullet\lambda$ is not, for any element $w \in W$, in the positive root monoid $R^+ = \{\sum_{\alpha\in\Phi^+} n_\alpha\alpha | n_\alpha \in \mathbb{N}\}$ and nonzero, then M_λ is irreducible.*

(3) *Every object in $\mathcal{O}$ is of finite length.*

(4) *The classes of the Verma modules M_λ (or, equivalently, their irreducible quotients L_λ) are a basis for the Grothendieck group $\mathbb{Z}[\mathcal{O}]$.*

Proof. (1) Since the action of the center is locally finite, we can decompose every object into a direct sum of generalized $Z(\mathfrak{g})$-eigenspaces. Any $\mathfrak{g}$-morphism commutes with the action of $Z(\mathfrak{g})$, so there are no $\mathfrak{g}$-morphisms between them.

(2) If M_λ is not irreducible, it contains a highest weight vector of weight $\mu < \lambda$ (in the same partial ordering as previously, i.e., $\mu \le \lambda$ means that $\lambda - \mu \in R^+$, the positive root monoid), hence there is a nontrivial map from M_μ to M_λ. By the decomposition of the category, on the other hand, such a nontrivial map can exist only if $\mu = w \bullet \lambda$ for some $w \in W$.

(3) By Lemma 6.2.3, it suffices to show that Verma modules are of finite length. Let K_λ be the kernel of the map $M_\lambda \to L_\lambda$. If nonzero, then K_λ admits a filtration as in Lemma 6.2.3, whose factors are surjective images

of modules M_μ with $\mu < \lambda$. But by the decomposition of categories, μ has to be a W-conjugate of λ (for the dot action), hence after a finite number of steps the module M_μ will be irreducible, by the previous statement.

(4) By the same argument, every object can be filtered by successive quotients of Verma modules, so they generate the Grothendieck group of the category. There cannot be a nontrivial relation between them,
$$\sum n_i[M_{\lambda_i}] = 0,$$
because for any $\lambda = \lambda_i$ which is maximal among the λ_i's in the partial ordering of weights, the λ-weight space of M_λ cannot be cancelled by another term. The fact that the simple modules L_λ also form a basis follows from the fact that the category is Artinian (every object is of finite length), and they are the only irreducible objects (non-isomorphic to each other).

□

As before, let C be the ring of formal sums $\sum_{\lambda\in\mathfrak{h}^*} c(\lambda)e^\lambda$, where $c_\bullet : \mathfrak{h}^* \to \mathbb{Z}$ is supported in a finite number of translates of the negative root monoid, and multiplication defined by $e^\lambda \cdot e^\mu = e^{\lambda+\mu}$.

Definition 6.6.3. The *Weyl denominator* is the following element of C:
$$\Delta = \prod_{\alpha>0}\left(e^{\frac{\alpha}{2}} - e^{-\frac{\alpha}{2}}\right) = e^\rho \prod_{\alpha>0}\left(1 - e^{-\alpha}\right)$$
(the product over all positive roots), thought of as a power series in elements of $\rho - R^+$ (where R^+ is the positive root monoid).

Notice that the weight ρ is integral because $\alpha = \rho - w_\alpha\rho = \langle\rho, \check{\alpha}\rangle\,\alpha$ for every simple root α. The name "Weyl denominator" is due to its appearance in the Weyl character formula, Theorem 6.6.5.

Proposition 6.6.4. *The character of the Verma module M_λ satisfies:*
$$\Delta \cdot ch_{M_\lambda} = e^{\lambda+\rho}.$$

Proof. As an $\mathfrak{h}$-module, $M_\lambda = U(\mathfrak{n}_-) \otimes \mathbb{C}_\lambda$, so $\mathrm{ch}(V) = \mathrm{ch}(U(\mathfrak{n}_-)) \cdot \mathrm{ch}(\mathbb{C}_\lambda) = \chi(U(\mathfrak{n}^-)) \cdot e^\lambda$. Therefore, it suffices to prove that the character of $U(\mathfrak{n}_-)$ is e^ρ/L, understood as a power series in the *negative* weight monoid.

By the Poincaré–Birkhoff–Witt theorem 4.3.6, as $\mathfrak{h}$-modules we have: $U(\mathfrak{n}_-) = \otimes_{\alpha>0}S(\mathfrak{g}_{-\alpha})$. The character of $S(\mathfrak{g}_{-\alpha})$ is $1 + e^\alpha + e^{2\alpha} + \cdots = \frac{1}{1-e^{-\alpha}}$, and this proves the proposition. □

Finally, we are ready to prove the *Weyl character formula*:

Theorem 6.6.5. *The character of the irreducible representation with heighest weight λ is given by the* Schur polynomial*:*
$$ch_{V_\lambda} = \frac{\sum_{w\in W} sgn(w)e^{w(\lambda+\rho)}}{\Delta}$$

Proof. By Proposition 6.3.4 and Theorem 6.6.2, we have $V_\lambda = L_\lambda$, and an equality of the form
$$[V_\lambda] = [M_\lambda] + \sum_{w\in W, w\neq 1} c_w[M_{w\bullet\lambda}]$$

in the Grothendieck group, for some integers c_w. Indeed, Proposition 6.3.4 represents L_λ as a quotient of M_λ by the image of a morphism

$$\bigoplus_{w\neq 1} M_{w\bullet\lambda} \to M_\lambda,$$

and by Theorem 6.6.2 the kernel of this morphism will have a finite composition series in terms of the $M_{w\bullet\lambda}$'s, necessarily with $w \neq 1$ as the weight λ does not appear in the kernel.

Hence,

$$\Delta \cdot \mathrm{ch}(V_\lambda) = e^{\lambda+\rho} + \sum_{w\in W, w\neq 1} c_w e^{w(\lambda+\rho)}.$$

On the other hand, again by Proposition 6.3.4, the character is W-invariant. Therefore the expression on the right should be (W, sgn)-equivariant. Therefore, $c_w = \mathrm{sgn}(w)$. □

6.7. Example: Irreducible representations of $\mathfrak{sl}_n$. Schur–Weyl duality.

Schur–Weyl duality refers to a correspondence between representations of symmetric groups and general linear groups (or their Lie algebras), which is realized inside the tensor powers $V^{\otimes^d}$ of a vector space. It is based on the following theorem from linear algebra:

Theorem 6.7.1. *If V is a finite-dimensional complex vector space, $A \subset End_B(V)$ is a semisimple subalgebra of operators, and $B = End_A(V)$ is its commutant, then*

(1) *B is semisimple.*
(2) *$A = End_B(V)$.*
(3) *There is a bijection $M_i \leftrightarrow N_i$ between isomorphism classes of simple A-modules and isomorphism classes of simple B-modules, and an isomorphism of $A \otimes B$-modules*

$$V = \bigoplus_i M_i \otimes N_i.$$

Proof. The proof uses the Artin–Wedderburn theorem which, in the case of the complex numbers, says that a finite-dimensional complex semisimple algebra is the direct sum of the endomorphism algebras of its simple modules.

Let M_i range over all isomorphism classes of simple A-modules. Since V is A-semisimple,

$$V = \bigoplus_i M_i \otimes \mathrm{Hom}_A(M_i, V). \tag{6.7.1.1}$$

Set $N_i = \mathrm{Hom}_A(M_i, V)$. Since the M_i's are non-isomorphic, we have, by Schur's lemma,

$$B = \mathrm{End}_A(V) = \bigoplus_i \mathrm{End}(N_i).$$

The unique isomorphism class of simple $\mathrm{End}(N_i)$-modules is N_i, hence B is semisimple, with its simple modules being precisely the N_i's, which are non-isomorphic. Applying now the same reasoning to (6.7.1.1), we see that

$$\mathrm{End}_B(V) = \bigoplus_i \mathrm{End}(M_i).$$

But A being semisimple means that it is isomorphic to $\bigoplus_i \mathrm{End}(M_i)$, thus $\mathrm{End}_B(V) = A$ and the map $M_i \mapsto N_i$ is a bijection of isomorphism classes of simple modules. $\square$

Now we apply this to S_d and $\mathfrak{gl}(V)$:

Theorem 6.7.2 (Schur–Weyl duality). *Consider the space $V^{\otimes^d}$ under the commuting actions of S_d and $\mathfrak{gl}(V)$, i.e., as a representation of the algebra $A \otimes B$, where $A = \mathbb{C}[S_d]$ and $B = U(\mathfrak{gl}(V))$. Then, the images $\bar{A}$, $\bar{B}$ of A and B in End$(V^{\otimes^d})$ are each others' commutants, that is,*

$$\bar{A} = \mathit{End}_B(V^{\otimes^d}), \text{ and}$$
$$\bar{B} = \mathit{End}_A(V^{\otimes^d}).$$

We have a decomposition

$$V^{\otimes^d} = \bigoplus_{\tau} \tau \otimes \theta(\tau), \tag{6.7.2.1}$$

where τ ranges all isomorphism classes of irreducible representations of S_d, and the $\theta(\tau) := \mathit{Hom}_{S_d}(\tau, V^{\otimes^d})$ are either zero, or distinct irreducible representations of $\mathfrak{gl}(V)$.

Notice that the action of $\mathfrak{gl}(V)$ on $V^{\otimes^d}$ is defined as we define tensor products of representations of Lie algebras, i.e., the image of $e \in \mathfrak{gl}(V)$ in $\mathrm{End}(V^{\otimes^d})$ is the element $S_d e := \sum_{i=1}^d 1 \otimes \cdots \otimes e$ (i-th factor) $\otimes \cdots \otimes 1$.

Proof. Since both subalgebras are semisimple (complete reducibility), by Theorem 6.7.1 it is enough to prove the second claim.

We have $\mathrm{End}_A(V^{\otimes^d}) = \mathrm{End}(V^{\otimes^d})^{S_d} = (\mathrm{End}(V)^{\otimes^d})^{S_d} = S^d\mathrm{End}(V)$, and the d-th symmetric power of any vector space E is spanned by the symmetric tensors $e \otimes \cdots \otimes e$, for $e \in E$. In this case, $E = \mathrm{End}(V) = B$. By the theory of symmetric polynomials, $e \otimes \cdots \otimes e$ is a polynomial in the elements $S_d(e^i)$, $i = 1, \ldots, d$, which are in the image of $\mathfrak{gl}(V)$. $\square$

Finally, we notice

Lemma 6.7.3. *Every irreducible representation of $\mathfrak{gl}_n$ restricts irreducibly to $\mathfrak{sl}_n$.*

Proof. We have $\mathfrak{gl}_n = \mathfrak{z} \oplus \mathfrak{sl}_n$, where $\mathfrak{z}$ is the center, but the center acts by a scalar, by Schur's lemma, so any $\mathfrak{sl}_n$-invariant subspace is also $\mathfrak{gl}_n$-invariant. $\square$

Therefore, the irreducible representations of $\mathfrak{gl}_n$ constructed in Theorem 6.7.9 are also irreducible for $\mathfrak{sl}_n$. We will now classify irreducible representations of the symmetric group, and make the correspondence explicit, observing, in particular, that when d is large enough, the decomposition 6.7.2.1 contains all irreducible representations of $\mathfrak{sl}_n$.

We follow, and reformulate, [**FH91**], where we point the reader for more fun, to describe irreducible representations of the symmetric group S_d.

For the Lie algebra $\mathfrak{gl}_n$ with the standard Cartan of diagonal elements and the standard Borel of upper triangular elements, the dominant, integral weights are of the form

$$\mathrm{diag}(z_1, z_2, \ldots, z_n) \mapsto \lambda_1 z_1 + \cdots + \lambda_n z_n,$$

with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ some integers.

For the Lie algebra $\mathfrak{sl}_n$, the positive, integral weights are described similarly, except that the λ_i's are determined modulo the operation of adding the same constant to all of them. To reduce ambiguity, we can always take $\lambda_n = 0$ (but won't be doing that yet).

On the other hand, if $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n \ge 0$ are integers, and $d = \sum_i \lambda_i$, the λ_i's describe a conjugacy class in the symmetric group S_d, namely, the conjugacy class of elements which can are products of disjoint cycles of lengths $\lambda_1, \lambda_2, \ldots, \lambda_n$. Notice that λ_n has been taken to be ≥ 0 here, which is slightly restrictive for $\mathfrak{gl}_n$, but not for $\mathfrak{sl}_n$; in fact, for $\mathfrak{sl}_n$, any integral, dominant weight defines a conjugacy class in any S_{d+kn}, for the minimal d determined by $\lambda_n = 0$.

We consider $G = S_d$ as the permutation group on the set $\Sigma = \{1, \ldots, d\}$, and for every $\lambda : \lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n \ge 0$, a λ-partition of Σ will be a disjoint decomposition $\Sigma = \bigsqcup_i \Sigma_i$, with $|\Sigma_i| = \lambda_i$. We can think of λ as a *Young diagram*, that is, the diagram consisting of a row of λ_1 squares stacked over λ_2 squares (aligned on the left), etc,[1] and we can also think of *Young tableaux*, which are ways to populate the squares of a given Young diagram with the elements of Σ (without repetitions). Then, the group $G = S_d$ acts on the space of Young tableaux of a given shape λ (we define this action as a right action), and the space Σ_λ of λ-partitions of Σ is the homogeneous space $P\backslash G$, where $P = G_\lambda$ is the stabilizer of the rows of the standard Young tableau (where the integers are placed in order).

The *dual partition* to λ is partition $\lambda^* : \lambda_1^* \ge \lambda_2^* \ge \ldots \lambda_m^* > 0$ of d counting the sizes of the columns of the Young diagram of λ. The space Σ_{λ^*} of λ^*-partitions of Σ is the homogeneous space $Q\backslash G$, where $Q = G_{\lambda^*}$ is the stabilizer of the columns of the standard Young tableau.

We will construct the irreducible representations of G by inducing the trivial and sign representation from the groups P and Q. None of them is irreducible, but they share a unique irreducible component.

The double quotient space $P\backslash G/Q$ can be identified with $(\Sigma_\lambda \times \Sigma_\mu)/G^{\mathrm{diag}}$, i.e., with the set of pairs (σ, τ), where σ is a λ-partition of Σ and τ is a μ-partition of Σ, *up to relabeling the elements of* Σ.

The classification of irreducible representations of S_d rests upon the following fundamental lemma:

Lemma 6.7.4. *If λ, μ are two partitions of d, and (σ, τ) is a pair consisting of a λ-partition σ of Σ and a μ^*-partition τ of Σ with no pair (k, l) of elements of Σ in the same subset of σ and of τ, then τ is the refinement of a λ^*-partition of Σ; in particular, $\mu \ge \lambda$ in the lexicographic ordering, i.e., $\mu = \lambda$ or at the first index i where $\mu_i \ne \lambda_i$ we have $\mu_i > \lambda_i$.*

If $\mu = \lambda$ then the only pairs (σ, τ) without a pair (k, l) in the same subset of σ and of τ are those where σ, τ are the rows, resp. columns, of a single Young tableau.

Proof. Exercise. $\square$

Now, we let M_λ, resp. A_λ, be the G-equivariant line bundles over the space Σ_λ which are induced, respectively, from the trivial, resp. sign character, of P. Explicitly, sections of M_λ are left-P-invariant functions on G, i.e., left-P-invariant elements of $\mathbb{C}[G]$, while sections of A_λ are functions on G which vary by the sign

[1] Since we allow some λ_i's to be zero, the Young diagram does not determine λ, unless n is known, but this will not play a role in what follows.

character under left translation by P. For notational simplicity, we will identify the bundles with their space of sections.

Proposition 6.7.5. *We have* $\dim \mathrm{Hom}_G(M_\lambda, A_{\lambda^*}) = 1$.
If $\lambda > \mu$*, we have* $\dim \mathrm{Hom}_G(M_\lambda, A_{\mu^*}) = 0$.

Proof. The space of G-morphisms $\mathrm{Hom}_G(M_\lambda, A_{\mu^*})$ can naturally be identified with the space of G^{diag}-invariant sections of $L := M_\lambda \otimes A_{\mu^*}$ over $\Sigma_\lambda \times \Sigma_{\mu^*}$, considered as kernel functions, i.e., the morphism T_K corresponding to a section K is

$$T_K(f)(y) = \sum_{x \in \Sigma_\lambda} f(x)K(x,y).$$

By Lemma 6.7.4, if $\lambda \geq \mu$, for a pair $(\sigma, \tau) \in \Sigma_\lambda \times \Sigma_{\mu^*}$ there is a transposition $t = (k, l)$ which stabilizes both σ and τ, unless $\lambda = \mu$ and the pair (σ, τ) corresponds to the rows and columns of a Young tableau. But then, t will act by -1 on the fiber of L over (σ, τ), which means that its orbit cannot support a G-invariant section of L. □

Theorem 6.7.6. *The image of a nonzero* G*-morphism* $M_\lambda \to A_{\lambda^*}$ *is an irreducible representation* V_λ*. For* λ, μ *different partitions,* V_λ, V_μ *are non-isomorphic, and these are all the irreducible representations of* $G = S_d$*.*

Proof. The image of a nonzero G-morphism $M_\lambda \to A_{\lambda^*}$ has to be an irreducible representation V_λ, because otherwise, the space $\mathrm{Hom}(M_\lambda, A_{\lambda^*})$ would have dimension > 1, by scaling the irreducible summands in the image by different scalars. This would contradict Proposition 6.7.5.

Let $\lambda \neq \mu$. Without loss of generality, assume that $\lambda > \mu$ in the lexicographic order. Then, again by Proposition 6.7.5, there are no G-morphisms from M_λ to A_{μ^*}. Therefore, V_λ cannot embed in A_{μ^*}, and is non-isomorphic to V_μ.

The number of partitions λ of d is equal to the number of conjugacy classes of S_d, so we have constructed all the isomorphism classes of irreducible S_d-representations. □

Explicitly, by the bijection between $P\backslash G/Q$ and $(\Sigma_\lambda \times \Sigma_\lambda^*)/G^{\mathrm{diag}}$, we can think of a G-invariant section of L as an element of $\mathbb{C}[G]$ which is left-P-invariant and varies by the sign character under right multiplication by Q. Then, a basis element for the space of invariant sections is the element

$$c_\lambda = a_\lambda b_\lambda,$$

where

$$a_\lambda = \sum_{g \in P} g,$$

$$b_\lambda = \sum_{g \in Q} \mathrm{sgn}(g) \cdot g.$$

Lemma 6.7.7. *Let* $a_\lambda, b_\lambda, c_\lambda \in \mathbb{C}[S_d]$ *be as above. Then, the irreducible representation* V_λ *of* S_d *is isomorphic to the module* $\mathbb{C}[S_d]c_\lambda$*, or equivalently to the module* $c_\lambda^*\mathbb{C}[S_d]$*, where* $c_\lambda^* = b_\lambda a_\lambda$*.*

For any vector space V*, the action of* a_λ *induces a surjective map onto the tensor product of symmetric powers*

$$V^{\otimes^d} \to S^{\lambda_1}V \otimes \cdots \otimes S^{\lambda_n}V,$$

while the action of b_λ induces a surjective map onto the tensor product of exterior powers

$$V^{\otimes^d} \to \bigwedge^{\lambda_1^*} V \otimes \cdots \otimes \bigwedge^{\lambda_m^*} V.$$

Moreover:

(1) $a_\lambda \cdot x \cdot b_\nu = 0$ *whenever* $\nu : \nu_1 \geq \dots \nu_r \geq 0$ *is a partition of* d *which is* smaller *than* λ *in the lexicographic ordering, i.e.,* $\nu_i = \lambda_i$ *for some* j *and all* $i < j$*, while* $\lambda_j > \nu_j$.

(2) c_λ *is the only element* $c \in \mathbb{C}[S_d]$*, up to scalar, with the property that* $pcq = sgn(q)c$ *for all* $p \in P$*,* $q \in Q$.

(3) $c_\lambda x c_\lambda$ *is a multiple of* c_λ*, for every* $x \in \mathbb{C}[S_d]$*. In particular,* c_λ *is an idempotent up to a scalar, i.e.,* $c_\lambda^2 = n_\lambda c_\lambda$ *for some* $n_\lambda \in \mathbb{C}$*. This scalar is* $n_\lambda = \frac{d!}{\dim V_\lambda}$.

Proof. We can consider $\mathbb{C}[G]c_\lambda$ as a submodule of $A_{\lambda^*} = \mathrm{Ind}_Q^G(\mathrm{sgn})$. The module $M_\lambda = \mathrm{Ind}_P^G(1)$ is generated by the characteristic function of $P1$, and its image in A_{λ^*} under c_λ, understood as a kernel function as in the proof of Proposition 6.7.5, is $1c_\lambda \in \mathbb{C}[G]c_\lambda \subset A_{\lambda^*}$. Equivalently, we can realize V_λ as the image of A_{λ^*} in M_λ under the adjoint operator (given by the same kernel), and then we obtain the submodule $\mathbb{C}[G]c_\lambda^* \subset M_\lambda$.

The actions of a_λ, b_λ on $V^{\otimes^d}$ are easy to describe from the definitions.

To prove $a_\lambda \cdot x \cdot b_\nu = 0$, it is enough to consider the basis elements $x = g \in S_d$, and then by renaming the elements it is enough to consider $g = 1$. If $\lambda > \nu$ (lexicographically), there are two elements k, l which belong to the same row in the Young diagram for λ and in the same column for ν. If $t = (k, l)$ then $a_\lambda \cdot t = a_\lambda$, $t \cdot b_\nu = -b_\nu$, hence $a_\lambda b_\nu = a_\lambda t \cdot t b_\nu = -a_\lambda b_\nu$, hence is zero.

The uniqueness (up to scalar) of c_λ with this property is a reformulation of Proposition 6.7.5, considering such elements, as in the proof of that proposition, as G^{diag}-invariant kernels.

Clearly, $c_\lambda x c_\lambda$ has this property, therefore is a multiple of c_λ. To determine the scalar n_λ, consider the operator of right multiplication by c_λ, as an endomorphism of $\mathbb{C}[S_d]$. It acts by n_λ on its image $\mathbb{C}[S_d]c_\lambda$, which is isomorphic to V_λ, hence its trace is $n_\lambda \dim[V_\lambda]$. On the other hand, the coefficient of the identity element in c_λ is 1, so its trace is equal to $\dim \mathbb{C}[S_d] = d!$. □

Definition 6.7.8. The element c_λ of the group algebra $\mathbb{C}[S_d]$ defined above is called the *Young symmetrizer* attached to the partition λ.

Theorem 6.7.9. *Let* V *be the standard representation of* $\mathfrak{sl}_n$*, and consider* $V^{\otimes^d}$ *as a representation of* $S_d \times \mathfrak{sl}_n$*. Denote by* Π_n*,* Π_d' *the sets of isomorphism classes of irreducible representations of* $\mathfrak{sl}_n$*, resp.* S_d*. There is a map* $\sigma : \Pi_d' \to \Pi_n \cup \{0\}$*, where* 0 *denotes the zero-dimensional representation, such that*

$$V^{\otimes^d} = \bigoplus_{\pi \in \Pi_d'} \pi \otimes \sigma(\pi)$$

as $S_d \times \mathfrak{sl}_n$*-modules.*

The map σ *is induced by the* Schur functor*, sending a vector space* V *to* $c_\lambda V^{\otimes^d}$ *(or, equivalently, to* $c_\lambda^* V^{\otimes^d}$*), where* c_λ *is the Young symmetrizer of Definition 6.7.8* *(and $c_\lambda^* = b_\lambda a_\lambda$ its adjoint). If $\lambda : \lambda_1 \geq \cdots \geq \lambda_m$ is a partition, and $\lambda^* : \lambda_1^* \geq \cdots \geq \lambda_r^*$ its dual, the space $c_\lambda V^{\otimes^d}$ is the image of the subspace*

$$\bigwedge^{\lambda_1^*} V \otimes \cdots \otimes \bigwedge^{\lambda_m^*} V$$

of $V^{\otimes^d}$ under the symmetrization map:

$$V^{\otimes^d} \to S^{\lambda_1} V \otimes \cdots \otimes S^{\lambda_m} V$$

(and, respectively, the space $c_\lambda^* V^{\otimes^d}$ is the image of the above product of symmetric powers in the above produce of alternating powers, under the antisymmetrization map). Here, we think of the factors of V as labelled by the boxes of a Young diagram of shape λ, with symmetric powers taken among the factors in the same row, and exterior powers taken among the factors in the same column.

The map σ takes the irreducible representation of S_d parametrized by the partition λ to the irreducible $\mathfrak{sl}_n$-module of heighest weight $\lambda_1 \geq \cdots \geq \lambda_m \geq 0 \geq \cdots \geq 0$, if $m \leq n$, or to zero, otherwise.

Consequently, the map σ is injective away from the fiber of zero, does not have zero in the image if $n \geq d$, and the resulting map $\sqcup_d \Pi'_d \to \Pi_n \cup \{0\}$ is surjective.

Proof. The existence of the map σ follows from the double centralizer theorem 6.7.1 and Theorem 6.7.2. Under this theorem, $\sigma(V_\lambda) = \mathrm{Hom}_{S_d}(V_\lambda, V^{\otimes^d})$. Realizing V_λ as $\mathbb{C}[S_d]c_\lambda$, by the idempotence of c_λ the space $\mathrm{Hom}_{S_d}(V_\lambda, V^{\otimes^d})$ can be identified with $c_\lambda V^{\otimes^d}$, by the map that assigns to a morphism the image of $1c_\lambda$. Equivalently, we can realize V_λ as $\mathbb{C}[S_d]c_\lambda$, and the same argument holds.

The description of $c_\lambda V^{\otimes^d}$, $c_\lambda^* V^{\otimes^d}$ follows from Lemma 6.7.7. We determine the highest weight, using the realization $\sigma V_\lambda = c_\lambda^* V^{\otimes^d}$. Let $x_1, \ldots, x_n$ be a basis for V, and consider the vector

$$\otimes^{\lambda_1} x_1 \otimes \otimes^{\lambda_2} x_2 \cdots \otimes \otimes^{\lambda_n} x_n \in S^{\lambda_1} V \otimes \cdots \otimes S^{\lambda_m} V.$$

Its image in $\bigwedge^{\lambda_1^*} V \otimes \cdots \otimes \bigwedge^{\lambda_m^*} V$ (where, recall, we are labeling the factors of V according to the boxes in a Young diagram, and antisymmetrize along the columns) is the vector

$$\mathcal{A}(x_1 \otimes x_2 \otimes \cdots \otimes x_{\lambda_1^*}) \otimes \cdots \otimes \mathcal{A}(x_1 \otimes x_2 \otimes \cdots \otimes x_{\lambda_r^*}),$$

where $\mathcal{A}$ denotes the antisymmetrization map.

Evidently, this is an eigenvector for the parabolic subgroup of GL_n (and, a fortiori, of SL_n) that stabilizes the flag

$$\mathrm{span}(x_1, x_2, x_{\lambda_1^*}) \supset \mathrm{span}(x_1, x_2, x_{\lambda_2^*}) \supset \cdots \supset \mathrm{span}(x_1, x_2, x_{\lambda_r^*}),$$

and its weight is $(\lambda_1, \lambda_2, \ldots, \lambda_n)$.

□

6.8. The Kazhdan–Lusztig conjectures

[TBA]

6.9. Other chapters

(1) Introduction
(2) Basic Representation Theory
(3) Representations of compact groups

(4) Lie groups and Lie algebras: general properties
(5) Structure of finite-dimensional Lie algebras
(6) Verma modules
(7) Linear algebraic groups
(8) Forms and covers of reductive groups, and the L-group
(9) Galois cohomology of linear algebraic groups
(10) Representations of reductive groups over local fields
(11) Plancherel formula: reduction to discrete spectra
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CHAPTER 7

Linear algebraic groups

[This chapter needs a lot of improvement.]

7.1. Diagonalizable groups

We pass to the study of linear algebraic groups, over a general field k. [The rest of this chapter is quite incomplete; it strives to become complete some day, though, so if you notice any omissions/gaps in the arguments, that are not noted, please notify!]

The definitions will follow a different order than in the case of Lie algebras, because we want to distinguish between the additive group $\mathbb{G}_a = \mathrm{Spec} k[T]$ and the multiplicative group $\mathbb{G}_m = \mathrm{Spec} k[T, T^{-1}]$, whose Lie algebras are the same. It is easy to see that *there are no non-trivial morphisms between these two groups.*

Definition 7.1.1. The *character group* $X^\bullet(G)$ of a linear algebraic group G is the group of morphisms $G \to \mathbb{G}_m$. A linear algebraic group G over an field k is called *diagonalizable* if $\bar{k}[G]$ is spanned, as a vector space, by the $\bar{k}$-rational characters: $\bar{k}[G] = \bar{k}[X_\bullet(G_{\bar{k}})]$. A *torus* is a connected diagonalizable group. A diagonalizable group G is said to be *split* if $X^\bullet(G) = X^\bullet(G_{\bar{k}})$.

Theorem 7.1.2. *Any character of a diagonalizable group G over the algebraic closure $\bar{k}$ is defined over a finite separable extension of k. If Γ denotes the Galois group of the separable closure k^s over k, the contravariant functor that assigns to any group its $\bar{k}$-character group gives rise to a contravariant equivalence of categories:*

$$\{\textit{diagonalizable } k\textit{-groups}\} \leftrightarrow$$

$$\{\textit{finitely generated abelian groups without } p\textit{-torsion, with a continuous } \Gamma\textit{-action}\},$$

where p is the characteristic exponent of k. Under this equivalence, tori correspond to torsion-free abelian groups.

Proof. Choose any embedding $G \subset \mathrm{GL}(V)$, for a finite-dimensional vector space V. For the first statement, it is enough to show that G can be diagonalized over (a finite subextension of) k^s. By functoriality of the Jordan decomposition, the image of G consists of semisimple elements. Therefore, the minimal polynomial of every $g \in G(k)$ has distinct roots, which are therefore defined over a finite separable extension. Therefore, G can be diagonalized over a finite separable extension.

The rest of the statements are left to the reader. $\square$

It is often useful to replace tori by *induced tori*:

Definition 7.1.3. An *induced torus* over a field k is a torus of the form $\mathrm{Res}_{E/k}\mathbb{G}_m$, where E/k is a finite étale extension, i.e., a finite product of separable field extensions.

Lemma 7.1.4. *For every torus T over k, there is a monomorphism $T \hookrightarrow S_1$ and an epimorphism $S_2 \twoheadrightarrow T$, where S_1, S_2 are induced tori.*

Proof. Consider the Γ-module $\Lambda = X^*(T)$, and let E be the fixed field of the kernel of the action of Γ on Λ. If $\Gamma_E \subset \Gamma$ denotes the Galois group of k^s/E, the identity morphism on Λ gives rise to a Γ-equivariant embedding $\Lambda \hookrightarrow \mathrm{Ind}_{\Gamma_E}^{\Gamma} \Lambda|_{\Gamma_E}$, and the latter is the character group of an induced torus $S_2 \simeq \mathrm{Res}_{E/k}\mathbb{G}_m^r$, where r is the rank of Λ. Vice versa, we can write Λ as the quotient of a free Γ/Γ_E-module M, which is then the character group of an induced torus $S_1 \simeq \mathrm{Res}_{E/k}^{r'}$, where r' is the rank of M. $\square$

7.2. Unipotent, solvable, semisimple, and reductive groups

A main goal in our discussion of linear algebraic groups will be to recover some of the structure of semisimple Lie algebras that holds in characteristic zero but fails in positive characteristic. The action of the group on its coordinate ring allows us to recover, e.g., the Jordan decomposition. We define notions of semisimplicity etc with respect to this action. We denote by L, resp. R, the left, resp. right action of G on $k[G]$; recall that this is a locally finite action, so for every $v \in k[G]$ there is a finite-dimensional stable subspace $V \subset k[G]$ containing v, such that L (or R) is a morphism of algebraic groups $G \to \mathrm{GL}(V)$.

Definition 7.2.1. Let G be a linear algebraic group over a field k. An element $g \in G(k)$ is called *semisimple* if $R(g)$ is semisimple, and unipotent if $R(g)$ is unipotent.

Theorem 7.2.2 (Jordan decomposition)**.** *Let G be a linear algebraic group over a field k in characteristic $p \geq 0$.*

Every element $g \in G(k)$ admits a unique decomposition $g = g_s g_u$ in $G(k^{p^{-\infty}})$, with g_s semisimple and g_u unipotent, commuting with each other.

Moreover, every element $X \in \mathfrak{g}(k)$ admits a unique decomposition $X = X_s + X_n$ in $\mathfrak{g}(k^{p^{-\infty}})$, with $R(X_s)$ semisimple and $R(X_n)$ nilpotent, commuting with each other.

If $G \to G'$ is a morphism of linear algebraic groups, the Jordan decompositions of elements in the group or the Lie algebra are preserved.

Notice, in particular, that for $\mathfrak{g}$ semisimple in characteristic zero, the Jordan decomposition in $\mathfrak{g}$ is the same as the one defined by the adjoint representation in Theorem 5.2.15.

Proof. Assume first that k is algebraically closed. The right action being locally finite, hence a sum of finite-dimensional representations $G \to \mathrm{GL}(V)$, we can apply the Jordan decomposition for $\mathrm{GL}(V)$, to conclude that $R(g) = R(g)_s R(g)_u$ for unique semisimple, resp. unipotent $R(g)_s$, $R(g)_u$ which are *polynomials in* $R(g)$ (in $\mathrm{End}(V)$). The image of G in $\mathrm{GL}(V)$ is closed [General fact about morphisms of algebraic groups, to be added]. If $B \subset k[\mathrm{GL}(V)]$ is the ideal defining its image, it is stable under the right action of G, hence of $R(g)_s$ and $R(g)_u$ (since they are polynomial in $R(g)$). But the subgroup of elements in $\mathrm{GL}(V)(k)$ stabilizing this ideal is equal to $G(k)$, since any other right coset of G is a different closed subvariety of $\mathrm{GL}(V)$. Thus, $R(g)_s$ and $R(g)_u$ are the images of unique elements g_s, g_u of $G(k)$.

If k is not algebraically closed, by uniqueness of the Jordan decomposition these elements are fixed under the Galois group, and therefore defined over the maximal inseparable extension $k^{-p^{\infty}}$.

The statements on Lie algebras follow similarly. □

Definition 7.2.3. The *derived series* of an algebraic group G is the series of normal subgroups

$$\mathcal{D}^0(G) = G$$
$$\mathcal{D}^{i+1}(G) = \mathcal{D}(\mathcal{D}^i G),$$

where $\mathcal{D}$ denotes the commutator subgroup.

An algebraic group G is called *solvable* if $\mathcal{D}^n G = 1$ for some n.

A linar algebraic group G is called *unipotent* if $g = g_u$ in terms of the Jordan decomposition of Theorem 7.2.2, for every $g \in G(\bar{k})$.

Theorem 7.2.4. *Let G be a unipotent algebraic group over a field k. The only (algebraic) irreducible representation of G is the trivial one. For any representation $\rho : G \to GL(V)$, there is a full flag in V with respect to which V is upper triangular. Any unipotent group is solvable.*

This is the group analog of Engel's theorem 5.1.10.

Proof. Let V^G be the subspace of G-fixed vectors: By definition, this is the maximal G-stable subspace of V with the property that the action morphism $G \times V \to V$ coincides with the projection to V. It is clearly defined over k, so if we show that $V^G(\bar{k}) \neq 0$ then $V^G \neq 0$. But $V^G(\bar{k})$ is easily seen to be equal to $V(\bar{k})^{G(\bar{k})}$, so it is enough to assume that k is algebraically closed.

By the functoriality of the Jordan decomposition, $\rho(g)$ is unipotent, for every representation ρ and any $g \in G(k)$. If the representation is irreducible and $k = \bar{k}$, by Burnside's irreducibility criterion, $\mathrm{End}_k(V)$ is generated as an algebra by $\rho(G(k))$. Since $\rho(g)$ is unipotent, we have $\rho(g) = 1 + x$ for some nilpotent endomorphism x. For every $g' \in G(k)$ we have

$$\mathrm{tr}(x\rho(g')) = \mathrm{tr}((\rho(g) - 1)\rho(g')) = \mathrm{tr}(\rho(gg')) - \mathrm{tr}(\rho(g')) = \dim(V) - \dim(V) = 0,$$

since both $\rho(gg')$ and $\rho(g')$ are unipotent. But the trace pairing is nondegenerate on $\mathrm{End}_k(V)$, hence $x = 0$, and the representation is trivial.

By induction on the dimension, for any representation (ρ, V) of G there is a filtration F^iV such that G acts trivially on $\mathrm{gr}^i(V)$, i.e., there is a flag with respect to which $\rho(G)$ is upper triangular.

If we consider any faithful representation $G \to \mathrm{End}(\mathfrak{g})$, the image is upper triangular, hence solvable. □

Definition 7.2.5. A solvable algebraic group over a field k is said to be *split* if it has a filtration over k whose graded pieces are isomorphic to $\mathbb{G}_m$ or $\mathbb{G}_a$.

Lemma 7.2.6. *If k is algebraically closed, every connected solvable group is split.*

Proof. Omitted. □

Proposition 7.2.7. *If G is a split solvable linear algebraic group over a field k, all maximal tori in G are $G(k)$-conjugate. If T is a maximal torus, then G is the semidirect product $G = T\mathcal{R}_u(G)$, and any semisimple element in G is conjugate to an element of T.*

Proof. This is a delicate argument using induction on the dimension of G, see [**Bor91**, Theorem 10.6]. □

The analog of Lie's theorem 5.1.12 is

Theorem 7.2.8 (Borel's fixed point theorem). *If G is a split, solvable, connected algebraic group over field k, acting on a proper k-scheme X with $X(k) \neq \emptyset$, then $X(k)^G \neq \emptyset$.*

The reason that this is the analog of Lie's theorem is that, if G acts on a vector space V, and X is the flag variety classifying full flags in G, then the fixed point corresponds to a flag fixed by G. Notice that, unlike the Lie algebra version, there is not requirement of characteristic zero here.

Proof. By induction on the dimension d of G, the case $d = 0$ being trivial. Let $G' \subset G$ be normal of codimension one with $G/G' \simeq \mathbb{G}_m$ or $\mathbb{G}_a$, then, by induction, the fixed-point subvariety $X^{G'}$ has a nonempty set of k-points, so we may replace X by $X^{G'}$ and G by G/G', reducing us to the case where $G = \mathbb{G}_m$ or $\mathbb{G}_a$. Choose $x \in X(k)$, obtaining an orbit map $G \ni g \mapsto gx \in X$. Then, thinking of G as embedded inside of $\mathbf{P}^1$, by the valuative criterion for properness the map extends to a map $\varphi : \mathbf{P}^1 \to X$. This extension is G-equivariant, because X is proper, hence separated, hence "limits are unique", i.e., writing $\lim \gamma$ for specialization of the extension to the spectrum of a valuation ring $\mathfrak{o}$ of a map γ from the spectrum of its quotient field K, if $\gamma : \mathrm{Spec\ K} \to G$ then $\varphi(\lim \gamma) = \lim(\varphi \circ \gamma)$. Therefore, for every $g \in G$ we have $g \cdot \varphi(\lim \gamma) = g \cdot \lim(\varphi \circ \gamma) = \lim(g \cdot \varphi \circ \gamma) = \lim(\varphi(g \cdot \gamma))$ (because φ is equivariant on G) $= \varphi(\lim(g \cdot \gamma))$.

The image of $\infty \in \mathbf{P}^1$ will be the desired G-fixed point on $X(k)$. $\square$

Definition 7.2.9. The *radical* $\mathcal{R}(G)$ of an algebraic group G is its maximal connected solvable normal subgroup. The *unipotent radical* $\mathcal{R}_u(G)$ of a linear algebraic group G is its maximal connected unipotent normal subgroup. A group is *reductive* if its unipotent radical over the algebraic closure, $\mathcal{R}_u(G_{\bar{k}})$, is trivial, and *semisimple* if $\mathcal{R}(G) = 1$.

Remark 7.2.10. If k is perfect, then $\mathcal{R}_u(G_{\bar{k}}) = \mathcal{R}_u(G)_{\bar{k}}$. However, for non-perfect fields, the absolute unipotent radical could be larger; for example, consider a (nontrivial) finite, purely inseparable extension k'/k, and let $G = \mathrm{Res}_{k'/k}\mathrm{GL}_1$. Its k-unipotent radical is trivial, but its k'-unipotent radical is not. [Exercise!]

7.3. One-parameter subgroups and the associated parabolics

Let $\lambda : \mathbf{G}_m \to G$ be a cocharacter — also called a "one-parameter subgroup". We let it $\mathbb{G}_m$ act on G by conjugation via this character: ${}^{\lambda(a)}g := \lambda(a)g\lambda(a)^{-1}$.

We let

$$P(\lambda) = \{g \in G | \lim_{t\to 0} {}^{\lambda(t)}g \text{ exists}\},$$
$$U(\lambda) = \{g \in G | \lim_{t\to 0} {}^{\lambda(t)}g = 1\},$$
$$L(\lambda) = G^{\lambda(\mathbb{G}_m)}, \text{ the centralizer of } \lambda.$$

A priori, these groups could be non-reduced, but the following proposition states that this is not the case:

Proposition 7.3.1. *The groups $P(\lambda)$, $U(\lambda)$, $L(\lambda)$ are smooth, connected if G is, and $P(\lambda) = L(\lambda)U(\lambda)$ is a semidirect product decomposition of $P(\lambda)$. The multiplication map*

$$U(\lambda^{-1}) \times P(\lambda) \to G$$

is an open immersion (embedding).

Proof. For the proof, including a careful discussion of the definitions of these groups, see §24, 25 of the course notes of [**Cona**]. □

This implies:

Proposition 7.3.2. *If G is a connected linear algebraic group, the centralizer of any torus is connected.*

Proof. We may assume that $k = \bar{k}$. If $T = T_1 T_2$, where the T_i's are tori of smaller dimension, then the centralizer of T in G is equal to the centralizer of T_1 inside of the centralizer of T_2. This way, the problem reduces to $\dim(T) = 1$. Let $\lambda : \mathbb{G}_m \to G$ be a nontrivial cocharacter, whose image is T. Then the claim follows from Proposition 7.3.1. □

Definition 7.3.3. A subgroup P of a linear algebraic group G is called *parabolic* if the quotient G/P is proper (equivalently: projective).

Definition 7.3.4. A *Levi decomposition* of a connected linear group G is a semidirect decomposition $G = L \cdot \mathcal{R}_u(G)$, where $\mathcal{R}_u(G)$ is the unipotent radical of G.

Levi decompositions always exist in characteristic zero, but not in positive characteristic.

Theorem 7.3.5. *If G is a connected reductive group, the parabolics are precisely the subgroups of the form $P(\lambda)$, and the decomposition $P(\lambda) = L(\lambda)U(\lambda)$ is a Levi decomposition.*
Proof. [Omitted, for now.] □

7.4. Density of points; Borel and Cartan subgroups

For the definitions that follow, there are slight variants in the literature, e.g., allowing Cartan subgroups to be disconnected when the group is disconnected. To avoid confusion, establish an easy-to-remember principle, and stay close to the theory of Lie algebras, we are imposing connectedness in our definitions.

Definition 7.4.1. A *Borel subgroup* of a linear algebraic group G over a field k is a subgroup $B \subset G$ over k such that, over an algebraic closure $\bar{k}$, $B_{\bar{k}}$ is a maximal connected solvable subgroup of $G_{\bar{k}}$. A *Cartan subgroup* is the identity component of the centralizer of a maximal torus.

Remark 7.4.2. If G is connected then, by Proposition 7.3.2, the centralizer of any torus is connected, so the word "connected" in the definition of Cartan subgroups is superfluous.

Theorem 7.4.3. *A connected solvable subgroup is Borel if and only if the quotient G/B is projective. All Borel subgroups are conjugate over the algebraic closure.*

We will eventually see that, in any connected reductive group G, all Borel subgroups defined over the base field k (if there are any) are conjugate under $G(k)$. (The statement remains true without the assumption of reductivity, but is harder to show if k is not perfect, because it requires structure theory for pseudo-reductive groups.)

Proof. We first prove that if B is of maximal dimension among connected solvable subgroups, then G/B is projective. We may and will assume that the field of definition k is algebraically closed. By Theorem 4.1.6, there is a linear representation

V of G such that B is the stabilizer of a line L. Applying the Borel fixed point theorem 7.2.8 on the flag variety of V/L, B stabilizes a full flag f in V whose first element is L; hence it is the stabilizer of that flag in G. Any other stabilizer of a flag in V is also solvable, and by the maximality of $\dim(B)$, the dimension of $G/B = G \cdot f$ is minimal among the dimensions of G-orbits on the flag variety of V. Hence, $G \cdot f = G/B$ is closed in the flag variety, hence projective.

Given that G/B is projective, if P is any other solvable connected subgroup of G, again by Borel's fixed point theorem it fixes a point on G/B, i.e., $P \subset B'$ for some conjugate B' of B. Thus, if P is maximal, it is also of maximal dimension, and G/P is projective, as already proven.

Vice versa, if G/P is projective, and choosing a maximal solvable connected B, Borel's fixed point theorem implies that B fixes a point on G/P, hence $P \supset B'$ for a conjugate B' of B. If P is solvable and connected, by the maximality of B, $P = B'$.

The above show that any two Borel subgroups are conjugate over an algebraically closed field. □

Proposition 7.4.4. *If G is a connected linear algebraic group that is not unipotent, over an algebraically closed field k, then G contains a nontrivial torus. Any Cartan subgroup C of G is a direct product $T \times U$, with T a torus and U unipotent, and is equal to the identity component of its normalizer. Its Lie algebra is a Cartan subalgebra of $\mathfrak{g}$.*

We will see in Theorem 7.4.7 that, even if the field is not algebraically closed, tori exist over k.

Proof. Assume $k = \bar{k}$, and let B be a Borel subgroup; then, by Proposition 7.2.7, it has a Levi decomposition as a semidirect product $B = TN$, with T a maximal torus and N its unipotent radical. Assume that T is trivial. If $G \to \mathrm{GL}(V)$ is a representation where N stabilizes a line L, because there are no nontrivial homomorphisms $N \to \mathbf{G}_m$, N stabilizes every point on the line, and we get an embedding $G/N \hookrightarrow V$. Since G/N is projective, by Theorem 7.4.3, and V is affine, G/N must be a point, i.e., G is unipotent.

For the second claim, if C is the connected centralizer of T, C/T cannot contain nontrivial tori (this would contradict the maximality of T), and therefore is unipotent. By the Levi decomposition of Proposition 7.2.7, $C = TU$ is a semidirect product of T with the unipotent radical, but T is in the center, so the product is direct. If N is the connected normalizer of C, then T is normal in N, hence N/T acts on T by automorphisms; but the automorphism group of T is discrete, and N is connected, so N centralizes T, hence is equal to C. The same argument proves that its Lie algebra is self-normalizing, hence (since it is nilpotent) a Cartan subalgebra. □

We prove a result on the existence of regular semisimple elements:

Lemma 7.4.5. *Assume that $\mathfrak{g}$ is the Lie algebra of an algebraic group over an infinite field k. Then, $\mathfrak{g}(k)$ contains regular semisimple elements.*

Proof. Since k is infinite, $\mathfrak{g}(k)$ is Zariski dense, hence meets the Zariski open subset of s-regular elements (Lemma 5.1.17) nontrivially. If $X = X_s + X_n$ is the Jordan decomposition (Theorem 7.2.2) of an s-regular element, then, if k is perfect, X_s and X_n are defined over k, in which case X_s is the regular semisimple element we seek. Otherwise, $\mathrm{char}(k) = p > 0$, in which case $\mathfrak{g}$ has the structure of a restricted Lie algebra (Definition 4.2.8 and Example 4.2.9), and then for some $r > 0$ we have $X_n^{[p]^r} = 0$, and $X^{[p]^r} = X_s^{[p]^r}$ is the regular semisimple element that we seek. □

Theorem 7.4.6. *If G is a linear algebraic group over an algebraically closed field k, all maximal tori of G, and all Cartan subgroups, are $G(k)$-conjugate.*

Proof. We first show that Cartan subgroups are centralizers of regular semisimple elements: Let T be a maximal torus, and C its connected centralizer, a Cartan subgroup. It is smooth, by Proposition 7.3.1. Since T consists of semisimple elements, the adjoint action of T on $\mathfrak{g}$ decomposes into a direct sum of eigenspaces (whose nontrivial eigencharacters are called *roots*), and $\mathrm{Lie}(C)$ is the zero-eigenspace. Since k is infinite, there is an element $s \in T(k)$ with $\alpha(s) \neq 1$ for all roots α.

By Proposition 7.2.7, any semisimple element $s' \in G(k)$ is conjugate to an element of $T(k)$. Thus, all Cartan subgroups are conjugate.

This reduces the statement on tori to the case $C = G$, i.e., the case where T is central in G. But then, T is the *unique* maximal torus in G, for any nontrivial torus T' not contained in T would lead to a larger torus TT', contradicting the maximality of T. □

Theorem 7.4.7. *If G is a connected linear algebraic group over a field k, then maximal k-tori $T \subset G$ remain maximal after passing to the algebraic closure. In particular, there exist maximal k-tori of $G_{\bar{k}}$ that are defined over k.*
Proof. [Omitted, for now] □

Proposition 7.4.8. *If G is reductive, the centralizer of any torus is reductive, and Cartan subgroups are the maximal tori.*

Proof. See [**Bor91**, §13.17, Corollary 2] or the handout "Basics of reductivity and semisimplicity" in [**Conb**]. The second claim follows from the first, given that Cartan subgroups are of the form $T \times U$. □

7.5. The (universal) Cartan, and the scheme of Borel subgroups

An important feature of Borel subgroups is that they are self-normalizing. At the level of Lie algebras, this was an easy corollary of the definition, see Lemma 5.4.4. At the level of algebraic groups, one needs to be more careful, because the normalizer is a group scheme, not guaranteed to be smooth (in positive characteristic). [Definitions of normalizers, centralizers, etc. are the natural ones, and left to the reader, for now.]

Theorem 7.5.1. *Let B be a Borel subgroup of a reductive group G over a field k. The normalizer subgroup scheme $\mathcal{N}_G(B)$ is equal to B.*

Proof. We have an embedding $B \hookrightarrow \mathcal{N}_G(B)$, so we need to show that it is an isomorphism. For that, it is enough to assume that k is algebraically closed.

First, we show that $B(k) = N_G(B)(k)$ (which is equal to $N_{G(k)}(B(k))$). Choose a maximal torus $T \subset B$. Since all maximal tori inside of B are conjugate (Proposition 7.2.7), if $g \in G(k)$ normalizes B, then $gTg^{-1} = bTb^{-1}$ for some $b \in B$, hence, replacing g by $b^{-1}g$, we may assume that g normalizes T.

The commutator map $t \mapsto gtg^{-1}t^{-1}$ is a homomorphism $T \to T$. There are two cases:

(1) The kernel of this map, i.e., the fixed-point set $S = T^g$, is finite. Then, for dimension reasons, the commutator map is surjective. In particular, if B' is the group generated by B and g, any character $B' \to \mathbb{G}_m$ is trivial on B (because $B = TU$, where U is its unipotent radical, and characters are trivial on unipotent groups and commutators). By 4.1.6, there is a representation $G \to \mathrm{GL}(V)$ where B' is the stabilizer of a line, acting it via a character $B' \to \mathbb{G}_m$. Since that character is trivial on B, we obtain a map $G/B \to V$, which has to be constant because G/B is projective (Theorem 7.4.3). Therefore, $G \subset B'$, which means that $g \in G = B$.
(2) The fixed-point set $S = T^g$ is infinite. Then we can replace G by a group of smaller dimension, which is either G/S° (if S° is central) or the centralizer of S° (if S° is not central), which is connected by Proposition 7.3.2. By an inductive argument, the proof is complete.

We have proved that $B(k) = N_G(B)(k)$, *without using the assumption of reductivity*. To finish the proof, we need to show that $N_G(B)$ is reduced. [Omitted for now, as it requires a lot of material that has not been written.] □

Now we show that, for a reductive group G over a field k, there is a smooth variety over k which parametrizes the set of Borel subgroups (whether there are any, or not). The result is established over more general bases in [**DG70**, XXII, Corollaire 5.8.3], cf. [**Con14**, Corollary 5.2.9].

Theorem 7.5.2. *Let G be a connected reductive group over a field k. The functor that assigns to any scheme S/k the set of Borel subgroups of G_S is representable by a smooth projective variety $\mathcal{B}$ over k. It comes equipped with a canonical line bundle L, described as follows: If $\mathbb{B} \to \mathcal{B}$ is the universal Borel subgroup, i.e., the subgroup scheme of $G \times \mathcal{B}$ whose pullback over any S-point $S \to \mathcal{B}$ is the parametrized Borel subgroup of G_S, then $L = \det(Lie(\mathbb{B}))^*$.*

By definition, a Borel subgroup of G_S is a smooth affine subgroup scheme over S whose geometric fibers are Borel subgroups.

Sketch of proof. The proof requires extension to an arbitrary basis of two results that we have already proven over a field k: Assuming the existence of a split maximal torus T, Borel subgroups containing T are in bijection with bases for the root system of T in G (Proposition **??**), and Borel subgroups are self-normalizing (Theorem 7.5.1). [Omitted for now.]

Let T be a maximal torus in G. By Theorem 7.4.7, it remains maximal over the algebraic closure, and by Theorem 7.1.2, it splits over the separable closure k^s. By Proposition **??**, there is a Borel subgroup B over k^s which contains T_{k^s}. By the conjugacy of Borel subgroups (Theorem 7.4.3), and their self-normalizing property (Theorem 7.5.1), the quotient $\mathcal{B}^s := G_{k^s}/B$ represents the functor of Borel subgroups over k^s.

We need to show that this scheme descends to k, which is where the ample line bundle will come into play. We first prove that the stated line bundle is ample: If d is the dimension of a Borel subgroup, consider the map

$$\mathcal{B}^s \to \mathrm{Gr}(\mathfrak{g}_{k^s}, d)$$

sending a Borel subgroup to its Lie algebra in the Grassmannian of d-dimensional subspaces of $\mathfrak{g}_{k^s}$. Then L is the pullback of the dual of the determinant line bundle on $\mathrm{Gr}(\mathfrak{g}_{k^s}, d)$, which is ample. Therefore, L is ample.

Finally, for the descent, we have an embedding $\mathcal{B}^s \to \mathbb{P}V^s$, where $V^s = H^0(\mathcal{B}^s, L^n)^*$ for some $n \gg 0$, and compatible semilinear actions of the Galois group Γ of k^s/k on $\mathcal{B}^s$ and on the k^s-vector space V^s. Therefore, $V^s = V \otimes_k k^s$, where $V = (V^s)^\Gamma$, and $\mathcal{B}^s$ is the base change of a closed subvariety $\mathcal{B} \subset \mathbb{P}V$. □

Now we pass to an important construction, which assigns, canonically, a torus $\mathbf{A}^G$ to every reductive group G.

Proposition 7.5.3. *Let G be a connected reductive group over a field k, $\mathbb{B} \to \mathcal{B}$ be the universal Borel over the variety of Borel subgroups (Theorem 7.5.2), and $\mathbb{A} \to \mathcal{B}$ the variety of their reductive quotients (i.e., the fiber of $\mathbb{A}$ is the fiber of $\mathbb{B}$ divided by its unipotent radical — hence, a torus).*

There is a torus $\mathbf{A}^G$ over k such that $\mathbb{A} = \mathbf{A}^G \times \mathcal{B}$.

Notice that, given such a torus $\mathbf{A}^G$, it is unique up to unique isomorphism. Indeed, since $\mathbb{B}$ is projective and tori are affine, any isomorphism $\mathbb{A}_1^G \times \mathcal{B} \simeq \mathbb{A}_2^G \times \mathcal{B}$ over $\mathcal{B}$ arises from a unique isomorphism $\mathbb{A}_1^G \simeq \mathbb{A}_2^G$.

Proof. Is is enough to prove the proposition over the separable closure k^s. Indeed, if $\mathbb{A}_{k^s} = \mathbb{A}^s \times \mathcal{B}_{k^s}$ over k^s, where $\mathbb{A}^s$ is some torus over k^s, then the k^s-semilinear action of the Galois group Γ on $\mathbb{A}_{k^s}$ is induced, necessarily, from its action on $\mathcal{B}_{k^s}$ and a k^s-semilinear action on $\mathbb{A}^s$ which, by the equivalence of categories between tori and abelian groups (Theorem 7.1.2), descends to a form $\mathbb{A} = \mathbf{A}^G$ over k.

Hence, assume that $k = k^s$. Let B_1^s, B_2^s be two Borel subgroups, and A_1^s, A_2^s their reductive quotients. There is a canonical isomorphism $A_1^s = A_2^s$, induced by the action of any $g \in G(k)$ with $gB_1^s g^{-1} = B_2^s$. Indeed, such a g exists since all Borel subgroups are conjugate (Theorem 7.4.3), and it is unique up to right multiplication by B_1^s, since Borel subgroups are self-normalizing (Theorem 7.5.1). But conjugation by an element of B_1^s is the identity on the reductive quotient A_1^s (which is connected), hence all such elements g induce the same isomorphism $A_1^s = A_2^s$. We can therefore set $\mathbf{A}^G = A_1^s$, and through these isomorphisms we get the isomorphism $\mathbb{A} = \mathbf{A}^G \times \mathcal{B}$ asserted in the proposition. □

Now let $\mathbf{A}^G$ be the torus of Proposition 7.5.3. It comes with a canonical quotient map $B \to \mathbf{A}^G$ for *any* Borel subgroup of G.

Lemma 7.5.4. *Let G be a connected reductive group over a field k, and B a Borel subgroup. Let $T \subset B$ be any maximal torus in B. The composition $T \to B \to \mathbf{A}^G$ is an isomorphism of tori over k, inducing a Γ-equivariant isomorphism of their absolute character groups $X^*(T_{\bar{k}}) \simeq X^*(\mathbf{A}^G_{\bar{k}})$. If we use this isomorphism to transfer the subsets $\Phi^+ \subset \Phi$ of B-positive roots, resp. roots, of T to $\mathbf{A}^G$, the resulting subsets $\Phi^+ \subset \Phi \subset X^*(\mathbf{A}^G_{\bar{k}})$ do not depend on the choices of B or T.*

Proof. Easy, and left to the reader. □

Definition 7.5.5. Given a connected reductive group G over a field k, the torus $\mathbf{A}^G$ of Proposition 7.5.3 is called the *universal Cartan group*, or *abstract Cartan group*, or simply *the Cartan group*[1] of G. The sets $\Phi^+ \subset \Phi$ of Lemma 7.5.4 are the *abstract (positive) roots* of G.

[1] But not *subgroup!*

We first discuss the existence of a torus that remains maximal over the algebraic closure. We follow [**Bor91**, Theorem 18.2], presenting the arguments only for the least degenerate cases.

The existence of a maximal torus can be shown inductively on the dimension of G, by considering centralizers, under the adjoint representation, of semisimple elements in the Lie algebra $\mathfrak{g}$, but there are some complications that can appear in positive characteristic, that we will deal with in the end. Assume the theorem to be proven for all dimensions smaller than the dimension of G (the case $\dim(G) = 0$ being trivial).

Assume at first that k is infinite, and the Lie algebra $\mathfrak{g}$ is not nilpotent. Recall from Lemma 7.4.5 that $\mathfrak{g}(k)$ contains a regular semisimple element Y. The centralizer G_Y of Y in G is reduced (exercise, or see the proof of this theorem in [**Bor91**, Theorem 18.2]), and of smaller dimension than G, since $\mathfrak{g}$ is not nilpotent. We claim that $G_{Y,\bar{k}}$ contains a maximal torus of $G_{\bar{k}}$. It is enough to show that Y is contained in the Lie algebra of some torus $T' \subset G_{Y,\bar{k}}$, because then T' is contained in a maximal torus, which centralizes Y. If T' is a maximal torus in $G_{Y,\bar{k}}$, its centralizer C is a Cartan subgroup, whose Lie algebra $\mathfrak{c}$ is the centralizer of $\mathfrak{t}'$, hence $Y \in \mathfrak{c}$. But $C^\circ = T' \times N'$ for some unipotent group N' by Proposition 7.4.4, and since Y is semisimple, it has to lie in the Lie algebra of T'. Thus, by induction, there is a maximal torus $T \subset G_Y$ which remains maximal over the algebraic closure, and such a torus is maximal in $G_{\bar{k}}$, by the conjugacy of maximal tori, Theorem 7.4.6.

Now let us assume that k is infinite, but the Lie algebra $\mathfrak{g}$ is nilpotent. [WRONG:] If k is perfect, then that means that G is unipotent, and contains no tori, so there is nothing to prove. Otherwise, see [**Bor91**, 17.8 and 18.2].

Finally, assume that k is finite, with q elements. The relative Frobenius $F_q : G \ni g \mapsto g^{(q)} \to G$ is the morphism over $\mathrm{Spec}(k)$ which, when G is realized as a subscheme of affine space, corresponds to raising the coordinates to the q-th power; precisely, when $G = \mathrm{Spec}(R)$, with R a k-algebra, F_q is induced by raising elements of r to the q-th power.

By the conjugacy of all maximal tori under the algebraic closure, for any maximal torus $T \subset G_{\bar{k}}$ there is an element $g \in G(\bar{k})$ such that $gT^{(q)}g^{-1} = T$. By Lang's theorem, $g = a^{-1}a^{(q)}$ for some element a, and then the torus $T' = aTa^{-1}$ is stable under the Frobenius morphism, hence defined over k.

This proves the existence of a maximal torus over k. To prove that any maximal k-torus of G remains maximal over $\bar{k}$, we proceed by induction on the dimension of G. Given any k-torus $S \subset G$, now, its centralizer is connected by Proposition 7.3.2, and therefore either S is central, or we are done by induction, replacing G by the centralizer. On the other hand, if S is central, proceed by induction on

7.6. Other chapters

(9) Galois cohomology of linear algebraic groups
(10) Representations of reductive groups over local fields
(11) Plancherel formula: reduction to discrete spectra
(12) Construction of discrete series
(13) The automorphic space
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CHAPTER 8

Forms and covers of reductive groups, and the L-group

8.1. Classification of reductive groups over a separably closed field

For this section we assume k to be separably closed (unless otherwise noted). [For now, this section is missing a lot of results, as one needs to establish the analogs of results that we proved for semisimple Lie algebras in characteristic zero, for reductive groups.]

We saw in Theorem 7.1.2 a simple combinatorial description for diagonalizable groups, and we would like to have a similar description for more general reductive groups. This is not possible in the sense of getting an equivalence of categories[1], but at least we can fully describe the isomorphism classes this way.

Let G be reductive (over k), and let T be a maximal torus in G. Let $X^\bullet(T)$, $X_\bullet(T)$ be the character and cocharacter groups of T. The adjoint action of T on $\mathfrak{g}$ is semisimple, and we have a decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha,$$

where the α's, here, are eigencharacters $\alpha : T \to \mathbb{G}_m$.

By Proposition 7.4.8, $\mathfrak{g}_0$ is just the Lie algebra of T, however, *to accommodate the case of positive characteristic*, we will not be using $\mathfrak{t}$ to denote this Lie algebra, but the real vector space

$$\mathfrak{t} = X_\bullet(T) \otimes \mathbb{R}$$

(and, similarly, $\mathfrak{t}^* = X^\bullet(T)\otimes\mathbb{R}$). For real algebraic groups, this $\mathfrak{t}^*$ can be identified with the dual of the Lie algebra, by the map that assigns to any character χ its differential $d\chi$ at the identity.

Definition 8.1.1 (Root datum)**.** A *root datum* is a quadruple $(X, \Phi, \check{X}, \check{\Phi})$, where $X, \check{X}$ are two lattices (finitely generated, torsion-free abelian groups) in duality, $\Phi \subset X$ and $\check{\Phi} \subset \check{X}$ are finite subsets, such that there exists a bijection $\Phi \ni \alpha \leftrightarrow \check{\alpha} \in \check{\Phi}$, satisfying:

(1) $\langle \alpha, \check{\alpha}\rangle = 2$;
(2) the endomorphisms of $X, \check{X}$ defined by $w_\alpha(x) := x - \langle x, \check{\alpha}\rangle\, \alpha$, $w_{\check{\alpha}}(\check{x}) = \check{x} - \langle \alpha, \check{x}\rangle\, \check{\alpha}$ preserve Φ and $\check{\Phi}$.

A *based root datum* is a root datum as above, together with a choice of positive roots $\Phi^+ \subset \Phi$ (in the sense of Definition 5.4.1).

[1] There is a good reason for it: To get an equivalence of categories one must consider the category of all G-representations, cf. Tannaka-Krein duality. For diagonalizable groups this category is described easily in terms of combinatorial data, this is no longer the case for other groups.

Remarks 8.1.2.
(1) The last axiom is equivalent to: the endomorphisms w_α preserve Φ and generate a finite group (the Weyl group W).
(2) The condition $\langle \alpha, \check{\alpha} \rangle = 2$ characterizes the bijection $\Phi \leftrightarrow \check{\Phi}$, so it need not be part of the data.
(3) The $\mathbb{R}$-span of Φ in $X^* \otimes \mathbb{R}$, together with Φ and the Weyl group of automorphisms generated by the w_α's, forms a root system, as can be easily verified; hence the definition of based root datum in terms of based root systems.

We now define the appropriate notion of morphisms between root data.

Definition 8.1.3. An *isogeny* of root data $(X, \Phi, \check{X}, \check{\Phi}) \to (X', \Phi', \check{X}', \check{\Phi}')$ is a homomorphism $f : X \to X'$ with the following properties:
(1) f is injective, and with finite cokernel — hence, so is its adjoint $f^* : \check{X}' \to \check{X}$;
(2) f and f^* induce bijections between the subsets of roots and coroots, respectively.

Proposition 8.1.4. *Given a connected reductive group G and a maximal torus T with set of roots $\Phi \subset X^*(T)$, and given $\alpha \in \Phi$, consider the subtorus $T_\alpha = \ker(\alpha)^\circ \subset T$. Let L_α be the centralizer of T_α (which is connected by Proposition 7.3.2, and L'_α its derived subgroup. Then L'_α is isomorphic to SL_2 or PGL_2, and therefore there is a unique cocharacter $\check{\alpha} : \mathbb{G}_m \to T \cap L'_\alpha$ with $\langle \alpha, \check{\alpha} \rangle = 2$. If w_α is the nontrivial element of the Weyl group of T inside of L'_α, then the elements w_α generate the full Weyl group of T in G, $W = \mathcal{N}_G(T)/T$.*

Proof. Omitted. □

Definition 8.1.5. In the notation of Proposition 8.1.4, the elements $\check{\alpha} \in X_*(T)$ associated to the roots $\alpha \in X^*(T)$ are the *coroots* of the torus T in G.

Definition 8.1.6. Let (G, T), (G', T') be two pairs consisting of a connected reductive group and a maximal torus. A *central isogeny* $(G', T') \to (G, T)$ is a morphism $G' \to G$ sending $T' \to T$, surjective, whose kernel is a finite central group scheme. A central isogeny $G' \to G$ is a morphism which induces a central isogeny $(G', T') \to (G, T)$ for some maximal tori T', T (or, equivalently, for any maximal torus T', and T the image of T').

Remark 8.1.7. A surjective morphism $(G', T') \to (G, T)$ with finite kernel gives rise to an injective map $X^*(T) \to X^*(T')$ with finite cokernel, and a bijection between the roots of T on G and the roots of T' on G'. The requirement that the kernel be a central group scheme can be reformulated as follows: the map induces isomorphisms between the corresponding root spaces of T' and T in $\mathfrak{g}'$, resp. $\mathfrak{g}$. This requirement is automatically satisfied in characteristic zero, but not in positive characteristic. For example, if G is defined over a finite field with q elements, the Frobenius morphism F_q (see the proof of Theorem 7.4.7) restricts to the Frobenius morphism on the subgroups $\mathfrak{g}_\alpha \simeq \mathbb{G}_a$, which is not an isomorphism. The kernel of Frobenius on $\mathfrak{g}_\alpha$ does not have any nontrivial points over the finite field, but it is not a central group scheme.

Now we define some categories of reductive groups with extra data that will be used in the classification.

Definition 8.1.8. We let *Red* be the category whose objects are connected reductive groups G over k, and whose morphisms are central isogenies $G' \to G$. We let Red_T be the category whose objects are pairs $(G \supset T)$ of reductive groups with maximal tori, and whose morphisms are central isogenies of pairs $(G', T') \to (G, T)$. We let $Red_{B,T}$ be the category whose objects are triples $(G \supset B \supset T)$ of reductive groups with Borel subgroups and maximal tori thereof, and whose morphisms are central isogenies of pairs $(G', T') \to (G, T)$ sending B' to B.

We let RD be the category whose objects are root data $(X, \Phi, \check{X}, \check{\Phi})$ and whose morphisms are central isogenies. We let RD^+ be the category whose objects are root data, together with a choice $\Phi^+ \subset \Phi$ of positive roots.

Definition 8.1.9. If G is a connected reductive group over a field k which is not necessarily separably closed, it is said to be quasisplit if there exists a Borel subgroup over k, and *split* if there exists a maximal torus which is split over k (hence also a Borel subgroup).

Theorem 8.1.10 (Classification over the separable closure)**.** *Assume k separably closed. Given a connected reductive group G and a maximal torus T, the quadruple $\Psi(G,T) = (X^*(T), \Phi, X_*(T), \check{\Phi})$, where $\Phi, \check{\Phi}$ denote, respectively, the sets of roots and coroots of T in G, is a root datum.*

The assignment $(G,T) \mapsto \Psi(G,T)$ is a functor $Red \to RD^{op}$, in the notation of Definition 8.1.8.

This functor is a bijection on isomorphism classes of objects, and every morphism $\Psi(G,T) \to \Psi(G',T')$ in RD lifts to a morphism (i.e., central isogeny) $(G',T') \to (G,T)$, uniquely up to precomposing with conjugation by elements of T', or post-composing with conjugation by elements of T.

More generally, the same statement holds if k is not separably closed, but we consider the full subcategory of such pairs (G,T) where T (and, hence, G) is split over k.

Proof. Omitted. See [**Spr79**] for more details and references. □

We will now see two variants of this theorem: One, where we choose more data in order to rigidify the functor, and another, where we make no choices (i.e., we do not choose tori).

If, in the context of Theorem 8.1.10, we choose a Borel subgroup $B \supset T$, it gives rise to a based root datum, with Φ^+ being the set of roots appearing in the Lie algebra of B. We will generally denote by Δ the subset of simple roots.

Definition 8.1.11. A *pinning* of a triple (G, B, T) consisting of a connected reductive group G, a Borel subgroup, and a maximal torus T is a choice of isomorphisms $p_\alpha : \mathfrak{g}_\alpha \simeq \mathbb{G}_a$, for every simple root $\alpha \in \Delta$.

Given a pair (G, B) of a connected reductive group and a Borel subgroup, whose unipotent radical we will denote by N, an *algebraic Whittaker datum* is a homomorphism $\ell : N \to \mathbb{G}_a$ which restricts to a nontrivial linear map on the one-parameter subgroup $G_\alpha \simeq \mathbb{G}_a$ associated to every simple root α. (In particular, its differential induces a pinning; vice versa, given a pinning as above, there is a unique such ℓ with $d\ell|_{\mathfrak{g}_\alpha} = p_\alpha$.)

We let Red_{PIN} denote the category whose objects are pinned reductive groups, and whose morphisms are central isogenies preserving the Borel, the torus, and the pinning.

Pinnings allow us to rigidify the morphisms lifted from morphisms of (based) root data:

Theorem 8.1.12. *The assignment* $(G,B,T,P) \to \Psi^+(G,B,T)$*, where* $\Psi^+(G,B,T)$ *denotes the based root datum consisting of* $\Psi(G,T)$ *(notation as in Theorem 8.1.10) with* Φ^+ *the positive roots associated to* B*, is an equivalence of categories:* $Red_{PIN} \to RD^+$*.*

Proof. Omitted. See, again, [**Spr79**]. ☐

Finally, a version of the classification for morphisms $G' \to G$, without any choices: Recall that to any reductive group G, we have associated its (universal) Cartan $\mathbb{A}^G$, endowed with sets of roots and positive roots $\Phi^+ \subset \Phi \subset X^*(\mathbb{A}^G)$. (Recall that in this section we are assuming the field to be separably closed.) We will denote by $\Psi^+(\mathbb{A}^G)$ this based root system.

Lemma 8.1.13. *The association* $G \to \mathbb{A}^G$ *is functorial in the category Red of reductive groups with central isogenies.*

Proof. If $f : G \to G'$ is a central isogeny, it sends a Borel subgroup $B \subset G$ onto a Borel subgroup $B' \to G'$, inducing morphisms of their reductive quotients: $\mathbb{A}^G = B/N \to \mathbb{A}^{G'} = B'/N'$. If we choose another Borel subgroup $B_1 \subset G$, there is a $g \in G$ with $gBg^{-1} = B_1$, hence $f(g)B'f(g)^{-1} = B_1' := f(B_1)$, and the morphism $\mathbb{A}^G \to \mathbb{A}^{G'}$ induced from $B_1 \to B_1'$ is the same as before, given how we have identified $\mathbb{A}^G$ as the quotient of *any* Borel subgroup. ☐

Theorem 8.1.14. *The assignment* $G \mapsto \Psi^+(\mathbb{A}^G)$ *is a functor* $Red \to (RD^+)^{op}$*, in the notation of Definition 8.1.8. It induces a bijection on isomorphism classes of objects, and for any morphism* $\Psi^+(\mathbb{A}^G) \to \Psi^+(\mathbb{A}^{G'})$ *there is a morphism* $G' \to G$*, unique up to inner automorphisms.*

Proof. This follows from Theorem 8.1.10, and the conjugacy of Cartan subgroups. ☐

8.1.15. Simply connected and adjoint groups.

Definition 8.1.16. Let $(X, \Phi, \check{X}, \check{\Phi})$ be a root datum, let $R \subset X$, $\check{R} \subset \check{X}$ be the subgroups spanned by the roots, resp. coroots, and let $P = \check{R}^*$, $\check{P} = R^*$ the dual lattices.

The root datum is called *semisimple* if $\mathcal{R}$ is of finite index in X.

Assume this to be the case, so that we have containments with finite quotients $P \supset X \supset R$ and $\check{P} \supset \check{X} \supset \check{R}$. We say that the root datum is *simply-connected* if $X = \mathcal{P}$, equivalently $\check{X} = \check{R}$, and *adjoint* if $X = \mathcal{R}$, equivalently $\check{X} = \check{P}$.

Definition 8.1.17. A reductive group is *adjoint* if it has trivial center, and *simply connected* if it admits no nontrivial central isogeny from a connected smooth group over k.

The relationship between this notion of being simply connected, and the algebrogeometric one, will become clear in Proposition 8.1.19 and Remark 8.1.20.

Remark 8.1.18. The "center" mentioned in Definition 8.1.17 needs to be understood as a group scheme. For example, the center of SL_p in characteristic p is the nontrivial group scheme μ_p of p-th roots of unity (which has only one point over $\overline{\mathbb{F}_p}$).

Proposition 8.1.19. *Let (G,T) be a connected reductive group with a maximal torus over k, and let $\Psi(G,T)$ be the associated root datum.*

(1) *G is semisimple iff $\Psi(G,T)$ is semisimple.*
(2) *G is adjoint iff $\Psi(G,T)$ is adjoint.*
(3) *G is simply connected (in the sense of Definition 8.1.17 iff $\Psi(G,T)$ is simply connected.*
(4) *In characteristic zero, G is simply connected as a scheme (or, equivalently, $G(\mathbb{C})$ is simply connected as a topological space when $k=\mathbb{C}$) iff $\Psi(G,T)$ is simply connected.*

Proof. (1) A reductive group is semisimple iff its center is finite. The center belongs to T, and coincides with the common kernel of all roots. In other words, in terms of the equivalence of diagonalizable groups and finitely generated abelian groups (Theorem 7.1.2), the character group of the center is the cokernel of the inclusion $R \to X$. Hence, it is finite iff R is of finite index in X.

(Notice that the center is not necessarily reduced; for example, the center of SL_p in characteristic p is the non-reduced group scheme μ_p of p-th roots of unity.)

(2) Continuing along the same lines, the center is trivial iff $R = X$.
(3) A central isogeny $G' \to G$ which is not an isomorphism, mapping some maximal tori $T' \to T$, induces an isogeny $\Psi(G,T) \to \Psi(G',T')$, which is completely determined by the map of cocharacter groups $X_*(T') \to X_*(T)$. This map is injective, of finite cokernel, and has to preserve coroot lattices, so if $X_*(T)$ is equal to the coroot lattice, the map is an isomorphism.
(4) By [**BS13**, Theorem 1], if p is the characteristic exponent of the field, every prime-to-p étale Galois cover $G' \to G$ is a central isogeny. In particular, in characteristic zero, G is simply connected in the sense of Definition 8.1.17 iff it is simply connected in the sense of étale topology. Moreover, if $k = \mathbb{C}$, the étale fundamental group is the profinite completion of the topological fundamental group of $G(\mathbb{C})$.

□

Remark 8.1.20. In positive characteristic, the étale fundamental group is always infinite, for a smooth affine scheme $X = \mathrm{Spec}(R)$ of positive dimension. For example, we have the Artin–Schreier $\mathbb{Z}/p$-covers, $X' = \mathrm{Spec} R[y]/(y^p - y - f)$, if $f \in R$ is chosen appropriately.

8.1.21. Automorphisms. Given an automorphism $f : X \to X$ of an affine variety X over a field k, its defining homomorphism $f^* : k[X] \to k[X]$ restricts to an automorphism $V \to V$ on a generating, finite-dimensional subspace V of $k[X]$. Therefore, the automorphism group $\mathrm{Aut}(X)$ has a natural structure as the ind-algebraic group

$$\lim_{\overrightarrow{V}} \mathrm{Aut}(X,V),$$

where V runs over all finite-dimensional, generating subspaces of $k[X]$, and $\mathrm{Aut}(X,V) \subset \mathrm{GL}(V)$ is the subgroup of those automorphisms of V (as a vector space) which induce an automorphism of X (as an algebraic variety).

Definition 8.1.22. The *group of inner automorphisms*, $\mathrm{Inn}(G)$, is the image of the natural morphism $G \to \mathrm{Aut}(G)$ given by the conjugation action of G on itself. It is also called the *adjoint group* of G, and denoted by G_{ad}.

The quotient $\mathrm{Aut}(G)/\mathrm{Inn}(G)$ is the group of *outer automorphisms* of G.

The kernel of the map $G \to \mathrm{Aut}(G)$ is the center of G, so the group $G_{\mathrm{ad}} = \mathrm{Inn}(G)$ is the quotient of G by its center.

For reductive groups the automorphism group is of finite type. This follows from the fact that, by the classification of reductive groups in terms of root data, the outer automorphism group is the group of automorphisms of based root data.

Proposition 8.1.23. *The functor from reductive groups to based root data of Theorem 8.1.14 gives rise to a short exact sequence of (ind-)algebraic groups:*

$$0 \to Inn(G) \to Aut(G) \to Aut\Psi^+(\mathbb{A}^G) \to 0. \tag{8.1.23.1}$$

In particular, $Out(G) = Aut\Psi^+(\mathbb{A}^G)$.

Moreover, any pinning on G (Definition 8.1.11) gives rise to a splitting $Out(G) \to Aut(G)$, characterized by the fact that its image consists of the automorphisms preserving the pinning.

Proof. Applying the functor of Theorem 8.1.14 to automorphisms of G, we get a homomorphism $\mathrm{Aut}(G) \to \mathrm{Aut}\Psi^+(\mathbb{A}^G)$. By Theorem 8.1.14, this homomorphism is surjective, and its kernel is precisely the group of inner automorphisms.

The existence of a unique splitting that fixes a given pinning follows from Theorem 8.1.12. □

8.2. Forms and Galois descent

Definition 8.2.1. Let G be a linear algebraic group over a field k or over an extension L/k. A *form* of G over k is a linear algebraic group G' over k, such that $G_L \simeq G'_L$.

The goal of this section is to explain how isomorphism classes of forms of G are described by the Galois cohomology set $H^1(\Gamma, \mathrm{Aut}(G_{k^s}))$, where $\Gamma = \mathrm{Gal}(k^s/k)$, the Galois group of the separable closure of k. There is nothing special about classification of linear algebraic groups here — the same arguments apply to any category of objects that satisfy *effective descent*. We introduce these notions in some generality, but not the full generality of faithfully flat descent (see [**Sta19**, Tag 0238]).

Let Aff_k be the category of affine schemes over the field k. A *k-groupoid* is a category with a functor $\mathcal{F} \to \mathrm{Aff}_k$ satisfying two axioms, which can be summarized as "base change exists", and "fiber categories are groupoids". To formulate them, denote by $\mathcal{F}_U$ the full subcategory living over an object $U \in \mathrm{Aff}_k$. The axioms are:

(1) Given $U \in \mathrm{Aff}_k$, $u \in \mathcal{F}_U$, and $f : V \to U$ a morphism, there exists a morphism $\tilde{f} : v \to u$ in $\mathcal{F}$ lying over f.
(2) For any pair of morphisms $\tilde{f} : u \to z$, $\tilde{g} : v \to z$ in $\mathcal{F}$, lying over $f : U \to Z$ and $g : V \to Z$ in Aff_k, and any $h : U \to V$ with $f = g \circ h$, there exists a unique lift $\tilde{h} : u \to v$ with $\tilde{f} = \tilde{g} \circ \tilde{h}$.

If v is as in the first axiom, we will feel free to write $v = u_V$, i.e., as a base change of u. Notice that, for every other choice v', the second axiom provides a canonical isomorphism $v' \to v$, corresponding to the identity map on V. For

$f : V \to U$ and $u \in \mathcal{F}_U$, one has a unique morphism $u_V \to u'_V$ over any morphism $\phi : u \to u'$; we will be denoting that by $f^*\phi$. For a composition $W \to V \to U$, we will feel free to identify $(u_V)_W$ with u_W — see [**Sta19**, Tag 02XN] for a clarification of these issues.

Now consider a Grothendieck topology on Aff_k, turning it into a site. For notational simplicity, we will represent every cover as a single morphism $U \to X$, instead of a family of morphisms. The groupoid $\mathcal{F} \to \mathrm{Aff}_k$ is called a *prestack* (of groupoids) if, for any $U \in \mathrm{Aff}_k$, and any $x, y \in \mathcal{F}_U$, the presheaf $V \mapsto \mathrm{Mor}(x_V, y_V)$ is a sheaf on the site Aff_U, and it is called a *stack* (of groupoids) if *descent data are effective*.

These are two conditions that need to be formulated for arbitrary covers, but we just explain their meaning for the case of covers corresponding to finite Galois extensions L/k with Galois group Γ; the generalization to arbitrary U and arbitrary covers is straightforward, see [**Sta19**, Tag 0268].

Denote $U = \mathrm{Spec}(L) \to X = \mathrm{Spec}(k)$, and notice that there is a canonical isomorphism $U \times_X U \simeq \Gamma \times U$, equivariant with respect to the action of Γ on the first copy, taking $U \times \{1\}$ to the diagonal. A *descent datum* for L/k consists of an object $u \in \mathcal{F}_U$, together with (iso)morphisms $\phi_\gamma : \gamma u \to u$ (over the *identity* of U), where γu is the base change of u via the map $\gamma^{-1} : U \to U$, and the ϕ_γ's are required to satisfy the cocycle condition $\phi_{\gamma_1\gamma_2} = \phi_{\gamma_1} \circ \gamma_1^* \phi_{\gamma_2}$. Equivalently, a descent datum is an isomorphism $\phi : u_1 \to u_2$, where u_i is the pullback via the i-th projection $U \times_X U \to U$, and the isomorphism is required to satisfy the natural cocycle condition with respect to the triple product $U \times_X U \times_X U$.

Let $\mathcal{F}_{L/k}$ denote the category of descent data (u, ϕ). (The morphisms in this category are the obvious ones, morphisms of the u's commuting with the ϕ's.) There is a canonical functor

$$\mathcal{F}_k \to \mathcal{F}_{L/k} \tag{8.2.1.1}$$

taking an object $x \in \mathcal{F}_k = \mathcal{F}_X$ to its base change x_L, together with the descent data induced from the Galois automorphisms of $U = \mathrm{Spec}(L)$ over k. (Notice that a morphism $\gamma u \to u$ over the identity in U is equivalent to a morphism $u \to u$ over the map $\gamma : U \to U$.)

Then, $\mathcal{F}_k$ satisfies *effective descent* (together with the prestack condition) with respect to the Galois cover $L \to k$, if the functor (8.2.1.1) *is an equivalence*. Here are two basic examples where it happens:

Example 8.2.2. The category $\mathcal{F}_k$ of vector spaces over k satisfies effective descent with respect to any separable extension L/k. (It can be extended to a k-stack $\mathcal{F}$ by considering vector bundles over an arbitrary basis.)

Example 8.2.3. The category $\mathcal{F}_k$ of algebras over k satisfies effective descent with respect to L/k. Indeed, let B be an L-algebra. The descent datum amounts to an isomorphism $\phi : B \otimes L \simeq L \otimes B$ (tensor products over k), and the cocycle condition states that the triangle consisting of $B \otimes L \otimes L$, $L \otimes B \otimes L$ and $L \otimes L \otimes B$, applying ϕ to get isomorphisms between them, commutes. We then take our algebra A over k to consist of those sections of B whose pullbacks to $L \otimes B \overset{\phi}{\simeq} B \otimes L$ (geometrically, in the notation above: whose pullbacks to $U \times_X U$ under both projections) coincide, i.e., $A = \ker(B \to L \otimes B)$, where the map is given by $b \mapsto 1 \otimes b - \phi(b \otimes 1)$. One checks that A is a k-algebra, and $B \simeq A \otimes L$ compatibly with descent data, see [**Sta19**, Tag 0244].

Example 8.2.4. Galois descent is also satisfied for any quasi-projective scheme, see [**BLR90**, §6.2, Example B].

Example 8.2.3 is the basic one for our purposes. It immediately extends to groupoids of affine varieties with extra structure given by "closed" conditions, such as algebraic groups.

Finally, descent data can be parametrized by torsors for the automorphism group:

Proposition 8.2.5. *Given $(u, \phi) \in \mathcal{F}(L/k)$, let $A = Aut_{\mathcal{F}_L}(x)$, considered as a Γ-module (possibly non-abelian) with action $\gamma a := \phi_\gamma \circ a \circ \phi_\gamma^{-1}$. Then, we have a natural equivalence of of categories*

$$\mathcal{F}(L/k)/u \xrightarrow{\sim} A - Tors^\Gamma,$$

where $\mathcal{F}(L/k)/u$ denotes the category of descent data (u', ϕ') with $u' \simeq u$, and $A-Tors^\Gamma$ denotes the category of (set-theoretic) right A-torsors T with a compatible left Γ-action, i.e., ${}^\gamma(t \cdot a) = {}^\gamma t \cdot {}^\gamma a$ for all $\gamma \in \Gamma, a \in A$, and $t \in T$.

The functors are:

- *$(u', \phi') \to Isom(u, u')$, and*
- *$T \to (u' = T \times^A u, \phi')$, where $\phi' = \{\phi'_\gamma\}$ is obtained by acting diagonally on T (via the given Galois action) and on u (via ϕ).*

Proof. Left to the reader. □

If Γ is a finite group, and A is a (possibly non-abelian) group with a Γ-action, then isomorphism classes of A-torsors with a Γ-action are naturally parametrized by the *1st cohomology set* $H^1(\Gamma, A)$, which is the pointed set of A-orbits on the set of 1-cocycles $Z^1(\Gamma, A) = \{c : \Gamma \to A | c(\gamma_1\gamma_2) = c(\gamma_1) \cdot {}^{\gamma_1}c(\gamma_2)\}$, where A acts by twisted conjugation $(a \cdot c)(\gamma) = ac(\gamma)^\gamma a^{-1}$. The parametrization takes a cocycle c to the A-torsor T which can be identified with A as a right-A-set, but with Galois action twisted by c, i.e., if we let a' be the element of T corresponding to an element a of A, then ${}^\gamma a' = c(\gamma) \cdot {}^\gamma a$.

Thus, we obtain:

Proposition 8.2.6. *The equivalence of 8.2.5 induces an isomorphism of pointed sets:*

$$(\mathcal{F}(L/k)/u)/\!\sim \xrightarrow{\sim} H^1(\Gamma, A),$$

where the left-hand side denotes isomorphism classes of descent data for u over L/k, with the chosen descent datum as the base point.

Proof. One just needs to check that isomorphism classes of Γ-equivariant A-torsors are indeed parametrized by $H^1(\Gamma, A)$. Left to the reader. □

Remark 8.2.7. Notice that, although Proposition 8.2.5 classifies a family of descent data depending only on the isomorphism class of u, the classification, and the Galois structure on the automorphism group A, depend on the chosen descent structure on u. This choice makes the isomorphism classes of descent data into a pointed set, which corresponds to the distinguished point (the class of the trivial torsor) in $H^1(\Gamma, A)$.

8.3. Forms of reductive groups

Lemma 8.3.1. *If G, G' are two reductive groups over a field k that are isomorphic over the algebraic closure $\bar{k}$, they are isomorphic over a finite separable extension $L \subset k^s$.*

Proof. This is a corollary of the existence of maximal $\bar{k}$-tori over k, Theorem 7.4.7, the fact that those split over a finite separable extension, Theorem 7.1.2, and the classification of split reductive groups in terms of root data, Theorem 8.1.14. □

By "continuous descent data" over a separable closure k^s, in a k-groupoid $\mathcal{F}$ as above, we will mean pairs (u, ϕ) consisting of $u \in \mathcal{F}_{k^s}$ and isomorphisms $\phi_\gamma : \gamma u \to u$ for $\gamma \in \Gamma = \mathrm{Gal}(k^s/k)$, such that *these descent data are induced from descent data over a finite Galois extension L/k*, in the obvious way (i.e., u is the base change of some object $u' \in \mathcal{F}_L$, and ϕ is obtained by extending scalars from some descent datum ϕ' over L).

Proposition 8.3.2. *The category of reductive groups over k is equivalent to the category of continuous descent data (G, ϕ), where G is a reductive group over a fixed separable closure k^s, and ϕ is an isomorphism $G \times_k k^s \simeq k^s \times_k G$.*

Given a reductive group G over k, the set of isomorphism classes of forms G' of G over k is in natural bijection with the Galois cohomology set $H^1_{cont}(\Gamma, Aut(G_{k^s}))$, defined as the set of $G(k^s)$-orbits of continuous *1-cocycles $\Gamma \to Aut(G_{k^s})$ (i.e., factoring through a finite extension).*

Proof. The first statement follows from Lemma 8.3.1, and effectiveness of descent for linear algebraic groups (an easy corollary of Example 8.2.3). The second follows from Proposition 8.2.6. □

Thus, we are led to study the cohomology of the automorphism group of G (over the separable closure). From now on, for every linear algebraic group over k, we will be writing simply $H^1(\Gamma, G)$ for the continuous cohomology set $H^1_{\mathrm{cont}}(\Gamma, G(k^s))$.

Proposition 8.3.3. *Given a reductive group G over a field k, there is an exact sequence*

$$H^1(\Gamma, Inn(G)) \to H^1(\Gamma, Aut(G)) \to H^1(\Gamma, Out(G)) \to 0. \tag{8.3.3.1}$$

Moreover, if we assume that G is split over k, and we fix a pinning $(G, B, T, \{p_\alpha\}_{\alpha\in\Delta})$ over *k (Definition 8.1.11), the resulting splitting $Out(G) \to Aut(G)$ of Proposition 8.1.23 induces a splitting*

$$H^1(\Gamma, Out(G)) \to H^1(\Gamma, Aut(G))$$

whose image corresponds to the quasisplit forms of G.

Moreover, the cohomology groups of (8.3.3.1), together with this splitting, classify isomorphism classes in the following sequence of categories:

$$\{G_{ad}\text{-torsors over } k\} \to \{\text{forms of } G \text{ over } k\} \rightleftarrows \{\text{quasisplit forms of } G \text{ over } k\},$$

compatibly with the inclusion functor on the right, and the functor that assigns to a right G-torsor T the G-automorphism group $Aut^G(T)$.

Proof. The sequence (8.3.3.1) follows immediately by applying the long exact sequence of cohomology to the short exact sequence of (8.1.23.1). To prove surjectivity of the map to $H^1(\Gamma, \mathrm{Out}(G))$, it is enough to assume that G is split, since the choice of a different form only affects the base point. (Such a form always exists by Theorem 8.1.10.) Then, the splitting $\mathrm{Out}(G) \to \mathrm{Aut}(G)$ of Proposition 8.1.23 gives rise to a splitting of the corresponding cohomology groups, and in particular proves surjectivity. Finally, the identification of cohomology groups with isomorphism classes of objects in the stated categories follows from effective descent and Proposition 8.2.6. In all cases, the stated groups are the groups of isomorphisms of a given object in the stated category (for example, $G_{\mathrm{ad}} = \mathrm{Inn}(G)$ is the automorphism group of a G_{ad}-torsor), except for the category of quasisplit forms of G over k, which has more automorphisms than the outer automorphisms of G, and therefore requires some explanation.

By Proposition 8.1.23, $\mathrm{Out}(G)$ can be identified with automorphisms of the based root datum $\Psi^+(\mathbb{A}^G)$. By Theorem 8.1.12, this can be identified with the automorphism group of a pinned quadruple (G, B, T, p) (over k^s) or, equivalently, a quadruple (G, B, T, ℓ), where ℓ is an algebraic Whittaker datum for B (Definition 8.1.11). Such quadruples make sense over k, as well (and its Galois extensions), thus, by Galois descent, the set $H^1(\Gamma, \mathrm{Out}(G))$ classifies isomorphism classes of such quadruples over k. But the forgetful map $(G, B, T, \ell) \to (G, B)$ is a bijection on isomorphism classes, because every pair (G, B) admits a maximal torus $T \subset B$ and a Whittaker datum ℓ, and any two such are conjugate by an element of $B_{\mathrm{ad}} \subset G_{\mathrm{ad}}$. [More details to be added.]

□

Definition 8.3.4. A *pure inner form* of an algebraic group G over k is a G-torsor T; the term is often used to refer to the G-automorphism group of T, but with the understanding that a G-torsor has been fixed. An *inner form* of G is a pure inner form R for the adjoint group $G_{\mathrm{ad}} = \mathrm{Inn}(G)$; the term is often used to refer to the form $R \times^{\mathrm{Inn}(G)} G$, but with the understanding that a G_{ad}-torsor has been fixed.

8.4. The L-group and the C-group

Let G be a reductive group over a field k, fix a separable closure k^s of k with Galois group Γ, and let $\Psi^+(G)$ be the associated based root datum of G_{k^s}, by Theorem 8.1.14. Since the universal Cartan $\mathbb{A}^G$ is defined over k (Proposition 7.5.3), and has a canonical set of positive roots, there is an action of the Galois group Γ on its character group which preserves the positive roots, hence an action

$$\Gamma \to \mathrm{Aut}(\Psi^+(G)). \tag{8.4.0.1}$$

Remark 8.4.1. By Proposition 8.1.23, we have a canonical isomorphism $\mathrm{Out}(G) \simeq \mathrm{Aut}(\Psi^+(G))$. Viewed as a Galois cocycle into $\mathrm{Aut}(\Psi^+(G))$ (with trivial Galois action, corresponding to the split form of G), the corresponding cohomology class in $H^1(\Gamma, \mathrm{Out}(G))$ is the one attached to the outer class of G by Proposition 8.3.3.

Notice that the universal Cartan groups for inner forms are canonically isomorphic, in an order-preserving way, so that the action (8.4.0.1), hence also the L-group that we are about to define, are identical for two groups that are inner forms of each other.

Definition 8.4.2. Let G be a reductive group over a field k, and fix a separable closure k_s with Galois group Γ. Let $\Psi^+(G)$ be the based root datum of G, and $\check{\Psi}^+(G)$ the dual based root datum, obtained by interchanging the character and the cocharacter lattices. The *Langlands dual group* $\check{G}$ is the pinned group (provided by the equivalence of categories of Theorem 8.1.12) with based root datum $\check{\Psi}^+(G)$. The *L-group* is the semidirect product ${}^L G = \check{G} \rtimes \Gamma$, where Γ acts through (8.4.0.1), by the pinned automorphisms provided by the equivalence of Theorem 8.1.12.

Notice that the L-group can be thought of as being defined over any field, in fact over $\mathbb{Z}$. The ring of definition that one uses for the L-group depends on the coefficients of representations of G that one considers.

Remark 8.4.3. The choice of separable closure, and the definition of the L-group as a semidirect product (hence, a distinguished splitting $\Gamma \to {}^L G$) are not completely justified. It is better to think of the L-group as *a sheaf of pinned reductive groups over the étale site of SpecF*. Hence, instead of talking about "the" based root datum of G (defined using its weight lattice over a fixed separable extension), we have based root data for every separable extension over which G splits, and isomorphisms between them for every morphism of such fields. In particular, the descent data give rise to the Galois action.

Although the Langlands program with complex coefficients is often formulated in terms of the L-group, this is not the most natural dual group to consider, and in particular does not work well with integer coefficients. A more natural choice is the *C-group*, which we introduce now, following [**BG14**] and ideas of Joseph Bernstein. The description of [**BG14**] is combinatorial, but, following Bernstein, we will describe the C-group in terms of the Langlands dual of a natural extension of G.

Lemma 8.4.4. *Let G be a connected reductive group, $\mathcal{B}$ its flag variety of Borel subgroups, Ω the canonical bundle (bundle of volume forms) on $\mathcal{B}$. Then, there is a canonical square root $\Omega^{\frac{1}{2}}$, i.e., a line bundle over $\mathcal{B}$ whose square is Ω, characterized by the fact that it admits a linearization for the action of the simply connected cover G_{sc} of the derived group of G.*

We recall that "linearization" means an action of the group on the bundle, i.e., on its sheaf of sections, compatible with its action on the base, i.e., an isomorphism of the two pullbacks under the projection and action maps $G_{sc} \times \mathcal{B} \rightrightarrows \mathcal{B}$ which is compatible with the group structure.

Proof. By Galois descent and the uniqueness of the square root with this property, it is enough to assume that the field is algebraically closed. If $B \in \mathcal{B}$ is a Borel subgroup, 2ρ is the sum of positive roots of G, and we use exponential notation when thinking about the corresponding character $e^{2\rho} : B \to \mathbb{G}_m$, then Ω is a G-linear bundle under the right action of G on $\mathcal{B}$, and B acts by the character $e^{2\rho}$ on its fiber [exercise!]. In other words, Ω is induced from the character $e^{2\rho}$ of B; its total space is $\mathbb{G}_{a,2\rho} \times^B G$, where $\mathbb{G}_{a,2\rho}$ is the line with an action of B by this character.

If B_{sc} is the corresponding Borel of G_{sc}, then $\mathcal{B} = B\backslash G = B_{sc}\backslash G_{sc}$, and the restriction of 2ρ to B_{sc} admits a unique square root, namely ρ. Thus, there is a unique G_{sc}-linear bundle $\Omega^{\frac{1}{2}}$ over $\mathcal{B}$ whose square is Ω. □

Definition 8.4.5. Let G be a connected reductive group, $\mathcal{B}$ its flag variety of Borel subgroups, $\Omega^{\frac{1}{2}}$ the canonical square root of the canonical bundle on $\mathcal{B}$ (Lemma 8.4.4). The *canonical extension* $\tilde{G}$ is the group of pairs (g, τ), where $g \in G$ and τ is an isomorphism: $g^*\Omega^{\frac{1}{2}} \xrightarrow{\sim} \Omega^{\frac{1}{2}}$.

Remark 8.4.6. Notice that we consider G as acting on the right on $\mathcal{B}$, so pullback is a left action on line bundles: $(g_1g_2)^*\Omega = g_1^*(g_2^*\Omega)$. The composition law on the extended group is $(g_1,\tau_1)\cdot(g_2,\tau_2) = (g_1g_2, \tau_1\circ(g_1^*\tau_2))$. The extended group in a central linear algebraic extension:

$$1 \to \mathbb{G}_m \to \tilde{G} \to G \to 1, \tag{8.4.6.1}$$

where the central $\mathbb{G}_m$ is the group of scalar automorphisms of the line bundle $\Omega^{\frac{1}{2}}$. It acts on $\mathcal{B}$ through the quotient G, and, by definition, *the line bundle $\Omega^{\frac{1}{2}}$ is canonically $\tilde{G}$-linear.*

On the other hand, if G is simply connected, then, by the definition of $\Omega^{\frac{1}{2}}$, it already has a (unique) G-linearization; hence, in this case, $\tilde{G} = \mathbb{G}_m \times G$, canonically.

Definition 8.4.7. For a reductive group G over a field k (in a fixed separable closure with Galois group Γ), the *extended Langlands dual group* $\check{\tilde{G}}$ of G is the Langlands dual group (Definition 8.4.2) of the canonical extension $\tilde{G}$ (Definition 8.4.5), and the *C-group* ${}^CG$ is the L-group of the canonical extension.

The C-group comes with a canonical character dual to the sequence (8.4.6.1):

$$1 \to {}^LG \to {}^CG \to \mathbb{G}_m \to 1, \tag{8.4.7.1}$$

which will be called the *cyclotomic character* of the C-group.

Remark 8.4.8. In terms of based root data, if X denotes the weight lattice in $\Psi^+(G)$, and $\check{X}$ the coweight lattice, the weight lattice $\tilde{X}$ of $\tilde{G}$ is generated inside of the vector space $X_{\mathbb{Q}} \oplus \mathbb{Q}$ by X and the element $\tilde{\rho} = (\rho, 1)$; its projection $\mathbb{Z} \subset \mathbb{Q}$ is the weight lattice of the central $\mathbb{G}_m$, and $\tilde{\rho}$ is the character from which the $\tilde{G}$-linear line bundle $\Omega^{\frac{1}{2}}$ is induced.

Notice that the element $e^{2\rho}(-1)$ is a canonical central element of the dual group $\check{G}$ (possibly trivial), thus defining a map $\mu_2 \to \check{G}$. The extended dual group $\check{\tilde{G}}$ is the group $\check{G} \times^{\mu_2} \mathbb{G}_m$.

8.5. The real case: compact Lie groups and Lie algebras

The main reference for this section is [**Bou05**, Ch. IX §1].

Definition 8.5.1. Let $\mathfrak{g}$ be a finite-dimensional real or complex Lie algebra. The *group of inner automorphisms* $\mathrm{Inn}(\mathfrak{g})$ is the connected immersed subgroup of $\mathrm{GL}(\mathfrak{g})$ (see 4.4.9) whose Lie algebra is $\mathrm{ad}(\mathfrak{g})$.

For compact or semisimple Lie groups, this coincides with the group $\mathcal{E}(\mathfrak{g})$ of Definition 5.5.1, but we will not prove that.

Proposition 8.5.2. *Let $\mathfrak{g}$ be a (finite-dimensional) real Lie algebra. The following conditions are equivalent:*

(1) *$\mathfrak{g}$ is isomorphic to the Lie algebra of a compact Lie group.*
(2) *The group Inn($\mathfrak{g}$) is compact.*
(3) *$\mathfrak{g}$ has a positive definite, invariant symmetric bilinear form.*
(4) *The adjoint representation of $\mathfrak{g}$ is semisimple, and for every $x \in \mathfrak{g}$, ad(x) is semisimple with purely imaginary eigenvalues.*
(5) *$\mathfrak{g}$ is a direct sum of its center and its maximal semisimple ideal, and the Killing form is negative semi-definite.*

If, moreover, $\mathfrak{g} = Lie(G)$ for some connected Lie group G, the above are equivalent to any of the following:

(6) *The group $Ad(G) \subset GL(\mathfrak{g})$ is compact.*
(7) *There is a Riemannian metric on G invariant under left and right translations.*

Finally, in that case the exponential map $\mathfrak{g} \to G$ is surjective.

Proof. (1) $\Rightarrow$ (2): If $\mathfrak{g} = \mathrm{Lie}(G)$ with G compact and connected, then $\mathrm{Inn}(\mathfrak{g}) = G/Z$, where Z is the center, hence is compact.

(2) $\Rightarrow$ (3): Choose any positive definite symmetric bilinear form on $\mathfrak{g}$, and average over the compact group $\mathrm{Inn}(G)$ to obtain an invariant one.

(3) $\Rightarrow$ (4): The orthogonal complement, with respect to a definite invariant form, of an invariant subspace is invariant, which proves semisimplicity. The operators $\mathrm{ad}(x)$ are then anti-self-adjoint, which implies that they are diagonalizable with purely imaginary eigenvalues.

(4) $\Rightarrow$ (5): If the adjoint representation is semisimple, $\mathfrak{g}$ is the direct sum of its center and its simple ideals. If B denotes the Killing form, then $B(x,x) = \mathrm{tr}(\mathrm{ad}(x)^2)$, and since the eigenvalues are all imaginary, this is ≤ 0.

(5) $\Rightarrow$ (1): The center of the Lie algebra is isomorphic to the Lie algebra of a compact real torus, so we are reduced to the case that $\mathfrak{g}$ is semisimple, with negative Killing form. Then, $\mathrm{Inn}(\mathfrak{g})$ is a closed [reference] subgroup of $\mathrm{GL}(\mathfrak{g})$, and since it preserves the negative definite form B, it belongs to the compact orthogonal subgroup $O(B)$, thus is compact (with Lie algebra $\mathfrak{g}$).

If $\mathfrak{g} = \mathrm{Lie}(G)$, with G connected, then $\mathrm{Ad}(G) = \mathrm{Inn}(\mathfrak{g})$, as one can confirm by comparing Lie algebras. The last claim follows from the rest by constructing an $\mathrm{Ad}(G)$-invariant positive definite form on $\mathfrak{g}$ (by the same averaging argument as above), and translating it by the left (or right) G-action. Vice versa, any invariant Riemannian metric, restricted to the tangent space at the identity, is such a form.

Finally, one can check [Exercise!] that, for a left- and right-invariant Riemannian metric, the group-theoretic exponential map coincides with the Riemannian exponential map, hence G is geodesically complete, and any two points can be joined by a length-minimizing geodesic (the *Hopf–Rinow theorem*). $\square$

Definition 8.5.3. A real Lie algebra $\mathfrak{g}$ satisfying the equivalent conditions of Proposition 8.5.2 is called a *compact Lie algebra*.

Theorem 8.5.4. *The Lie algebra of a connected Lie group G is compact if and only if G is the surjective image of a morphism $V \times T \times K \to G$ with finite kernel, where V is a vector space, T is a compact real torus, and K is a compact semisimple Lie group.*

Proof. The direction $\Leftarrow$ follows from Proposition 8.5.2.

As seen in Proposition 8.5.2, a compact Lie algebra is the direct sum of its center and a semisimple compact Lie algebra, and the connected component of the center has to be equal to a group of the form $V \times T$ as above, therefore we are reduced to the case when $\mathfrak{g}$ is compact and semisimple.

In that case, the center Z is discrete, and $G/Z = \mathrm{Ad}(G)$ is compact by Proposition 8.5.2, hence it suffices to prove that the group $G_{\mathrm{ad}} = G/Z$ cannot be the image of a central isogeny with infinite kernel from a connected Lie group. Let $H \to G_{\mathrm{ad}}$ be a central isogeny of finite degree. By Theorem 4.7.5, H and G_{ad} coincide with the real points of algebraic groups over $\mathbb{R}$, and the morphism between them is algebraic. Thus, we can apply the analysis of central isogenies of Theorem 8.1.10, and deduce that, since $\mathfrak{g}$ is semisimple, the degree of any central isogeny is bounded by the index of the coroot lattice in the cocharacter lattice. Thus, G is compact. $\square$

Proposition 8.5.5. *Any semisimple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ has a unique split real form up to (Inn($\mathfrak{g}$)-)conjugacy, and a unique compact form up to conjugacy.*

Proof. Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra, and let $\mathfrak{h}_0$ be the real subspace spanned by the images of $\mathbb{R}$ under the cocharacters $\check{\Phi} \ni \check{\alpha} : \mathbf{G}_a \to \mathfrak{h}$. The Chevalley construction (omitted) shows that it extends to a real form $\mathfrak{g}_0$ of $\mathfrak{g}$, such that, for every root α, $\mathfrak{g}_{\alpha,0} := \mathfrak{g}_0 \cap \mathfrak{g}_\alpha$ is a real form of the root space $\mathfrak{g}_\alpha$. Those real forms, for α in a basis Δ of the root system, determine the form, and in turn are determined by a pinning (Definition 8.1.11) relative to Δ. All such pinnings are conjugate under the image of $\exp(\mathfrak{h})$ in $\mathrm{Inn}(\mathfrak{g})$, hence by the conjugacy of Cartan subalgebras we conclude that all split forms are conjugate.

Starting now with such a split form, and a pinning $\mathfrak{g}_\alpha \simeq \mathbb{G}_a$ for $\alpha \in \Delta$, let $X_\alpha \in \mathfrak{g}_\alpha$ be the element corresponding to $1 \in \mathbb{G}_a$, and let $X_{-\alpha} \in \mathfrak{g}_{-\alpha}$ be the element forming an $\mathfrak{sl}_2$-triple $(h_\alpha, X_\alpha, X_{-\alpha})$, with $h_\alpha \in \mathfrak{h}$ the coroot. We can similarly choose elements $X_\alpha \in \mathfrak{g}_\alpha$ for all roots, such that $[X_\alpha, X_\beta] = N_{\alpha,\beta} X_{\alpha+\beta}$ whenever $\alpha + \beta$ is also a root, with constants $N_{\alpha,\beta}$ satisfying $N_{\alpha,\beta} = N_{\beta,\alpha}$. (This is part of the Chevalley construction, see [**Bou05**, Ch. VIII §2].) Then, we define another real form of $\mathfrak{g}$ by

$$\mathfrak{g}_c = i\mathfrak{h}_0 \oplus \bigoplus_{\{\pm\alpha\} \in \Phi/\{\pm 1\}} \left(\mathbb{R}(X_\alpha + X_{-\alpha}) \oplus i\mathbb{R}(X_\alpha - X_{-\alpha})\right).$$

One checks that this is a compact form, [**Bou05**, Ch. IX §3.2]. Obviously, this depends only on the pinning, not on the choices of X_α for α not simple, and is generated by the summands corresponding to simple roots α.

For any other compact form $\mathfrak{g}_c'$, we will show that it is of this form; by the conjugacy of pinnings, we will then conclude that $\mathfrak{g}_c'$ is G-conjugate to $\mathfrak{g}_c$.

To lighten notation we may already use the conjugacy of Cartan subalgebras and assume that $\mathfrak{g}_c'$ contains a form $\mathfrak{h}_c'$ of $\mathfrak{h}$. By one of the equivalent conditions of Proposition 8.5.2, the eigenvalues for the adjoint action of $\mathfrak{h}_c'$ on $\mathfrak{g}$ have to be purely imaginary, so we must have $\mathfrak{h}_c' = i\mathfrak{h}_0$. Consider the semilinear automorphism $\tau : \mathfrak{g} \to \mathfrak{g}$ corresponding to complex conjugation with respect to $\mathfrak{g}_c'$. It acts by -1 on $\mathfrak{h}_0$, hence interchanges the spaces $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$. If, for a simple $\alpha \in \Delta$ and X_α, $X_{-\alpha}$ as above, we have $\tau(X_\alpha) = cX_{-\alpha}$, then we claim that $c > 0$.

Indeed, if B is the Killing form, $0 > B(X_\alpha, X_{-\alpha}) = c^{-1} B(X_\alpha, \tau(X_{-\alpha}))$, and the quadratic form $X \mapsto B(X, \tau(X))$ is negative definite on $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathfrak{g}$, since B is a $\mathbb{C}$-linear quadratic form on $\mathfrak{g}$ that is negative on the fixed space $\mathfrak{g}_c'$ of the involution τ.

If we set $X_\alpha' = c^{-\frac{1}{2}} X_\alpha$, $X_{-\alpha}' = c^{\frac{1}{2}} X_\alpha'$, we obtain another $\mathfrak{sl}_2$-triple $(h_\alpha, X_\alpha', X_{-\alpha}')$ with $\tau(X_\alpha) = X_{-\alpha}$. Then, the form $\mathfrak{g}_c'$ is generated by the elements $X_\alpha' + X_{-\alpha}'$ and $i(X_\alpha' - X_{-\alpha}')$ with $\alpha \in \Delta$, hence is of the stated form. $\square$

We deduce that any complex, connected semisimple group has a unique compact form up to conjugacy, and, in fact, arrive at the following strengthening of Theorem 4.7.5, whose formulation is taken from class notes of Brian Conrad:

Theorem 8.5.6. *The functor $G \mapsto G(\mathbb{R})$ is an equivalence between: the category of $\mathbb{R}$-anisotropic reductive $\mathbb{R}$-groups whose connected components have $\mathbb{R}$-points, and the category of compact Lie groups. If G is such an $\mathbb{R}$-group then $G^0(\mathbb{R}) = G(\mathbb{R})^0$. The $\mathbb{R}$-group G is semisimple if and only if $G(\mathbb{R})$ has finite center, and in such cases G^0 is simply connected in the sense of algebraic groups if and only if $G(\mathbb{R})^0$ is simply connected in the sense of topology.*

Proof. [To be added] $\square$

8.6. Classification of real forms in terms of Cartan involutions

Let G be a connected, complex reductive group. Proposition 8.5.5 implies that it has a unique, up to conjugacy, split real form. [Details omitted for now — need to explain lift from Lie algebra to group.] Fixing that form, by Proposition 8.3.3 we obtain a bijection between isomorphism classes of real forms of G, and $H^1(\Gamma, \mathrm{Aut}(G))$, where $\Gamma \simeq \mathbb{Z}/2$. Equivalently, forms correspond bijectively to sections

$$\mathbb{Z}/2 \to \mathrm{Aut}(G) \rtimes \mathbb{Z}/2$$

up to $\mathrm{Aut}(G)$-conjugacy.

Here, we will discuss a more classical (and useful) description of real forms, in terms of actual homomorphisms

$$\mathbb{Z}/2 \to \mathrm{Aut}(G)$$

up to $\mathrm{Aut}(G)$-conjugacy. This gives rise to an extremely important correspondence between real forms of G and *symmetric spaces*, that is, spaces of the form G/G^θ, where θ is an involution (automorphism of order two) of G.

Definition 8.6.1. If $\mathfrak{g}_0$ is a semisimple Lie algebra over $\mathbb{R}$, with complexification $\mathfrak{g}$ and associated antiholomorphic involution σ, a *Cartan involution* for $\mathfrak{g}_0$ is a holomorphic involution θ, which commutes with σ, such that $\mathfrak{g}^{\theta\sigma}$ is a compact form of $\mathfrak{g}$ (Definition 8.5.3).

Given such a Cartan involution, the associated *Cartan decomposition* is the pair $(\mathfrak{k}, \mathfrak{p})$ of complementary vector subspaces of $\mathfrak{g}_0$, where $\mathfrak{k} = \mathfrak{g}_0^\theta$ (in particular, $\mathfrak{k}$ is a compact Lie subalgebra) and $\mathfrak{p} = \mathfrak{g}_0^{-\theta}$. Since the pair $(\mathfrak{k}, \mathfrak{p})$ obviously determines the involution θ, we will say that $(\mathfrak{k}, \mathfrak{p})$ is a Cartan decomposition, without reference to θ.

A *Cartan decomposition* for a real Lie group G is a pair $(K, \mathfrak{p})$, where $K \subset G$ is a compact subgroup, and $\mathfrak{p} \subset \mathfrak{g}$ is an $\mathrm{Ad}(K)$-stable subspace such that the map $K \times \mathfrak{p} \ni (k, X) \mapsto k \exp(X)$ is a diffeomorphism.

Given such a Cartan decomposition, the map $\theta(k \exp(X)) = k \exp(-X)$ (where $k \in K$, $X \in \mathfrak{p}$) is an involution of G, called the *Cartan involution*. Since θ determines $K = G^\theta$ and $\mathfrak{p} = \mathfrak{g}_0^{-\theta}$ uniquely, we will refer to such a θ as a Cartan involution, without reference to $(K, \mathfrak{p})$.

Example 8.6.2. Let $G = \mathrm{GL}_n$, $K = O(n)$, the compact orthogonal group of the standard inner product on $\mathbb{R}^n$, $\mathfrak{k} = \mathrm{Lie}(K)$ and $\mathfrak{p} =$ the subspace of symmetric matrices in $\mathfrak{g}_0 = \mathrm{Lie}(G) = \mathfrak{gl}(n, \mathbb{R})$.

Then, the pair $(\mathfrak{k}, \mathfrak{p})$ is a Cartan decomposition of $\mathfrak{g}_0$, and the pair $(K, \mathfrak{p})$ is a Cartan decomposition of G. Indeed, the quotient space G/K can be identified with the space of positive definite quadratic forms on $\mathbb{R}^n$, each represented by a symmetric matrix A with positive eigenvalues, and each such matrix A has a unique square root B of the same type, which in turn is the exponential of a unique symmetric matrix X, so $A = B^2 = \exp(X)^2 = \exp(X)\exp(X)^t$, uniquely.

The Cartan involution on G is $g \mapsto g^{-t}$, on $\mathfrak{g}$ it is $X \mapsto -X^t$, and the associated compact form of $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$ is the Lie algebra of the unitary group for the standard inner product on $\mathbb{C}^n$.

Proposition 8.6.3. *If G is a connected semisimple Lie group (i.e., with semisimple Lie algebra) with Lie algebra $\mathfrak{g}_0$, $(\mathfrak{k}, \mathfrak{p})$ is a Cartan decomposition of $\mathfrak{g}_0$, and $K = \exp(\mathfrak{k})$. If G has finite center, then K is a compact subgroup, and $(K, \mathfrak{p})$ is a Cartan decomposition of G.*

Proof. The proof reduces to the analogous statement for $G = \mathrm{GL}_n(\mathbb{R})$, $K = O(n)$, as follows:

First of all, let K' be the immersed Lie subgroup of G with Lie algebra $\mathfrak{k}$, see Proposition 4.4.9. Since $\mathfrak{k}$ is compact, by Proposition 8.5.2, the exponential map $\mathfrak{k} \to K'$ is surjective, hence $K' = K$.

Let θ be the associated Cartan involution, let B be the Killing form, and consider the quadratic form $q : X \mapsto B(X, \theta X)$, which is negative definite on $\mathfrak{g}_0$. One easily checks that, with respect to this form, $\mathrm{ad}(X)$ is

- symmetric, if $X \in \mathfrak{p}$;
- skew-symmetric, if $X \in \mathfrak{k}$.

Thus, $\mathrm{Ad}_{\mathfrak{g}}(K) \subset O(q) \subset \mathrm{GL}(\mathfrak{g})$, and by Example 8.6.2 the map $\mathrm{Ad}_{\mathfrak{g}}(K) \times \mathfrak{p} \ni (g, X) \mapsto g \exp(\mathrm{ad}(X)) \in \mathrm{GL}(\mathfrak{g})$ is a closed embedding. (Notice that ad has no kernel, by semisimplicity.) Its image is both open and closed in $\mathrm{Ad}(G)$, and by connectedness it is equal to $\mathrm{Ad}(G)$. Therefore, the map $K \times \mathfrak{p} \to G$ is an open and closed embedding, and again by connectedness it is an isomorphism. Note that K is compact, because by assumption the kernel of $K \to \mathrm{Ad}_{\mathfrak{g}}(K)$ is finite. $\square$

We will eventually see that any reductive Lie group admits a Cartan decomposition, but for now we restrict our attention to complex groups (considered, by restriction of scalars, as real Lie groups).

Proposition 8.6.4. *Let G be a complex, connected semisimple algebraic group, and $\mathfrak{k} \subset \mathfrak{g}$ a compact form of $\mathfrak{g}$, which exists, by Proposition 8.5.5, uniquely up to conjugacy. Let $\mathfrak{p} = i\mathfrak{k} \subset \mathfrak{g}$, and set $K = \exp(\mathfrak{k})$. Then, $(K, \mathfrak{p})$ is a Cartan decomposition of $G(\mathbb{C})$ (viewed as a real Lie group), in the sense of Definition 8.6.1. Moreover, the normalizer of K is $K \exp(i\mathcal{Z}(\mathfrak{k}))$, where $\mathcal{Z}(\mathfrak{k})$ denotes the center of $\mathfrak{k}$.*

Proof. Indeed, one immediately checks that, since the Killing form $B_{\mathfrak{g}}$ is negative definite on the real form $\mathfrak{k}$ of $\mathfrak{g}$, the Killing form $B_{\mathfrak{g}\otimes_{\mathbb{R}}\mathbb{C}}$ is negative definite on the real form $\mathfrak{k} \oplus i\mathfrak{p}$ of $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$, hence this is a compact form. The first claim now follows from Proposition 8.6.3, taking into account that complex semisimple groups have finite fundamental groups, and therefore finite center (see the argument in the proof of Theorem 8.5.4).

For the normalizer $\mathcal{N}_G(K)$ of K, it suffices to show that $\mathcal{N}_G(K) \cap \exp(\mathfrak{p}) = \exp(i\mathfrak{z}(\mathfrak{k}))$. If $X \in \mathfrak{p}$ is such that $\exp(X)$ normalizes K, then $[X, \mathfrak{k}] \subset \mathfrak{k}$, but on the other hand $[\mathfrak{p}, \mathfrak{k}] \subset \mathfrak{p}$, so $[X, \mathfrak{k}] = 0$, i.e., $X \in \mathfrak{p}^{\mathfrak{k}} = (i\mathfrak{k})^{\mathfrak{k}} = i\mathcal{Z}(\mathfrak{k})$. $\square$

Now we have the following, see also [**AT18**]:

Proposition 8.6.5. *Let G be a connected complex semisimple algebraic group. Then:*

(1) *Any holomorphic, resp. antiholomorphic, involution of $G(\mathbb{C})$ is algebraic, i.e., induced by an algebraic involution of G, resp. a conjugate-linear involution of $Res_{\mathbb{C}/\mathbb{R}}G$.*

For the rest of the statements, write G for $G(\mathbb{C})$.

(2) *If G is abelian, it admits a unique compact antiholomorphic involution τ (i.e., such that G^τ is compact). It commutes with all holomorphic or antiholomorphic involutions of G.*

(3) *For any antiholomorphic involution σ of G there exists a holomorphic involution θ, commuting with θ, such that $G^{\theta\sigma}$ is compact, and vice versa: for every such σ there exists such a θ.*

Moreover, θ is unique up to $(G^\sigma)^0$-conjugacy, and σ is unique up to $(G^\theta)^0$-conjugacy.

Proof. For the first statement, see [**AT18**, Lemma 3.1].

If G is abelian, it is a torus, and it is easy to see that the only compact antiholomorphic involution τ is the one which, on $\mathfrak{g}$, acts by -1 on the real subspace E generated by the differentials of cocharacters, and by $+1$ on iE. Any other holomorphic or antiholomorphic involution has to preserve the cocharacter lattice, hence the eigenspaces E and iE for τ, hence commutes with τ.

Any involution of G induces involutions on its derived group G_{der} and its center $\mathcal{Z}(G)$, and since $G = \mathcal{Z}(G)G_{\text{der}}$, the rest of the statements are now reduced to G_{der}. Thus, we may assume that G is semisimple.

By Proposition 8.5.5, the set M of compact forms of $\mathfrak{g}$ is a homogeneous space under G, and if we fix a compact form G_c, it can be identified with the quotient $G/G_c \exp(i\mathcal{Z}(\mathfrak{g}_c))$. The tangent space at the point $x = G_c$ can be identified with the quotient $\mathfrak{p}_{\text{ad}} = \mathfrak{p}/i\mathcal{Z}(\mathfrak{g}_c)$ of $\mathfrak{p} = i\mathfrak{g}_c$, and the Cartan decomposition, Proposition 8.6.3, shows that there is a well-defined exponential map

$$\exp_x : \mathfrak{p}_{\text{ad}} = T_xM \to M,$$

descending from the exponential map on $\mathfrak{p}$, which is an isomorphism.

Given σ, it induces an automorphism of order 2 of M, and our goal is to find a fixed point. Start with any point $x = G_c$, then σ induces an isomorphism σ_* between T_xM and $T_{\sigma(x)}M$. We claim:

> σ commutes with the exponential map, i.e., $\exp_{\sigma(x)}(\sigma_* X) = \sigma(\exp_x(X))$, for every $X \in T_xM$.

Indeed, since σ is antiholomorphic, it maps $\mathfrak{p} = i\mathfrak{g}_c$ to $i\sigma(\mathfrak{g}_c)$, which is the space analogous to $\mathfrak{p}$ for $\sigma(x)$.

Now, let $X \in T_xM$ be the unique element with $\sigma(x) = \exp_x(X)$, choose a preimage $\tilde{X}$ of X in $\mathfrak{p}$, and set $g = \exp(\tilde{X})$ (the exponential in the group, here). Hence, $\sigma(x) = gx$, and translation by g identifies the tangent space of x with that of $\sigma(x)$, by a map which we will denote by g_* (and its inverse by g^*).

We claim that $g^*\sigma_*(X) = -X$. Indeed, $x = \sigma(\sigma(x)) = \sigma(\exp_x(X)) = \exp_{\sigma(x)}(\sigma_* X)$ by the claim above, and on the other hand $x = g^{-1}(\sigma(x)) = \exp(-\tilde{X})(\sigma(x)) = \exp_{\sigma(x)}(g_*(-X))$, and by the fact that the map $\exp_{\sigma(x)}$ is an isomorphism, we deduce that $\sigma_* X = g_*(-X)$, or equivalently $g^*\sigma_* X = -X$.

Hence, the point $\exp_x(\frac{X}{2}) = \exp(\frac{\tilde{X}}{2})x$ is fixed under σ, because

$$\sigma(\exp_x(\frac{X}{2})) = \exp_{\sigma(x)}(\sigma_*\frac{X}{2}) = \exp_{\sigma(x)}(-g_*\frac{X}{2}) =$$
$$= \exp(-\frac{\tilde{X}}{2})\sigma(x) = \exp(-\frac{\tilde{X}}{2})\exp(\tilde{X})x = \exp(\frac{\tilde{X}}{2})x.$$

The proof for θ in place of σ is identical.

Now we show that all points of M^σ lie in the same $(G^\sigma)^0$-orbit. Given two such points x, x', we can write $x' = \exp_x(X)$, for a unique element $X \in T_xM$. It is enough to show that $X \in (T_xM)^\sigma$, because then (by semisimplicity) it can be lifted to $\tilde{X} \in \mathfrak{p}^\sigma$, and then $\exp(\tilde{X})$ will belong to $(G^\sigma)^0$. Again, by the σ-equivariance of the exponential map, we get that $\exp_x(X) = x' = \sigma(x') = \sigma(\exp_x(X)) = \exp_x(\sigma_*X)$, and by the fact that $\exp_x$ is an isomorphism we deduce that $\sigma_*X = X$.

The proof or θ is, again, identical.

□

This leads to the following two theorems which are the main results of this subsection:

Theorem 8.6.6. *(The group of $\mathbb{R}$-points of) any connected reductive algebraic group G over $\mathbb{R}$ admits a Cartan decomposition $(K, \mathfrak{p})$, see Definition 8.6.1. The group K is a maximal compact subgroup of $G = G(\mathbb{R})$, and all maximal compact subgroups (and all Cartan decompositions) are $G(\mathbb{R})^0$-conjugate.*

Proof. If σ denotes the antiholomorphic involution of $G(\mathbb{C})$ which fixes $G(\mathbb{R})$, by Proposition 8.6.5 there exists a commuting holomorphic involution θ such that $\sigma\theta$ is compact, unique up to $G(\mathbb{R})^0$-conjugacy. The restriction of θ to $G(\mathbb{R})$ induces a Cartan decomposition $(K, \mathfrak{p})$ with $K = G(\mathbb{R})^\theta$ and $\mathfrak{p} = \mathrm{Lie}(G(\mathbb{R}))^{-\theta}$. The group K is maximal compact, because if $K' \sup K$ were compact, and $g \in K'$ did not belong to K, then by the Cartan decomposition $g = k\exp(X)$ for some $X \neq 0$ in $\mathfrak{p}$, but then $\exp(X)$ belongs to K', but the powers of $\exp(X)$ have no accumulation point, a contradiction. □

Remark 8.6.7. Not every non-algebraic semisimple connected Lie group (i.e., with semisimple Lie algebra) admits a Cartan decomposition. For example, by the Cartan decomposition, $K = S^1$ is a deformation retract of $G = \mathrm{SL}_2(\mathbb{R})$, hence $\pi_1(\mathrm{SL}_2) \simeq \mathbb{Z}$, and its fundamental cover $\widetilde{\mathrm{SL}_2}$ is a semisimple Lie group with infinite center, no nontrivial compact subgroups (because the preimage of K is isomorphic to $\mathbb{R}$), and since the exponential map is not bijective, it does not admit a Cartan decomposition.

Theorem 8.6.8. *Let G be a connected complex reductive group. Let Ant denote the space of antiholomorphic involutions on G, and Hol the space of holomorphic involutions. There is a canonical bijection*

$$Ant/G \leftrightarrow Hol/G$$

induced by the distinguished G-orbit on $Ant \times Hol$ of those pairs (σ, θ) such that σ commutes with θ, and $\mathfrak{g}^{\sigma\theta}$ is compact.

If $\Gamma = Gal(\mathbb{C}/\mathbb{R})$, and we fix a compact real form (an antiholomorphic involution) τ on G, this bijection gives rise to bijections of pointed sets

$$H^1(\Gamma, Aut(G)) \xrightarrow{\sim} H^1(\mathbb{Z}/2, Aut(G)),$$

where $\mathbb{Z}/2$ acts trivially on $Aut(G)$, i.e.,
$H^1(\mathbb{Z}/2, Aut(G)) = \mathrm{Hom}(\mathbb{Z}/2, Aut(G))/Aut(G) - conj$*, as well as*

$$Z^1(\Gamma, Aut(G))/Inn(G) \xrightarrow{\sim} \mathrm{Hom}(\mathbb{Z}/2, Aut(G))/Inn(G),$$

where Z^1 denotes the set of 1-cocycles.

Proof. Indeed, Proposition 8.6.5 states that the set of those pairs (σ, θ) such that σ commutes with θ, and $\mathfrak{g}^{\sigma\theta}$ is compact, forms a unique G-orbit which surjects onto both Ant and Hol.

If τ is the antiholomorphic involution corresponding to a compact form, used to define the Galois action on $\mathrm{Aut}(G)$ by pre- and post-composition, for any $\kappa \in Z^1(\Gamma, \mathrm{Aut}(G))$, identified with the image in $\mathrm{Aut}(G)$ of complex conjugation, the composition $\sigma = \tau\kappa$ is also an antiholomorphic involution. This identifies $Z^1(\Gamma, \mathrm{Aut}(G)) \simeq Ant$, equivariantly under $\mathrm{Aut}(G)$, while $\mathrm{Hom}(\mathbb{Z}/2, \mathrm{Aut}(G)) = Hol$. The rest of the assertions now follow by descending the distinguished G-orbit (which is also an $\mathrm{Aut}(G)$-orbit) on $Ant \times Hol$. Note that $\tau \in Ant$ corresponds to the trivial involution in Hol, hence these bijections are indeed pointed. $\square$

Remark 8.6.9. Two 1-cocycles of Γ into $\mathrm{Aut}(G)$ are conjugate by $\mathrm{Inn}(G)$ if and only if they correspond to the same inner class, see 8.3.4. Thus, the quotient $Z^1(\Gamma, \mathrm{Aut}(G))/\mathrm{Inn}(G)$ is the union of all isomorphism classes of inner forms of all outer forms of G.

8.7. Other chapters

CHAPTER 9

Galois cohomology of linear algebraic groups

[This chapter needs a lot of work. For now, we only summarize the results needed in other sections.]

Let G be a linear algebraic group over a field k and k^s be a separable closure of k. There is a natural action of $Gal(k^s/k)$ on the k^s-points of G, and we can define the Galois cohomology group $H^1(k, G)$. In this chapter, we discuss this cohomology group for k is finite, local, or number field. A good reference for this chapter is Chapter 6 of [**PR94**].

9.1. Galois cohomology over a finite field

In this section, k is a finite field. The absolute Galois group $Gal(k^s/k) \cong \hat{\mathbb{Z}}$ is a procyclic group. Let φ be the arithmetic Frobenius in $Gal(k^s/k)$. Then φ is a topological generator of $Gal(k^s/k)$ and it induces a k-scheme endomorphism $\varphi_G := id_G \times \varphi$ of G_{k^s}.

Theorem 9.1.1 (Lang's theorem)**.** *If G is a connected algebraic group over k, then $H^1(k, G) = 1$.*

Proof. We define a k-scheme morphism $f : G \to G$ by

$$f(g) = g^{-1}\varphi(g).$$

To prove Lang's theorem, it suffices to prove that f is surjective.

Considering the action of G on itself by $g.a = g^{-1}a\varphi(g)$. If we fix a, this define a k-scheme endomorphism of G denoted by f_a. In particular, $f_e = f$. We claim that the map f_a is separable and its image is open and closed for any a. Then f is surjective since G is connected. This would complete the proof of Lang's theorem.

Let me prove the claim. We have

$$d_e f_a(X) = -Xa + d_e\varphi_G(X) = -Xa$$

for $X \in T_e(G)$. So the differential map $d_e f_a : T_e(G) \to T_a(G)$ is an isomorphism of the tangent spaces. As a consequence, f_a is dominant and separable. In particular, the orbit $f_a(G)$ contains a nonempty open subset of G, hence is open by homogeneity. Since this holds for any a, $f_a(G)$ are also closed. We are done. □

Lang's theorem has several important corollaries.

Proposition 9.1.2. *If G is a connected reductive group over k, then G is quasisplit.*

Proof. Let B be a Borel subgroup of G_{k^s}, and let B^φ be the Borel subgroup obtained by applying φ. Then $aB^\varphi a^{-1} = B$ for some a in $G(k^s)$. By the proof of Theorem 9.1.1, $a = g^{-1}\varphi(g)$ for some g in $G(k^s)$. Now, we consider the Borel subgroup $H = gBg^{-1}$. We can check that H is defined over k by the following computation.

$$H^{\varphi} = \varphi(g)B^{\varphi}\varphi(g)^{-1} = gaB^{\varphi}a^{-1}g^{-1} = H.$$

This completes the proof of this proposition.

□

Proposition 9.1.3 (Lang's isogeny theorem)**.** *If G and H are connected k-groups and $f : G \to H$ is a k-isogeny, then $|G(k)| = |H(k)|$.*

Proof. Let F be the kernel of f. We have a short exact sequence

$$1 \to F(k^s) \to G(k^s) \to H(k^s) \to 1$$

of groups. This exact sequence is compatible with the natural Galois action. Then we obtain an exact sequence

$$\{*\} \to F(k) \to G(k) \to H(k) \to H^1(k, F) \to H^1(k, G)$$

of pointed set. By Theorem 9.1.1, we have

$$\frac{|H(k)|}{|G(k)|} = \frac{|H^1(k, F)|}{|F(k)|}.$$

So it suffices to show that $|H^1(k, F)| = |F(k)|$. Note that

$$|H^1(k, F)| = \varinjlim H^1(Gal(k_n/k), F(k_n))$$

where k_n is the degree n extension of k. So it is enough to prove that, for each n, $|H^1(Gal(k_n/k), F(k_n))| = |F(k)|$. This is a property of Herbrand quotient (see [**AW67**, Proposition 11]). This completes the proof. □

Remark 9.1.4. Lang's theorem can also be used to classify connected reductive groups over a finite field. Let G be a split connected reductive group over k. We fix a based root system Ψ^+ of G. By Theorem 9.1.1, we have $H^1(k, Aut(G_{k^s})) \cong H^1(k, Aut(\Psi^+))$. So the k-forms of G are classified by the elements of $H^1(k, Aut(\Psi^+))$. We have the following conclusions.

(1) Any k-group of type B_n, C_n, E_7, E_8, F_4 or G_2 is split.
(2) There are exactly two nonisomorphic k-groups of type A_n (where $n > 1$) and D_n (where $n > 4$). The nonsplit one is split over a quadratic extension of k.
(3) There are exactly three nonisomorphic k-groups. The two non-split ones become split over a quadratic and a cubic extension of k respectively.

Theorem 9.1.5. *If G is a connected algebraic group over a number field K, then G_{K_v} is quasisplit for almost all finite places v of K.*

Proof. See [**PR94**, Theorem 6.7]. □

Using Lang's theorem we can also deduce a result on Galois cohomology of groups over the ring of integers of a local field. Let K be a local field with ring of integers $\mathcal{O}_K$. Let $G_{\mathcal{O}_K}$ be an algebraic group defined over $\mathcal{O}_K$ and let L be a finite Galois extension of K. The Galois group $Gal(L/K)$ acts naturally on the $\mathcal{O}_L$-point of $G_{\mathcal{O}_K}$. We can define the Galois cohomology group $H^1(L/K, G_{\mathcal{O}_K})$.

Theorem 9.1.6. *If a connected group $G_{\mathcal{O}_K}$ has a connected smooth reduction $\underline{G_{\mathcal{O}_K}}$ and the extension L/K is unramified, then $H^1(L/K, G_{\mathcal{O}_K}) = 1$.*

Proof. See [**PR94**, Theorem 6.8]. □

Remark 9.1.7. Let L be a finite extension of a number field K. By the above theorem, if G is a connected algebraic group over K, then for almost all finite places v of K and any w a place of L above v, we have $H^1(L_w/K_v, G_{\mathcal{O}_{K_v}}) = 1$.

9.2. Tate–Nakayama duality for tori

9.3. Cohomology of reductive groups over local fields

Lemma 9.3.1. *If G is an algebraic group over a local field F, then $H^1(F, G)$ is finite.*

Proof. □

Theorem 9.3.2. *If G is a (connected) simply connected, semisimple group over a non-Archimedean field F, then $H^1(F, G)$ is trivial. For an arbitrary connected reductive group over a local field F, there is a canonical surjective map [Kottwitz], which in the non-Archimedean case is a bijection.*

Proof. □

9.4. Cohomology of reductive groups over global fields; the Hasse principle

Definition 9.4.1. Let G be an algebraic group over a global field k. The kernel of the natural map $H^1(k, G) \to \prod_v H^1(k_v, G)$ is the *Tate–Shafarevich group* of G, denoted $\mathrm{Sha}(G)$. We say that G satisfies the *Hasse principle* if $\mathrm{Sha}(G) = 1$.

Theorem 9.4.2. *If G is an algebraic group over a number field, then Sha(G) is finite.*

Proof. □

The following is the Hasse principle for algebraic groups, due to Kneser, Harder (who proved it for groups without E_8 factors) and Chernousov (who completed the E_8 case) over number fields, and to Harder over function fields.

Theorem 9.4.3. *If G is (connected and) simply connected or adjoint over a global field, then Sha$(G) = 1$.*

Proof. See [**PR94**, Theorems 6.6] for the number field case, and [**PR94**, Theorems 6.22] for the reduction of the adjoint case to the simply connected case. The proof for number fields involves a difficult case-to-case analysis. For function fields, there is a general proof due to Harder, [**Har75**]. □

Proposition 9.4.4. *If G is a connected algebraic group over a global field k, then $H^1(k, G) \to \prod_{v|\infty} H^1(k_v, G)$ is surjective.*

Proof. See [**PR94**, Proposition 6.17]. □

9.5. Other chapters

(4) Lie groups and Lie algebras: general properties
(5) Structure of finite-dimensional Lie algebras
(6) Verma modules
(7) Linear algebraic groups
(8) Forms and covers of reductive groups, and the L-group
(9) Galois cohomology of linear algebraic groups
(10) Representations of reductive groups over local fields
(11) Plancherel formula: reduction to discrete spectra
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CHAPTER 10

Representations of reductive groups over local fields

[This chapter, especially the theory of asymptotics, is under construction and has not been proofread. Some statements may be slightly imprecise.]

In this chapter, we discuss representations of a group of the form $G(F)$, when F is a local (locally compact) field, and G is a reductive algebraic group over F. We treat the Archimedean and non-Archimedean cases in parallel, highlighting similarities. For economy of language, such a group will be called a "real or p-adic reductive group" — the p-adic case including non-Archimedean local fields in equal characteristic, $F = \mathbb{F}_q((t))$. The "real" case includes the case when $F = \mathbb{C}$ — notice that the complex structure plays no role in the representation theory, and we can think instead of $G(\mathbb{C})$ as $\mathrm{Res}_{\mathbb{C}/\mathbb{R}} G(\mathbb{R})$. Everything in this chapter also applies to finite central extensions of reductive groups of the form $G(F)$, like the *metaplectic group*, which are not necessarily algebraic; however, the notation is mostly adapted to the algebraic case. When it is clear from the context, the group $G(F)$ will simply be denoted by G. If the word "reductive" is omitted, a "real group" will be a Lie group, and a "p-adic group" will be a p-adic analytic group (although most statements will be true for arbitrary totally disconnected, locally compact groups, in this case).

Remark 10.0.1. A common misunderstanding, when $G = G(\mathbb{C})$ is a complex group, and (π, V) is a smooth complex representation of G, is that the (complex) Lie algebra $\mathfrak{g}$ acts by complex-linear endomorphisms on V; it does not! Instead, G should be treated as a real Lie group; for any smooth, complex representation of a real Lie group, we have an action of the complexified Lie algebra $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ on V by complex-linear automorphisms.

10.1. Various categories of representations

10.1.1. Smooth and SF-representations. The notion of a continuous, in particular of a Banach representation of a topological group was introduced in Definition 2.2.1. We also introduced an F-representation (or Fréchet representation of moderate growth) in Definition 2.6.1, which is a Fréchet representation that is a countable limit of Banach representations.

Definition 10.1.2. A *smooth vector* in a representation (π, V) of a real or p-adic group, resp. an *analytic vector*, in the real case, is a vector $v \in V$ such that the action map $G \ni g \mapsto \pi(g)v \in V$ is smooth (resp., analytic).[1] In particular, in the

[1]In the p-adic case, "smooth" means locally constant, so the definition is equivalent to requiring that v have an open stabilizer.

real case, for a smooth vector v and any element D of the universal enveloping algebra $U(\mathfrak{g}_{\mathbb{C}})$, the element $\pi(D)v$ is defined.

The *space of smooth vectors* of a representation (π, V) is denoted by V^∞, and considered as a topological space, in the p-adic case with the direct limit topology over the subspaces V^J as J varies over open compact subgroups, and in the real case with the topology of convergence of all $\pi(D)v$, where D ranges over elements of the universal enveloping algebra $U(\mathfrak{g}_{\mathbb{C}})$.

A *smooth representation* (π, V) is a representation such that $V = V^\infty$ as topological vector spaces.

An *SF-representation*, or *smooth representation of moderate growth* of a real or p-adic group G is a smooth F-representation.

Lemma 10.1.3. *If V is a Fréchet representation of a Lie or p-adic group G, the subspace V^∞ of smooth vectors is dense.*

Proof. By Proposition 2.3.3, the algebra $M_c^\infty(G)$ of smooth, compactly supported measures (= smooth, compactly supported functions times a Haar measure) acts on V. The image of the action is clearly in V^∞, and by an approximation of the identity, one sees that the image is dense. $\square$

A much stronger, and important, statement is true: the *Dixmier–Malliavin theorem* states that the image of the action of $M_c^\infty(G)$ is all of V^∞:

Theorem 10.1.4. *Let V be a Fréchet representation of a Lie group or p-adic group G. The action map*

$$M_c^\infty(G) \otimes V \to V^\infty$$

is surjective.

Notice that the tensor product here is not completed! The theorem means that every smooth vector can be written as a finite linear combinations of smooth, compactly supported measures acting on other vectors. (Also, without loss of generality, one might assume that $V = V^\infty$, if desired.)

Proof. The p-adic case is trivial, since every J-invariant vector (where J is an open compact subgroup) is fixed by the action of $e_J =$ the probability Haar measure on J. The real case is the theorem of Dixmier–Malliavin, see [**DM78**], or [**Cas11**]. $\square$

Remark 10.1.5. Outside of the realm of F-representations (Fréchet representations of moderate growth), the notion of smooth representation leads to counter-intuitive examples, e.g., the space of distributions on a Lie group G is a smooth representation. We will only be considering smooth Fréchet representations of moderate growth from now on.

Lemma 10.1.6. *If V is an F-representation of a real group, then V^∞ is an SF-representation.*

Proof. First of all, notice that the topology on V^∞ is also given by a countable set of G-continuous seminorms: If ρ_n is a sequence of G-continuous seminorms on V, definining its topology, and we fix, for every $d \ge 0$, a basis $(D_{d,i})_i$ of the d-th filtered part of the universal enveloping algebra $U(\mathfrak{g}_{\mathbb{C}})$, then the seminorms $\rho_{d,n}(v) = \max_i \rho_n(D_{d,i}v)$ define the topology on V^∞ as n and d vary, and are G-continuous, because $\rho_{d,n}(gv) = \max_i \rho_n(g \cdot \mathrm{Ad}(g^{-1})(D_{d,i})v) \ll \rho_{d,n}(v)$ (locally uniformly in G), since the adjoint representation preserves the filtration.

The content of the lemma, then, is that the topological vector space V^∞ is complete. One shows that the action map $g \mapsto \pi(g)v$ gives rise to a morphism $V^\infty \to C^\infty(G,V)$, where G acts by the right regular representation on $C^\infty(G,V)$, and that this is an isomorphism onto the closed subspace $C^\infty(G,V)^G$ of functions that are invariant under the simultaneous action: $g \cdot f(x) := \pi(g)f(g^{-1}x)$. □

10.1.7. Unitary representations. Unitary representations have been introduced in §2.7. Their Plancherel decomposition was discussed in §2.8. Here, we will just add the uniqueness of the Plancherel decomposition, for reductive real or p-adic groups. [LATER]

10.1.8. $(\mathfrak{g},K)$-modules. Topological representations of Lie groups do not form an abelian category. This is sometimes cumbersome; to make the theory more algebraic, we sometimes work with $(\mathfrak{g},K)$-modules.

Definition 10.1.9. Let $\mathfrak{g}$ be a complex Lie algebra, and H a Lie group, with an embedding $\mathfrak{h}_\mathbb{C} \hookrightarrow \mathfrak{g}$, and a representation $\mathrm{Ad} : H \to \mathrm{GL}(\mathfrak{g})$, extending the adjoint action on $\mathfrak{h}_\mathbb{C}$, whose differential coincides with the adjoint action of $\mathfrak{h} \subset \mathfrak{g}$. (For example, $\mathfrak{g}$ is the complexified Lie algebra of a Lie group containing H.) A $(\mathfrak{g},K)$-*module* is a vector space V with actions of both $\mathfrak{g}$ and H, such that:

(1) the action of H is locally finite;
(2) the differential of the action of H coincides with the action of $\mathfrak{h}$, considered as a subalgebra of $\mathfrak{g}$;
(3) $h \cdot X \cdot h^{-1} \cdot v = \mathrm{Ad}(h)(X) \cdot v$, for all $h \in H$, $X \in \mathfrak{g}$, $v \in V$.

This notion is most often (but not exclusively!) used when $H = K$ is a maximal compact subgroup of a Lie group G (with complexified Lie algebra $\mathfrak{g}$).

Lemma 10.1.10. *Let (π,V) be a representation of a Lie group G, and $H \subset G$ a subgroup. The subspace $V_{H\text{-fin}}$ of H-finite vectors is stable under the action of $\mathfrak{g}_\mathbb{C}$.*

Proof. For every $v \in V_{H\text{-fin}}$, the image of the action map $\mathfrak{g} \otimes \mathrm{span}(Hv) \to V$ is finite-dimensional, and contains the element $h \cdot X \cdot v$ for all $X \in \mathfrak{g}$ and $h \in H$, since $h \cdot X \cdot v = \mathrm{Ad}(h)(X) \cdot h \cdot v$. □

Recall also from Theorem 3.4.4 that if $H = K$ is compact, and the representation is Fréchet, the space of K-finite vectors is dense.

Definition 10.1.11. Let G be a reductive Lie group, and $K \subset G$ a maximal subgroup; use $\mathfrak{g}$ to denote the complexified Lie algebra of G. The $(\mathfrak{g},K)$-*module of a Fréchet representation* (π,V) of G is the $(\mathfrak{g},K)$-module $V^\infty_{K\text{-fin}}$ of K-finite smooth vectors in V.

Two representations V_1, V_2 are said to be *infinitesimally equivalent* if their $(\mathfrak{g},K)$-modules are isomorphic.

Remark 10.1.12. Infinitesimal equivalence captures more of the essence of representation theory than isomorphisms of representations. For example, all Banach representations $L^p(\mathbb{R}^\times)$ $(p \ge 1)$ of the group $\mathbb{R}^\times$ are infinitesimally equivalent, although they are not isomorphic as topological vector spaces. On the other hand, the "globalization" theorem of Casselman and Wallach [**Cas89, Wal92, BK14**]) says that any finitely generated, *admissible* (see Definition 10.1.15) $(\mathfrak{g},K)$-module admits a unique "globalization" to a smooth Fréchet representation of moderate growth. The proof of this theorem relies on the subrepresentation theorem (see

Theorem 10.4.5), realizing irreducible $(\mathfrak{g}, K)$-modules as submodules of parabolically induced representations.

Lemma 10.1.13. *If V is a Fréchet representation of a reductive Lie group G, and $K \subset G$ a maximal subgroup, its $(\mathfrak{g}, K)$-module $V^\infty_{K\text{-}fin}$ is dense in V. In particular, if the $(\mathfrak{g}, K)$-module $V^\infty_{K\text{-}fin}$ is irreducible, so is V.*

Proof. This follows from Lemma 10.1.3 and Proposition 3.4.5. □

The converse is true in the category of *admissible* representations (Theorem 10.1.20).

10.1.14. Admissibility.

Definition 10.1.15. A $(\mathfrak{g}, K)$-module (π, V) (in the real case), or a smooth K-module V (in the p-adic case) is called *admissible* if all irreducible representations of K appear with finite multiplicity, i.e., $\dim \mathrm{Hom}_K(\tau, V) < \infty$ for every irreducible representation τ of K.

A (topological) representation (π, V) of a real or p-adic reductive group G is admissible if the $(\mathfrak{g}, K)$-module (resp. K-module, in the p-adic case) $V^\infty_{K\text{-fin}}$ is admissible. Here, K is any maximal compact subgroup of G, in the real case, and any compact open subgroup of G, in the p-adic case.

Remark 10.1.16. The property of being admissible, for a representation of G, does not depend on the choice of K; indeed, in the real reductive case, all Cartan subgroups are conjugate, by Theorem 8.6.6. In the p-adic case, the independence follows from the lemma below.

Lemma 10.1.17. *In the p-adic case, a representation (π, V) is admissible if and only if, for every compact open $J \subset G$, we have $\dim V^J < \infty$.*

Proof. First of all, observe that $V^\infty_{K\text{-fin}} = V^\infty$ for every compact open $K \subset G$.

If a (smooth) irreducible representation τ of K appears with infinite multiplicity, then, obviously, $\dim V^J = \infty$ for all J with $\tau^J \neq 0$.

Vice versa, given K, for every open compact $J \subset K$, the set of (isomorphism classes of) irreducible representations τ of K with $\tau^J \neq 0$ is finite. Indeed, to prove this claim, we can replace J with the intersection of all its K-conjugates, which is still open and compact, but also normal. Then, if $\tau^J \neq 0$ and τ is irreducible, we have $\tau = \tau^J$, hence τ is an irreducible representation of the finite group K/J, and there are only finitely many such. Thus, admissibility according to Definition 10.1.15 implies that V^J is finite-dimensional, for every J. □

Definition 10.1.18. The *contragredient* of a $(\mathfrak{g}, K)$-module V, in the real case, or a smooth G-representation V, in the p-adic case, is the $(\mathfrak{g}, K)$-module, resp. smooth G-representation $\tilde{V} := (V^*)_{K\text{-fin}}$ of K-finite vectors in the linear dual of V.

Lemma 10.1.19. *Assume that V is an admissible $(\mathfrak{g}, K)$-module V, in the real case, or an admissible smooth G-representation, in the p-adic case. Then, $\tilde{\tilde{V}} = V$.*

If V is irreducible, any automorphism of V (as a $(\mathfrak{g}, K)$-module, resp. as a G-representation) is scalar.

Proof. The module is a direct sum over its K-types, and those are finite-dimensional. The contragredient, as a representation of K, will be the direct sum of the duals, and any automorphism preserves the isotypic spaces. The result, now, follows easily from the finite-dimensional case. □

The converse to Lemma 10.1.13 holds, for admissible representations:

Theorem 10.1.20. *If V is an irreducible admissible Fréchet representation of moderate growth of a reductive Lie group G, its $(\mathfrak{g}, K)$-module is irreducible, and all K-finite vectors are automatically analytic (in particular, smooth).*

Proof. First of all, since $V^\infty_{K\text{-fin}}$ is dense (Lemma 10.1.13) in V, it is also dense in $V_{K\text{-fin}}$. For any K-type τ, there is a measure μ_τ on K whose action on any Fréchet module is a projection onto the τ-isotypic component. Therefore, the τ-isotypic subspace $V^{\infty,\tau}$ is dense in V^τ. But the former is finite-dimensional, therefore the two coincide, i.e., every K-finite vector is smooth.

Suppose that $V_0 \subset V_{K\text{-fin}}$ is a nonzero $(\mathfrak{g}, K)$-submodule. We claim that the closure of V_0 is G-stable. This requires the "big hammer" of elliptic regularity to prove, so we only give a couple of steps, followed by references.

First, we notice that the action of the center $\mathfrak{z}(\mathfrak{g})$ of the universal enveloping algebra of the (complexified) Lie algebra $\mathfrak{g}$ on $V_{K\text{-fin}}$ is locally finite: indeed, it preserves the finite-dimensional, K-isotypic subspaces.

Elliptic regularity, now, implies that all vectors in $V_{K\text{-fin}}$ are analytic; see [**Wal88**, 3.4.9].[2] And, the closure of a $\mathfrak{g}$-stable subspace of analytic vectors in V is stable under the identity component of G: simply apply the exponential map $\mathfrak{g}_\mathbb{R} \to G$, whose image generates the identity component.

Since V_0 is not only $\mathfrak{g}$-stable, but also K-stable, and K meets all connected components of G, V_0 is G-stable. Since V is irreducible, V_0 is dense. But, again, applying projectors to the K-types, this means that for any K-type τ, the τ-isotypic subspace V_0^τ is dense in V^τ. Since these spaces are finite-dimensional, $V_0^\tau = V^\tau$ for all τ, hence $V_0 = V_{K\text{-fin}}$. □

10.2. Schwartz and Harish–Chandra Schwartz spaces

10.2.1. Schwartz space defined by a scale function. We follow [**BK14**, §2].

Definition 10.2.2. A *scale* on a locally compact group G is a function $s : G \to \mathbb{R}^+$ such that:

- s and s^{-1} are locally bounded,
- s is submultiplicative, i.e., $s(gh) \le s(g)s(h)$ for all $g, h \in G$.

A scale function s' *dominates* a scale function s, if there exist positive constants C, N such that $s \le Cs'^N$. They are *equivalent* if each dominates the other.

A *scale structure* on G is an equivalence class of scale functions.

In other words, a scale function is the exponential of a radial function, Definition 2.5.1.

In 2.5.2 we saw the "natural radial function" r_{nat} (and hence its exponential, the "natural scale function" s_{nat}, denoted $\|g\|$ there) of a compactly generated group; recall that $r_{\text{nat}}(g)$ counts how many times we need to multiply a compact generating neighborhood of the identity by itself in order to produce a set containing g.

[2] In Wallach's book, the argument is formulated for representations on Hilbert spaces, but it holds verbatim for Banach spaces, and hence for Fréchet representations of moderate growth. Note that a function $G \to V$, where V is a Banach space, is (real) analytic iff it is *weakly* analytic, i.e., iff its composition with any continuous functional $v^* : V \to \mathbb{C}$ is analytic.

Definition 10.2.3. Let G be a group equipped with a scale structure $[s]$ (Definition 10.2.2), with associated radial function $r = \log s$. The associated *Schwartz space* $\mathcal{S}_{[s]}(G)$ is the space of smooth vectors in the left and right regular F-representation on measures f on G which satisfy $f \cdot s^n \in L^1(G)$ for all $n \in \mathbb{N}$.

The *natural Schwarz space* $\mathcal{S}_{\mathrm{nat}}(G)$ is the one defined by the class of natural scale functions.

Equivalently, the natural Schwartz space is the space of smooth vectors in the space of rapidly decaying measures of Definition 2.5.6.

Example 10.2.4. For the additive group $G = \mathbb{G}_a(F)$, we have $\mathcal{S}_{nat}(G) =$ the Schwartz space of smooth functions (times a Haar measure) which, together with their derivatives (in the real case), are of superexponential decay. On the other hand, for $G = \mathbb{G}_m(F)$, they coincide with smooth functions f (times a Haar measure) such that $f(x) \cdot |x|^n$ is bounded for all $n \in \mathbb{Z}$, and similarly for all derivatives (in the real case).

Remark 10.2.5. There is some clumsiness in trying to deal with the real and p-adic cases at the same time, which is due to the fact that the notion of "smooth" in the p-adic case is not quite analogous to that of "smooth" in the real case; for example, smooth vectors in an F-representation of a p-adic group do not produce Fréchet spaces. There is a notion of "almost smooth" vectors in the p-adic case, which is a better analogy to "smooth" in the real case, see [**Sak13**], but it is not very useful in practice. Because of the strong definition of smoothness (=local constancy), and because we tend to forget about the topology on spaces of smooth vectors of representations of p-adic groups, the "rapid decay" Schwartz spaces that we are defining here are not suitable for p-adic groups; in the next subsection, we will discuss algebraically defined Schwartz spaces using compactifications, where the definitions in the real and p-adic cases coincide, and produce compactly supported functions/measures in the p-adic case. [But, note for the future: Maybe we can expand the notion of SF-representation to the p-adic case, to include the LF-spaces of smooth vectors in an F-representation; or, include a full discussion of "almost smooth" vectors, for the sake of uniformity.]

Proposition 10.2.6. *Let G be a real Lie group. The categories of smooth Fréchet representations of moderate growth of G, and of nondegenerate continuous algebra representations of $\mathcal{S}_{nat}(G)$ on Fréchet spaces, are equivalent.*

Proof. If (π, V) is any F-representation, the action of G extends to a continuous representation of the algebra of rapidly decaying measures by Proposition 2.5.7, in particular, to the natural Schwartz space.

A theorem of Dixmier and Malliavin [**DM78**] states that, if (π, V) is a smooth Fréchet representation of a real Lie group G, then the action map $M_c^\infty(G) \otimes V \to V$ is surjective. Hence, so is the map $\mathcal{S}_{\mathrm{nat}}(G) \otimes V \to V$, i.e., V is nondegenerate.

Vice versa, if V is a nondegenerate continuous Fréchet $\mathcal{S}_{\mathrm{nat}}(G)$-module, that is, it is nondegenerate and the action map $\mathcal{S}_{\mathrm{nat}}(G) \times V \to V$ is continuous, this action extends to the *projective tensor product* $\mathcal{S}_{\mathrm{nat}}(G) \hat{\otimes}_\pi V \to V$, which is also a Fréchet space, and this gives a topological identification of V as a quotient of the projective tensor product. Quotients of SF-representations are SF-representations, see [**BK14**, Lemma 2.9 and Proposition 2.20] for more details. $\square$

10.2.7. Schwartz space of a semi-algebraic manifold. If G denotes the points of a linear algebraic group over a local field, we also define another scale function, that depends on the algebraic structure. (The same definition can be given for finite covers thereof, by passing to the algebraic quotient.)

Definition 10.2.8. Let G be a linear algebraic group, and fix a closed embedding $G \hookrightarrow \mathbb{A}^r$, with coordinates $x_1, \ldots, x_r$. The corresponding *algebraic scale function* of $G(F)$, where F is a local field, is

$$s_{\mathrm{alg}}(g) = \max_i |x_i(g)|.$$

It is easy to prove that any two algebraic scale functions are equivalent.

Lemma 10.2.9. *If G is a reductive group, the natural and algebraic scale functions on G are equivalent.*

Proof. The statement is easily seen to be true for a torus. For a general reductive group, it reduces to the case of tori by the Cartan decomposition $G = KA^+K$. $\square$

This leads to a notion of "algebraic Schwartz space" according to Definition 10.2.3, but in the p-adic case we would like a stricter definition that coincides with the space of compactly supported smooth measures. In this subsection, we will provide a uniform such for arbitrary real or p-adic (smooth) varieties (or semialgebraic spaces).

[Definition of Schwartz space $\mathcal{S}(X)$ on a Nash manifold X here. In the p-adic case, it coincides with $M_c^\infty(X)$. In particular, in the p-adic group case, $\mathcal{S}(G) = \mathcal{H}(G) =$ the Hecke algebra.]

Proposition 10.2.10. *For both real and p-adic reductive groups, there is an equivalence of categories between SF-representations (in the real case), or smooth representations without topology (in the p-adic case), and nondegenerate $\mathcal{S}(G)$-modules.*

Proof. In the real case, this is just Proposition 10.2.6, together with the equivalence of natural and algebraic scale structures, Lemma 10.2.9.

In the p-adic case, the proof is similar (but simpler). Notice that the analogous statement holds, more generally, for any locally compact, totally disconnected group. $\square$

10.2.11. Harish-Chandra Schwartz space. We follow [**Ber88**]. We start by defining a notion of radial function for a homogeneous space of a locally compact group; everything in this section applies to a finite union of homogeneous spaces, as well.

Definition 10.2.12. Let X be a homogeneous space for a locally compact group G. A *radial function* is a locally bounded function $r_X : X \to \mathbb{R}_+$, such that:

(1) for every $R \in \mathbb{R}_+$, the "ball" $B(R) = \{x \in X | r_X(x) \le R\}$ is relatively compact in X;
(2) for any compact $\Omega \subset G$, there is a constant $C > 0$ such that $|r_X(gx) - r_X(x)| < C$.

We say that r_X, r'_X are two *equivalent radial functions* if there is a constant C such that $C^{-1}(1 + r_X) \le (1 + r'_X) \le C(1 + r_X)$.

We say that the space X is of *polynomial growth* (with respect to a given equivalence class of radial functions) if there is a $d \ge 0$ such that, for one (any) compact neighborhood Ω of the identity in G, and some positive constant C, the ball $B(R)$ can be covered by $\le C(1+R^d)$ orbits of Ω.

The equivalence class of *natural radial functions* on X is the equivalence class of the function $r_X(x) = \inf\{r(g)|x_0 \cdot g = x\}$, where r is a natural radial function on G, and x_0 is some fixed point on X.

Now we assume that X is a homogeneous real or p-adic manifold, of polynomial growth, under the action of a real or p-adic group G, or a finite union of such. [Fact, to be added: the space $X(F)$ of points of a spherical G-variety X over a local field F, under the action of the group $G(F)$, are such.]

Definition 10.2.13. Let X be a homogeneous real or p-adic manifold, under the action of a real or p-adic group G, of polynomial growth with respect to the natural scale. The *Harish-Chandra–Schwartz space* of X is the space $\mathcal{C}(X)$ of smooth vectors in the Fréchet space of half-densities f on X with $f \in \lim\limits_{\substack{\leftarrow \\ d>0}} L^2(X, (1+r)^d)$, where r is a natural scale function on X.

The notation $L^2(X, (1+r)^d)$ stands for the Hilbert space of half-densities f with norm equal to the square root of $\int_X |f|^2(1+r)^d$.

Remark 10.2.14. The space $\mathcal{C}(X)$ is a nuclear Fréchet space, in the real case, and a countable direct limit over the nuclear Fréchet spaces of J-invariants, as J ranges over a basis of open compact subgroups, in the p-adic case.

Remark 10.2.15. If X has an invariant measure dx, or, more generally, a positive G-eigenmeasure with (positive) G-eigencharacter η, one can think of half-densities as functions, by dividing by $(dx)^{\frac{1}{2}}$, but the action of G on those functions is twisted by the square root of η, that is:

$$(g \cdot \Phi)(x) = \eta^{\frac{1}{2}}(g)\Phi(x \cdot g). \tag{10.2.15.1}$$

In other words, if $\mathcal{F}(X)$ denotes functions and $\mathcal{D}(X)$ denotes half-densities, division by $(dx)^{\frac{1}{2}}$ defines an equivariant isomorphism $\mathcal{D}(X) \xrightarrow{\sim} \mathcal{F}(X) \otimes \eta^{\frac{1}{2}}$.

For example, consider the pre-flag variety $X = U\backslash G_0$, where $U \subset P \subset G_0$ is the unipotent radical of a parabolic subgroup. Considering it as a homogeneous space for the product $G = L \times G_0$, where L is the Levi quotient of P, it possesses G-eigenmeasure, which is invariant under G_0, but δ_P-equivariant under L, where δ_P is the modular character of P. Thus, half-densities on X can be identified (after a choice of such a measure, unique up to scalar), with functions on X, with the action of L on the latter twisted by δ_P.

Definition 10.2.16. The space of *tempered half-densities* on X is the dual of the topological vector space $\mathcal{C}(X)$. If a G-eigenmeasure dx on X is chosen (always to be taken G-invariant, if possible), the dual of the space $(dx)^{-\frac{1}{2}}\mathcal{C}(X)$ of Harish-Chandra–Schwartz functions is the space of *tempered measures*, and the dual of the space $(dx)^{\frac{1}{2}}\mathcal{C}(X)$ of Harish-Chandra–Schwartz measures is the space of *tempered generalized functions*.

The space of *tempered smooth half-densities* (and, correspondingly, functions or measures in the presence of an eigenmeasure) is the space of smooth vectors in the contragredient of the F-representation $\bigcap_d L^2(X, (1+r)^d)$ (Definition 2.6.5), that is, in the direct limit of Hilbert spaces $\lim\limits_{\substack{\rightarrow \\ d>0}} L^2(X, (1+r)^d)$.

Here is the main result of [**Ber88**]:

Theorem 10.2.17. *The inclusion $\mathcal{C}(X) \hookrightarrow L^2(X)$ is fine; that is, for any morphism from $L^2(X)$ to a direct integral $H = \int H_z\mu(z)$ of Hilbert spaces, the composition $\mathcal{C}(X) \to H$ is pointwise defined (Definition 2.8.12).*

Proof. This is [**Ber88**, Theorem 3.2], applied to the setting of [**Ber88**, §3.5, 3.7]. $\square$

Definition 10.2.18. An admissible smooth representation π of a real or p-adic reductive group is called *tempered* if its matrix coefficients are tempered, i.e., have image in the space $C^\infty_{\mathrm{temp}}(G)$ of smooth, tempered functions (Definition 10.2.16).

More generally, if X is a homogeneous G-space of polynomial growth, a morphism $\ell : \pi \to C^\infty(X)$ is called *tempered* if the image lies in $C^\infty_{\mathrm{temp}}(X)$.

10.3. Asymptotics

10.3.1. General setup. When $X = H\backslash G$ is a homogeneous G-space, and π a smooth representation of G, a morphism $m : \pi \to C^\infty(X)$ is sometimes called a *generalized matrix coefficient*; the reason is that any such morphism is equivalent (by Frobenius reciprocity) to an H-invariant functional ℓ, so $m(v)(x) = \langle \pi(g)v, \ell\rangle$ is a "matrix coefficient", where the covector ℓ is allowed to be non-smooth. In this section, we compare generalized matrix coefficients of certain representations of G on a spherical variety X, with generalized matrix coefficients on the boundary degenerations.

There are similarities, but also differences, between the real and p-adic cases. The main difference, in the real case, is that we need to restrict to admissible modules. (A general theory of asymptotics for smooth representations would be very desirable, but has not yet been developed! The naive tranlation of statements from the p-adic to the real case does not hold, in general.)

For the remainder of this section, G is a real or p-adic reductive group, and K is a maximal compact subgroup, if G is real. We compare generalized matrix coefficients on X and X_Θ by choosing some reasonable (but noncanonical) identification of the spaces "close to infinity":

Definition 10.3.2. Let Z be the closure of a G-orbit in a toroidal embedding $\bar{X}$ of X. An *approximate exponential map* is an analytic map $\phi : U_Z \to \bar{X}(F)$, where U_Z is a neighborhood of Z in the F-points of the normal bundle $N_Z\bar{X}$, with the property that the partial differential of ϕ induces the identity between on the normal bundle, and ϕ maps the intersection of every G-orbit with U_Z to the corresponding G-orbit on $\bar{X}$. The *exponential bundle* $\mathrm{Exp}_Z\bar{X}$ over Z is the fiber bundle of germs, over Z, of approximate exponential maps.

Note that $\mathrm{Exp}_Z\bar{X}$ is a torsor for the group bundle $\mathrm{Exp}_Z N_Z\bar{X}$ of germs of approximate exponential maps from the normal bundle to itself (defined the same way).

Proposition 10.3.3. *Assume that F is non-Archimedean. Using the notation of Definition 10.3.2, let $\phi : U_Z \to \bar{X}(F)$ be an approximate exponential map for some orbit closure $Z \subset \bar{X}$. Then, given an open compact subgroup $J \subset G$, there is a J-invariant neighborhood $U'_Z \subset U_Z$ of Z, with J-invariant image $U'_X \subset \bar{X}(F)$, such that ϕ descends to a bijection: $U'_Z/J \to U'_X/J$. Moreover, any two approximate exponential maps descend to the same bijection, if the neighborhood U'_Z is taken sufficiently small.*

Proof. See [**SV17**, Proposition 4.3.1]. The reader is encouraged to check it directly in the baby case of $\bar{X} = \mathbb{A}^1$, $Z = \{0\}$, $G = \mathbb{G}_m$. $\square$

Now, the normal bundle to Z contains some open G-orbit, which we have called the boundary degeneration; let's denote it by X_Θ. This proposition implies that, for any J-invariant functions f, f_Θ on X and X_Θ, respectively, there is a well-defined notion of the functions being asymptotically equal:

Definition 10.3.4. Assume that F is non-Archimedean. Let X be a spherical variety, and X_Θ an asymptotic cone thereof, obtained as the open G-orbit in the normal bundle of some orbit Z in a toroidal embedding. If $f \in C^\infty(X)$, $f_\Theta \in C^\infty(X_\Theta)$, we say that f is *asymptotically equal* to f_Θ, written $f \sim f_\Theta$, if there is an approximate exponential map $\phi : U_Z \to \bar{X}(F)$ (Definition 10.3.2), where U_Z is a neighborhood of Z, such that, after possibly replacing U_Z by a smaller neighborhood, $\phi^* f|_{U_Z} = f_\Theta|_{U_Z}$.

Notice that, by Proposition 10.3.3, this notion does not depend on the choice of approximate exponential.

In the real case, things are finer, since smooth functions are not locally constant. Therefore, any such attempt to identify f and f_Θ will depend on the choice of approximate exponential. Instead of looking at arbitrary smooth functions, here, we will restrict our attention to "functions that look like generalized characters" (of the tori A_Θ) at infinity—we will call such functions "asymptotically finite". The following baby example captures the essence of such functions:

Example 10.3.5. Let $\bar{X} = \mathbb{A}^1 \supset X = \mathbb{A}^1 \smallsetminus \{0\}$, over $F = \mathbb{R}$. Let $Z = \{0\}$; then, $N_Z\bar{X} = \mathbb{A}^1$. Here, we want to think of $\bar{X}$ simply as a variety (without a group action), while $N_Z\bar{X}$ has a $\mathbb{G}_m$-action. Any analytic map $\phi : U_Z \to \mathbb{R}$, where U_Z is a neighborhood of zero, fixing zero and inducing the identity on its tangent space, is an asymptotic exponential. Explicitly, such a ϕ is given by a power series of the form $\phi(x) = x + \sum_{n=2}^\infty a_n x^n$, convergent within some radius.

An "asymptotically finite" function f on X is a function with the property that $\phi^* f = \sum_\lambda f_\lambda \cdot h_\lambda$, a finite sum indexed by characters of the multiplicative group, where f_λ is a generalized $\mathbb{G}_m$-eigenfunction with generalized eigencharacter λ, and $h_\lambda \in C^\infty(U_Z)$. The reader should check [exercise!] that this notion does not depend on the choice of approximate exponential ϕ.

Definition 10.3.6. Let F be real or non-Archimedean, and let X be a spherical variety over F. An *asymptotically finite* function on X is a smooth function f with the property that, for some toroidal compactification $\bar{X}$, in a neighborhood of any point $z \in \bar{X}$ (belonging to a G-orbit Z whose normal bundle is the boundary degeneration X_Z), and for any approximate exponential ϕ defined in a neighborhood U of z, the function $\phi^* f$, restricted to a neighborhood $U' \subset U$ of z, is equal to

$$\sum_\lambda f_\lambda \cdot h_\lambda, \tag{10.3.6.1}$$

a finite sum indexed by characters of A_Z, where f_λ is a generalized A_Z-eigenfunction with generalized eigencharacter λ, and $h_\lambda \in C^\infty(U')$.

We let $\mathrm{Fin}_Z(\bar{X})$ denote the bundle of germs, over a G-orbit Z, of asymptotically finite functions defined in a neighborhood of Z in $\bar{X}$, and call the image (germ) of such a function f in $\mathrm{Fin}_Z(\bar{X})$ the *asymptotic expansion* of f at Z. Equivalently, if $\mathrm{Fin}_Z(N_Z\bar{X})$ denotes the space of germs, at Z, of functions of the form (10.3.6.1) defined in a neighborhood of Z in $N_Z\bar{X}$, the asymptotic expansion of f is the induced map

$$\mathrm{Exp}_Z(\bar{X}) \to \mathrm{Fin}_Z(N_Z X)$$

from germs of approximate exponential functions (see Definition 10.3.2), which is equivariant for the group bundle $\mathrm{Exp}_Z(N_Z\bar{X})$.

The characters λ in an expansion (10.3.6.1) will always be assumed to be such that no quotient of two of them extends to a smooth function on U'. Under that assumption, the *dominant term* of an asymptotically finite function of the form (10.3.6.1) is the sum $f_Z = \sum_\lambda f_\lambda \in C^\infty(U')$; when U' contains the entire orbit of z, f_λ extends uniquely as a generalized A_Z-eigenfunction to X_Z, and we will consider the dominant term as a function on X_Z. (This depends on the orbit of z, not just the isomorphism class of X_Z!) We write $f \sim f_Z$ to indicate that f_Z is the dominant term of f.

Remark 10.3.7. Notice that, in the non-Archimedean case, the functions h_λ in the asymptotic expansion (10.3.6.1) are not needed, since they are constant in a neighborhood of z; hence, an asymptotically finite function is exactly equal to an A_Z-eigenfunction in a neighborhood of z.

Lemma 10.3.8. *The dominant term f_Z of an asymptotically finite function along a G-orbit is independent of the choice of an approximate exponential function used to define it.*

Proof. [Easy; will be added.] □

In the real case, asymptotically finite functions with respect to a given compactification $\bar{X}$, set E of "exponents" λ, and bounded degree for the generalized eigenfunctions f_λ have a natural structure of a Fréchet space. [Details are left to the reader, for now.]

Remark 10.3.9. The following is expected to be true for every spherical variety:

Expected theorem:

Let X denote the points of a homogeneous spherical G-variety, and let X_Θ be a boundary degeneration.

If π is any smooth representation of G, in the p-adic case, and an admissible SF representation of G, in the real case, then for any morphism $\ell : \pi \to C^\infty(X)$, there is a unique morphism $\ell_\Theta : \pi \to C^\infty(X_\Theta)$, such that $\ell(v) \sim \ell_\Theta(v)$ for all $v \in \pi$. (In particular, in the admissible case, $\ell(v)$ is asymptotically finite.)

In fact, one can make a stronger statements, where the neighborhood of infinity, or the rate of convergence of asymptotic expansions, is determined by a compact open subset by which v is invariant, resp. a continuous seminorm of v. This theorem has not appeared in the literature in complete generality. In the next subsections we will formulate (some of) the cases that are known.

10.3.10. Asymptotics in the non-Archimedean case.

Theorem 10.3.11. *Let X denote the points of a homogeneous spherical G-variety over a non-Archimedean field, and let X_Θ be a boundary degeneration. Under the following assumptions:*

- *G is split and X is of* wavefront *type (see* [**SV17**, §2.1]*), OR*
- *X is symmetric,*

the following is true: There is a unique morphism

$$e_\Theta : \mathcal{S}(X_\Theta) \to \mathcal{S}(X)$$

with the property that, whenever X_Θ is realized in the normal bundle of an orbit Z in a smooth toroidal compactification of X, and ϕ is an approximate exponential map (Definition 10.3.2), for every open compact subgroup J there is a J-stable neighborhood U'_Z of Z as in Proposition 10.3.3— in particular, ϕ induces a bijection $U'_Z/J = U'_X/J$, where U'_X is the image of U'_Z in X— such that, for $f \in \mathcal{S}(U'_Z)^J$, $e_\Theta(f) = \phi_*(f)$, its pushforward to U'_X/J through this identification.

In particular, the adjoint morphism $e^*_\Theta : C^\infty(X) \to C^\infty(X_\Theta)$ has the property that $e^*_\Theta f|_{U'_Z} = \phi^* f|_{U'_Z}$, for every $f \in C^\infty(X)^J$.

The theorem is expected to hold without these assumptions on X.

In particular, if $\ell : \pi \to C^\infty(X)$ is any morphism of smooth representations, we obtain the asymptotic morphism ℓ_Θ of the "Expected Theorem" of Remark 10.3.9 as $\ell_\Theta = e^*_\Theta \circ \ell$.

Proof. See [**SV17**, Theorem 5.1.1] and [**Del18**, Theorem 1]. □

10.3.12. Asymptotics in the real case.

Theorem 10.3.13.
(1) *Let $X = H$, a (connected) reductive group over $\mathbb{R}$, under the $G = H \times H$-action. Let τ be an admissible smooth Fréchet representation of moderate growth of H, and $\tilde\tau$ its contragredient. Then, for every class P of parabolics in H, there exists a finite set E of A_P-exponents and a degree d, depending on τ, such that all matrix coefficients*

$$f_{v,\tilde v}(g) := \langle \tau(g)v, \tilde v\rangle$$

are asymptotically finite with exponents $\lambda \in E$ and degree bounded by d in a neighborhood of P-infinty, and the map from $\tau\hat\otimes\tilde\tau$ to the corresponding Fréchet space $Fin_P^{E,d}$ of asymptotic expansions is continuous.

In particular, considering only leading terms, there is a morphism $\ell_P : \tau\hat\otimes\tilde\tau \to C^\infty(X_P)$ such that $f_{v,\tilde v} \sim \ell_P(v \otimes \tilde v)$ in a neighborhood of P-infinity.

Moreover, if $\ell_P = 0$ (i.e., the matrix coefficients of τ are of rapid decay), for any P, then $\tau = 0$.

(2) *Let X be any real spherical variety for a reductive group G, and π an admissible representation with a* tempered *morphism $\ell : \pi \to C^\infty_{temp}(X)$, and let X_Θ denote a boundary degeneration, identified with the open G-orbit in the normal bundle of some orbit in a toroidal compactification. Then, there exists a tempered morphism $\ell_\Theta : \pi \to C^\infty_{temp}(X_\Theta)$, an A_Θ-eigenfunction h on X_Θ with real positive eigencharacter which is < 1 on $\exp(\mathfrak{a}^+_\Theta)$, and a continuous seminorm q, such that, for any approximate exponential map ϕ, $|\phi^*\ell(v) - \ell_\Theta(v)| \le h \cdot q(v)$ in a neighborhood of Θ-infinity.*

Proof. For the group case, see [**Wal88**, 4.4]. For the tempered case, see [**DKS19**]. □

Definition 10.3.14. Let τ be an arbitrary smooth representation of a p-adic reductive group H, or an admissible smooth representation of moderate growth of a real reductive group H. For every class P of parabolics in H, let H_P be the corresponding boundary degeneration. The *asymptotic matrix coefficient* morphism associated to P is the morphism

$$m_P : \tau\hat{\otimes}\tau \to C^\infty(H_P),$$

where $m_P = \ell_P$ in the notation of Theorem 10.3.13, in the real case, and $m_P = e_P^* \circ m$, where m is the matrix coefficient map, and $e_P^* : C^\infty(H) \to C^\infty(H_P)$ is the asymptotics map of Theorem 10.3.11, in the p-adic case.

10.4. Consequences of the asymptotics

10.4.1. Supercuspidals.

Proposition 10.4.2. *For an admissible smooth representation of a real or p-adic Lie group H, the following are equivalent:*

(1) *The matrix coefficients of τ are of rapid decay (in the real case) or compactly supported (in the p-adic case) modulo the center.*
(2) *The asymptotic matrix coefficient morphisms m_P (Definition 10.3.14 are zero for every class P of proper parabolics in H.*

In particular, in the real case, if the matrix coefficients are of rapid decay modulo the center, then $\tau = 0$.

Proof. Follows immediately from Theorems 10.3.11 and Theorem 10.3.13, together with the fact that, in the real case, if the asymptotic expansion at infinity is zero, then the function is of rapid decay (modulo center). □

Definition 10.4.3. Let H be a p-adic reductive group. An irreducible admissible representation τ of H is called *supercuspidal* if its matrix coefficients are compactly supported modulo the center.

10.4.4. The subrepresentation theorem.

Theorem 10.4.5. *Any irreducible admissible representation τ of a real reductive group H, is infinitesimally equivalent to a submodule of an irreducible representation induced from a minimal parabolic; that is, there exists an irreducible (finite-dimensional, necessarily) representation σ of the Levi quotient L of the minimal parabolic subgroup P of H, and an embedding of $(\mathfrak{g}, K)$-modules $\tau_{K\text{-fin}} \hookrightarrow I_P(\sigma)_{K\text{-fin}}$, where $I_P(\sigma) = Ind_P^H(\sigma\delta_P^{\frac{1}{2}})$ is the (normalized) induced representation.*

Proof. This relies on the statement of Theorem 10.3.13, that the asymptotics of matrix coefficients in any direction have to be nontrivial. In particular, for the minimal direction we have a non-zero map, which by irreducibility has to be an embedding, $\tau \otimes \tilde{\tau} \to C^\infty(H_P) = I_{P\times P^-}C^\infty(L)$, whose image consists of A_P-finite functions. By projecting to an A_P-eigenquotient of the image, we may assume that the image is in an eigenspace, with respect to some character χ of A_P. Notice that L/A_P is compact; hence, the space $C^\infty(L/A_P, \chi)$ has a dense subspace of L-finite vectors, which are spanned by matrix coefficients of irreducible representations. Thus, restricting to K-finite vectors, there is an morphism (necessarily an embedding) of $(\mathfrak{g}, K)$-modules $(\tau\otimes\tilde{\tau})_{K\times K\text{-fin}} \hookrightarrow I_{P\times P^-}(\sigma\otimes\tilde{\sigma})_{K\times K\text{-fin}} = I_P(\sigma)_{K\text{-fin}}\otimes I_{P^-}(\tilde{\sigma})_{K\text{-fin}}$, for some irreducible representation σ of L, and by fixing a vector in $\tilde{\tau}_{K\text{-fin}}$, we get the embedding claimed in the theorem. □

10.5. The Langlands classification

Definition 10.5.1. Let G be a reductive real or p-adic group, let $P \to L$ be a parabolic subgroup with its Levi quotient, and let $\nu : L \to \mathbb{C}^\times$ be a character. We will say that ν is *P-dominant* if $\log(\nu) \in \mathfrak{a}_P^{*,+}$, and *strictly P-dominant* if $\log(\nu) \in \mathfrak{a}_P^{*,+}$. Here, $\mathfrak{a}_P^* = \mathrm{Hom}(L, \mathbb{G}_m) \otimes \mathbb{R}$, log is the map that sends the absolute value of an algebraic character to its image in $\mathfrak{a}_P^*$, and $\mathfrak{a}_P^{*,+}$, $\mathfrak{a}_P^{*,+}$ are those characters which are non-negative (resp. strictly positive) on coroots $\check{\alpha}$ corresponding to roots in the unipotent radical of P, i.e., $|\nu(e^{\check{\alpha}}(x))| = |x|^\epsilon$ for some $\epsilon \geq 0$ (resp. $\epsilon > 0$).

Equivalently, $\mathfrak{a}_P^*$ is identified with a subspace of $\mathfrak{a}^*$ (spanned by the F-rational characters of the universal Cartan), and $\mathfrak{a}_P^{*,+}$ (resp. $\mathfrak{a}_P^{*,+}$) is just the corresponding wall (resp., relative interior of the wall) of the dominant Weyl chamber.

Theorem 10.5.2 (The Langlands quotient theorem)**.** *Let G be a reductive real or p-adic group, let $P \to L$ be a parabolic subgroup with its Levi quotient, and let τ be an irreducible tempered representation of L. For any character $\nu : L \to \mathbb{C}^\times$ which is strictly P-dominant, the (normalized) induced representation $I_P^G(\tau\nu)$ has a unique irreducible quotient $\pi_{P,\tau\nu}$, and every irreducible representation of G is of this form, for a unique (up to conjugacy) pair $(P, \tau\nu)$. Moreover, $\pi_{P,\tau\nu}$ is the image of the standard intertwining operator $M_{P^-|P}(\tau\nu) : I_P(\tau\nu) \to I_{P^-}(\tau\nu)$.*

Proof. [Later] $\square$

Example 10.5.3. The trivial representation, for a quasisplit group, is equal to $\pi_{B,\delta^{\frac{1}{2}}}$, where B is a Borel subgroup, and δ is its modular character.

Remark 10.5.4. The Langlands quotient theorem reduces the classification of irreducible representations to the case of irreducible tempered representations, offering an invaluable link between the "smooth" and the "L^2 theory/Plancherel formula" of irreducible representations. It is also supposed to be compatible with the parametrization provided by the local Langlands conjecture: If $\phi_\tau : \Gamma \to {}^L L$ and $\phi_\nu : \Gamma \to {}^L L$ are Langlands parameters for τ and ν (where Γ, here, denotes the appropriate version of the Weil, or Weil–Deligne group), then $\phi_\tau \cdot \phi_\nu : \Gamma \to {}^L L \hookrightarrow {}^L G$ is a Langlands parameter for $\pi_{P,\tau\nu}$. (Notice that ϕ_τ and ϕ_ν commute, because ν is a character.)

For example, the Langlands parameter (or rather, its projection to $\check{G}$) of the trivial representation of a quasisplit group is given by $\Gamma \to \mathbb{C}^\times \to \check{G}$, where $\Gamma \to \mathbb{C}^\times$ is the "cyclotomic"/absolute value character, and $\mathbb{C}^\times \to \check{G}$ is given by $e^{2\rho} : \mathbb{G}_m \to \check{A} \subset \check{G}$ (where $\check{A}$ is the dual of the universal Cartan).

10.6. The Satake isomorphism

Definition 10.6.1. A reductive group over a local non-Archimedean field F is said to be *unramified* if it is quasisplit, and splits over an unramified extension.

Proposition 10.6.2. *For a connected reductive group G over F, the following are equivalent:*

1. *G is unramified (Definition 10.6.1);*
2. *G admits a reductive model of the ring of integers $\mathfrak{o}$ (i.e., a smooth model with connected reductive geometric fibers).*

Moreover, the integral model over $\mathfrak{o}$ is unique up to $G_{ad}(F)$-conjugacy, that is, for any two reductive $\mathfrak{o}$-groups $\mathcal{G}_1$, $\mathcal{G}_2$ with general fiber identified with G, there is an isomorphism $\mathcal{G}_1 \simeq \mathcal{G}_2$ that restricts to an inner automorphism (over F) on G.

[This proposition should be moved to the chapter on algebraic groups.]

Proof. For the direction from the second to the first, see [**Con14**, Corollary 5.2.14]. The opposite direction follows from the classification in terms of root data with Galois actions. [To be added.] For the uniqueness, see [**Con14**, Theorem 7.2.16]. □

Definition 10.6.3. A hyperspecial subgroup of $G(F)$, where G is an unramified connected reductive group over F, is a subgroup of the form $K = \mathcal{G}(\mathfrak{o})$, where $\mathcal{G}$ is a reductive integral model.

Hyperspecial subgroups are unique, up to conjugacy, for adjoint groups, as follows from the uniqueness statement of Proposition 10.6.2. This does not need to be true when $G(F)$ does not surject onto $G_{\mathrm{ad}}(F)$.

Proposition 10.6.4. *A hyperspecial subgroup (Definition 10.6.3) is maximal.*

Proof. [Omitted for now.] □

From now on, G will denote $G(F)$. Fix a hyperspecial subgroup $K = \mathcal{G}(\mathfrak{o})$, corresponding to an $\mathfrak{o}$-model $\mathcal{G}$, and consider the *integral* unramified ("spherical") Hecke algebra $\mathcal{H}(G, K)$ of $\mathbb{Z}$-valued, K-biinvariant functions on G. If we consider them as functions on the discrete space G/K, using the counting measure on this space we can identify them as measures, and this defines their convolution and, more generally, their action on the K-invariant vectors of any representation V (with arbitrary coefficients!). Explicitly, if $v \in V^K$, the characteristic function of a double coset KgK acts as

$$1_{KgK} \cdot v = \sum_{\gamma \in [KgK/K]} \gamma \cdot v.$$

The goal of this section is to establish the integral Satake isomorphism. For this purpose, let $\mathcal{Y}$ be "the" full pre-flag variety of $\mathcal{G}$ over $\mathfrak{o}$, $\mathcal{Y} \simeq \mathcal{N}\backslash\mathcal{G}$, where $\mathcal{N}$ is the unipotent radical of a Borel subgroup. We do not really choose a Borel subgroup, but the choice of integral model matters, as it endows $\mathcal{Y}$ with a distinguished K-orbit, equal to $\mathcal{Y}(\mathfrak{o})$, that will serve as our base point. As for the group, will use Y etc. to denote F-points. We let $T \supset T_0$ denote the universal Cartan $T = B/N$, and its maximal compact subgroup $T_0 = \mathcal{T}(\mathfrak{o})$; we reserve the letter A for the maximal split torus in T.

We let $\mathcal{S}(Y/K)$ denote the space of $\mathbb{Z}$*-valued*, compactly supported, K-invariant functions on $\mathcal{S}$. It is a module for $\mathcal{H}(G, K)$ (under the right action of G on Y) and for T (under the "left" action of T on Y). To be clear, the action of an element $t \in T$ on functions is defined as translation by t, not t^{-1}, and it is not normalized by any modular character — which is not defined over $\mathbb{Z}$: $(t \cdot f)(y) = f(ty)$; this way, the center of G acts the same, whether it is considered as a subgroup of G or of T.

Lemma 10.6.5. *Every element of $\mathcal{S}(Y/K)$ is T_0-invariant, hence the action of T factors through the quotient T/T_0; in particular, we have an action of the Hecke* *algebra $\mathcal{H}(T, T_0)$. Under this action, $\mathcal{S}(Y/K)$ is a free module of rank one, and the element $e_0 = 1_{\mathcal{Y}(\mathfrak{o})}$ is a generator.*

Proof. The Iwasawa decomposition [needs to be added] $G = NTK$ shows that $G = \bigsqcup_{t \in T/T_0} NtK$, and in particular:

- every $N\backslash G/K$-coset is left invariant by T_0;
- the group T/T_0 acts simply transitively on the cosets.

One of these cosets is equal (modulo N) to $\mathcal{Y}(\mathfrak{o})$. □

Let $\Lambda = T/T_0$. We start with the Satake isomorphism for tori:

Proposition 10.6.6. *Let T be an unramified torus over F, and let $A \subset T$ be the maximal split subtorus. If $\varpi \in F$ is a uniformizer, the map $\Lambda := X_*(A) \ni \lambda \mapsto \lambda(\varpi) \in A(F)$ descends to an isomorphism $X_*(A) \simeq T/T_0$.*

Moreover, let $\check{T}$ be the dual torus to T, understood as a group scheme over $\mathbb{Z}$, with an action of the unramified Galois group $\Gamma = \langle\sigma\rangle$, where σ denotes the Frobenius element. Then, the dual $\check{A}$ of A is the maximal torus quotient of $\check{T}$ where Γ acts trivially, and the natural maps induce isomorphisms of algebras

$$\mathbb{Z}[\check{T}\sigma]^{\check{T}} = \mathbb{Z}[\check{A}] = \mathbb{Z}[\Lambda] = \mathcal{H}(T, T_0), \tag{10.6.6.1}$$

where the "coset" $\check{T}\sigma$ is the space $\check{T}$ equipped with the σ-twisted conjugation of $\check{T}$, $x\sigma \cdot t = (t \cdot {}^\sigma t^{-1})x\sigma$ (and the notation $\mathbb{Z}[\cdot]$ is used both for group rings and coordinate rings — it should be clear which is which).

Proof. Consider the quotient of algebraic group schemes over $\mathfrak{o}$:

$$1 \to \mathcal{A} \to \mathcal{T} \to \mathcal{T}/\mathcal{A} \to 1.$$

Since A is split, this induces (by Hilbert 90) surjections at the level of F- and $\mathbb{F}_q$-points, hence (by smoothness) at the level of $\mathfrak{o}$-points. On the other hand, T/A is anisotropic, and $T/A(F) = \mathcal{T}/\mathcal{A}(\mathfrak{o})$. Therefore, $T(F) = A(F)\mathcal{T}(\mathfrak{o})$, and since $A(F) \cap \mathcal{T}(\mathfrak{o}) = A(\mathfrak{o})$, and $A(F)/A(\mathfrak{o}) = \Lambda$, this shows the bijection $\Lambda \xrightarrow{\sim} T/T_0$.

For the coordinate rings, if Λ' is the $\bar{F}$-cocharacter group of T, then the torus A is spanned by the images of Galois-stable cocharacters, hence $\Lambda = (\Lambda')^\Gamma$. On the dual side, the embedding $\Lambda \hookrightarrow \Lambda'$ induces a morphism of dual tori $\check{T} \to \check{A}$, which identifies $\check{A}$ with the quotient of $\check{T}$ by the subtorus of all elements of the form $(t \cdot {}^\sigma t^{-1})$, $t \in \check{T}$, hence $\mathbb{Z}[\check{T}\sigma]^{\check{T}} = \mathbb{Z}[\check{A}] = \mathbb{Z}[\Lambda]$. □

Now denote by t_λ a representative for $\lambda \in \Lambda = T/T_0$ in T, and let $\delta_\lambda = 1_{Nt_\lambda K} = t_\lambda^{-1} \cdot 1_{\mathcal{Y}(\mathfrak{o})}$; thus, the elements δ_λ form a basis for $\mathcal{S}(Y/K)$. By Lemma 10.6.5, the action map $\mathcal{H}(T, T_0) \ni h \mapsto h \cdot 1_{\mathcal{Y}(\mathfrak{o})}$ identifies the spaces $\mathcal{S}(Y/K)$ and $\mathcal{H}(T, T_0) \simeq \mathbb{Z}[\Lambda]$; notice, however, that the characteristic function $t_\lambda T_0$ in $\mathcal{H}(T, T_0)$ corresponds to $\delta_{-\lambda}$.

Theorem 10.6.7 (Satake isomorphism)**.** *Let $S : \mathcal{H}(G, K) \to \mathcal{H}(T, T_0)$ be the map given by the action map $\mathcal{H}(G, K) \ni h \mapsto h \cdot 1_{\mathcal{Y}(\mathfrak{o})} \in \mathcal{S}(Y/K)$ and the identification of $\mathcal{S}(Y/K)$ with $\mathcal{H}(T, T_0)$ (again through the analogous action map), i.e., $S(h)$ is that element of $\mathcal{H}(T, T_0)$ such that $h \cdot 1_{\mathcal{Y}(\mathfrak{o})} = S(h) \cdot 1_{\mathcal{Y}(\mathfrak{o})}$. Then S gives rise to an isomorphism of algebras:*

$$\mathcal{H}(G, K) \simeq \mathcal{H}(T, T_0)_{\mathbb{Z}} \cap \mathcal{H}(T, T_0)_{\mathbb{Q}}^{W,\bullet} = \mathbb{Z}[\Lambda] \cap \mathbb{Q}[\Lambda]^{W,\bullet}, \tag{10.6.7.1}$$

where the $\bullet$-action of the (relative) Weyl group W is defined by $(w \bullet f)(t_\lambda) = q^{\langle\lambda, \rho - w\rho\rangle} f(t_{w^{-1}\lambda})$ (where q is the degree of the residue field).

Explicitly, the element $h \in \mathcal{H}(G,K)$, is mapped to the element of $\mathcal{H}(T,T_0)$ whose value at t is equal to

$$h \cdot 1_{\mathcal{Y}(\mathfrak{o})}(Nt^{-1}) = \int_G h(g) 1_{NK}(Nt^{-1}g)dg = \int_N h(tn)dn, \tag{10.6.7.2}$$

where the Haar measure on N gives volume 1 to $\mathcal{N}(\mathfrak{o})$. The way that the Satake isomorphism is usually defined in the literature is through the equation (10.6.7.2), multiplied by $\delta_B(t)^{\frac{1}{2}}$ (where δ_B is the modular character of the Borel subgroup), in order to replace the $\bullet$-action of W by the usual action of W — but this modification is not defined over $\mathbb{Z}$.

Notice also that, the characteristic function of $t_\lambda T_0$ in $\mathcal{H}(T,T_0)$ corresponds to the element $\delta_{-\lambda}$ of $\mathcal{S}(Y/K)$, image of the action map will be invariant under the following Weyl group action on $\mathcal{S}(Y/K)_{\mathbb{Q}}$:

$$w \bullet \delta_\lambda = q^{\langle \lambda, \rho - w^{-1}\rho\rangle} \delta_{w\lambda}. \tag{10.6.7.3}$$

Proof. First of all, we notice that the map S is a homomorphism of algebras, because the actions of G and T on Y commute, and $\mathcal{H}(T,T_0)$ is abelian: if $h_i \in \mathcal{H}(G,K)$ and $h_i' \in \mathcal{H}(T,T_0)$ are such that $h_i \cdot 1_{\mathcal{Y}(\mathfrak{o})} = h_i' \cdot 1_{\mathcal{Y}(\mathfrak{o})}$, then $(h_1h_2)\cdot 1_{\mathcal{Y}(\mathfrak{o})} = h_1 \cdot (h_2 \cdot 1_{\mathcal{Y}(\mathfrak{o})}) = h_1(h_2' \cdot 1_{\mathcal{Y}(\mathfrak{o})}) = h_2'(h_1 \cdot 1_{\mathcal{Y}(\mathfrak{o})}) = (h_2'h_1')\cdot 1_{\mathcal{Y}(\mathfrak{o})} = (h_1'h_2')\cdot 1_{\mathcal{Y}(\mathfrak{o})}$.

Next, we claim that the image lies in $\mathbb{Q}[\Lambda]^{W,\bullet}$. We will present two proofs for that, one of them only for the split case [but it can be generalized — to do].

The first proof, following [**Car79**], uses the explicit expression (10.6.7.2) of the Satake transform as an integral, and replaces it by an orbital integral for the action of G on itself by conjugacy. Namely, choose a lift of the quotient map $B \to T$, thus identifying T as a subtorus of B. If $t \in T$ is a *regular* element (i.e., has trivial stabilizer under the action of the Weyl group), then we have the following formula:

$$\int_N h(tn)dn = |\det(\mathrm{Ad}_{\mathfrak{n}}(t^{-1}) - 1)| \int_N f(n^{-1}tn)dn = |\det(\mathrm{Ad}_{\mathfrak{n}}(t^{-1}) - 1)| \int_{T\backslash G} f(g^{-1}tg)dg,$$

where $\mathrm{Ad}_{\mathfrak{n}}$ denotes the left adjoint action of T on the Lie algebra $\mathfrak{n}$, and the invariant measure on $T\backslash G$ is normalized so that the total measure of K-orbits represented by $T\backslash TN(\mathfrak{o})$ is 1. This formula follows easily by considering the map $N \to tN$ given by $n \mapsto n^{-1}tn$, and representing the measures as absolute values of volume forms; at the last step, one uses the K-invariance of H to represent $T\backslash G$ by NK.

Hence, for $w \in W$ and t a *regular* element in T (any class in T/T_0 has such representatives),

$$\frac{\int_N h(n^w t)dn}{\int_N h(nt)dn} = \frac{|\det(\mathrm{Ad}_{\mathfrak{n}}({}^w t^{-1}) - 1)|}{|\det(\mathrm{Ad}_{\mathfrak{n}}(t^{-1}) - 1)|} = \left| \prod_{\alpha>0} \frac{1 - e^{-w^{-1}\alpha}(t)}{1 - e^{-\alpha}(t)} \right|$$
$$= \left| \prod_{\alpha>0, w\alpha<0} \frac{1 - e^{\alpha}(t)}{1 - e^{-\alpha}(t)} \right| = \left| \prod_{\alpha>0, w\alpha<0} e^{\alpha}(t) \right| = |e^{\rho - w\rho}(t)|,$$

which amounts to the stated invariance property. (We have not assumed G to be split, for this calculation: the terms inside the absolute values are algebraic functions, and therefore it is valid to manipulate them over the algebraic closure — where all roots are defined.)

[Another proof with Fourier transforms to be added.]

Finally, we prove that the map $\mathcal{H}(G,K) \to \mathbb{Z}[\Lambda] \cap \mathbb{Q}[\Lambda]^{W,\bullet}$ is an isomorphism. We will argue by identifying the space on the right (call it M) as a subspace of

$\mathcal{S}(Y/K)$ — the $\bullet$-action of W is given by (10.6.7.3). Notice that M is a free $\mathbb{Z}$-module with generators m_λ, indexed by dominant cocharacters λ into A, given by $m_\lambda = \sum_{\lambda'\sim\lambda} q_{\lambda'}\delta_{\lambda'}$, where $\lambda' \sim \lambda$ means that $\lambda' = w\lambda$ for some $w \in W$, and in this case we set $q_{\lambda'} = q^{\langle\lambda,\rho-w^{-1}\rho\rangle}$. We define a filtration of this module with respect to the partial ordering by coroots, $\mu \geq \lambda$ if $\mu-\lambda$ is a sum of positive coroots. Similarly, $\mathcal{H}(G,K)$ has a basis consisting of the characteristic functions of the cosets $Kt_{-\lambda}K$ (with λ dominant, again), and we use it to define a filtration of $\mathcal{H}(G,K)$ indexed by dominant weights. We claim that the map $\mathcal{H}(G,K) \to M$ respects these filtrations:

$$F^\lambda\mathcal{H}(G,K) \to F^\lambda M,$$

and that the generator of the λ-th graded piece, represented by the function $1_{Kt_\lambda K}$, maps to the generator of the λ-th graded piece, represented by m_λ. These statements follow from the following fundamental fact:

For λ dominant, we have

$$Kt_\lambda K \subset \bigcup_{\mu\leq\lambda} Nt_\mu \cdot K, \tag{10.6.7.4}$$

and $Kt_\lambda K \cap Nt_\lambda K = \mathcal{N}(\mathfrak{o})t_\lambda K$.

[The proof of this will be added together with the proof of the Cartan and Iwasawa decompositions.]

This implies that $1_{Kt_{-\lambda}K}\cdot\delta_0 = \delta_\lambda + \sum_{\mu<\lambda} c_{\mu,\lambda}\delta_\mu$ for some coefficients $c_{\mu,\lambda} \in \mathbb{N}$. We leave it to the reader to check that this is equivalent to the claim.

□

10.7. Langlands parameters

Let G be a (connected) reductive group over a local field F. We will write G for $G(F)$. The L-group and the C-group of G have been defined in Section **??**. We denote by Γ_F the Galois group of F (of a fixed separable extension[3]), and by $\mathcal{W}_F$ its Weil group. For definitions, see [**Tat79**]. We only remind here that the Weil group comes with isomorphisms $\mathcal{W}_F/\mathcal{W}_E = \Gamma_F/\Gamma_E = \mathrm{Hom}(E,F^s)$ for every separable extension E of F, and $\mathcal{W}_F^{ab} \xrightarrow{\sim} F^\times$, compatible with the isomorphism $\Gamma_F \xrightarrow{\sim} \widehat{F^\times}$ (profinite completion) of class field theory. As in [**Tat79**], we will normalize the isomorphism of class field theory so that a Frobenius element maps to the inverse of a uniformizer, i.e., a *geometric Frobenius* element maps to a uniformizer. In particular, we have a norm map $|\bullet| : \mathcal{W}_F \to F^\times \to \mathbb{R}_+^\times$, sending a Frobenius element to q: the degree of the residue field.

We also remind of the modification of the Weil group that is needed in order to pass from l-adic to complex representations:

Definition 10.7.1. Let F be a non-Archimedean field. The *Weil–Deligne group* $\mathcal{W}_F'$ is the semidirect product $\mathcal{W}_F \ltimes \mathbb{G}_a$, with $wxw^{-1} = |w|x$ for $w \in \mathcal{W}_F$ and $x \in \mathbb{G}_a$. A *representation of the Weil–Deligne group* over a field E of characteristic zero is a pair (ρ, N) consisting of a representation of $\mathcal{W}_F$ *with open kernel* on a finite-dimensional vector space V over E, and a nilpotent endomorphism N of V, satisfying $\rho(w)N\rho(w)^{-1} = |w|\dot{N}$.

[3]See Remark 8.4.3: it is better not to fix a separable extension, and to translate these definitions to sheaves over the étale site of F.

The "open kernel" condition is the important one here; it makes irrelevant the topology of $\mathrm{GL}_E(V)$. The Weil–Deligne group is a convenient way to de-topologize the l-adic representations of the Weil group that show up "in nature" (in étale cohomology), and translate them among different l's, or to the complex numbers:

Proposition 10.7.2. *Let l be a prime different from the residual characteristic p of (a non-Archimedean field) F, and let E be a finite extension of $\mathbb{Q}_l$. There is a canonical bijection between isomorphism classes of (continuous) finite-dimensional E-representations $\phi : \mathcal{W}_F \to GL(V)$ and representations (ρ, N) of the Weil–Deligne group over E (Definition 10.7.1), characterized by the property that*

$$\phi(\Phi\sigma) = \rho(\Phi\sigma)\exp(t_l(\sigma)N), \tag{10.7.2.1}$$

for some Frobenius element $\Phi \in \mathcal{W}_F$, any element σ of the inertia subgroup, and t_l a choice of isomorphism of the pro-l-quotient of (tame) inertia with $\mathbb{Z}_l$.

Recall that the tame inertia quotient is generated by n-th roots of a uniformizer, for $(n,p) = 1$, and is isomorphic (up to a choice of topological generator) to $\widehat{\mathbb{Z}}^p = \prod_{l\neq p} \mathbb{Z}_l$.

Proof. See [**Tat79**, §4.2] for references. □

Definition 10.7.3. A *Langlands parameter* into the L-group of G is a morphism $\mathcal{W}'_F \to {}^LG$ over Γ.

The local Langlands conjecture posits the existence of a canonical finite-to-one map:

$$\{\text{irreducible admissible representations of } G\}/\sim \to \{\text{Langlands parameters into } {}^LG\}/\sim .$$

10.8. Other chapters

CHAPTER 11

Plancherel formula: reduction to discrete spectra

We need at least one reference [**Som**] in each chapter.

11.1. Other chapters

CHAPTER 12

Construction of discrete series

We need at least one reference [**Som**] in each chapter.

12.1. Other chapters

CHAPTER 13

The automorphic space

13.1. The automorphic quotient, and basic examples

Automorphic representations are the representations that appear when we perform harmonic analysis on the homogeneous space $G(k)\backslash G(\mathbb{A})$, where k is a global field and $\mathbb{A}$ is its ring of adeles.

Let G be a linear algebraic group over k. Since G is affine, the subgroup $G(k)$ of $G(\mathbb{A})$ is discrete and the space $[G] := G(k)\backslash G(\mathbb{A})$ is a locally compact space, homogeneous under the action of $G(\mathbb{A})$. It carries an invariant measure under $G(\mathbb{A})$.

Definition 13.1.1. If G is a linear algebraic group over k, the space $G(k)\backslash G(\mathbb{A})$ is called the *automorphic space* of G, and denoted by $[G]$.

This term is not completely standard, but there is no other name for it. Here we study properties of this space, discuss the adelic and the classical picture, and some relevant arithmetic issues. We fix throughout a global field k (either a number field, or the function field of a curve over a finite field), and all groups are linear algebraic groups defined over k. The letters S, Σ will always denote finite sets of places of k, $\mathbb{A}^S$ will denote the adeles outside of S, i.e. the restricted product $\prod'_{v\notin S} k_v$, and $\mathbb{A}_S$ will denote the product $\prod_{v\in S} k_v$. For a variety X over S we will denote: $X_k := X(k)$, $X_{\mathbb{A}} := X(\mathbb{A})$, $X^S := X(\mathbb{A}^S)$ and $X_S := X(\mathbb{A}_S)$. The (finite) set of archimedean places will be denoted by ∞. If k is a function field, we pick a place that we denote by ∞. We let $\mathfrak{o}$ be the ring of integers of k, if k is a number field, and the ring of integers away from the chosen place ∞, if k is a function field. We let $\mathbb{A}_f = \mathbb{A}^\infty$, the ring of finite adeles, when k is a number field, and the ring of adeles away from ∞, when k is a function field.

13.1.2. The additive group.

Proposition 13.1.3. *Let $G = \mathbb{G}_a$. The automorphic space $[G]$ is compact, and for any non-empty set S of places of k, the embedding $k \hookrightarrow \mathbb{A}^S$ is dense. In particular, for every open compact $K \subset \mathbb{A}_f$, the group k_∞ acts with a unique orbit on the quotient space $[G]/K$, which is isomorphic to $k_\infty/\mathfrak{o}_K$ as a k_∞-space, for a subgroup $\mathfrak{o}_K$ of k_∞ that is finitely generated over the integers of the base field (i.e., over $\mathbb{Z}$ or $\mathbb{F}_q[t]$).*

Proof. By restriction of scalars, $\mathrm{Res}_{k/\mathbb{Q}}\mathbb{G}_a = \mathbb{G}_a^{(k:\mathbb{Q})}$, the problem reduces to the base field $k = \mathbb{Q}$ or $k = \mathbb{F}_q(t)$. We present only the case of $k = \mathbb{Q}$, $S = \{\infty\}$, leaving the general case as an exercise to the reader. In this case, we have $\mathbb{A}^S = \mathbb{A}_f = \widehat{\mathbb{Z}}\otimes\mathbb{Q}$, and the density statement follows from the density of $\mathbb{Z}$ in its profinite completion.

The stabilizer for the action of k_∞ on $[G]/K$ is the intersection $k\cap K k_\infty$. This is a submodule for $\mathbb{Z}$ or $\mathbb{F}_q[t]$, because every open compact subgroup K is (exercise!).

It is finitely generated, because this is the case when $K = N \cdot \prod_v \mathfrak{o}_v$ for some $N \in k^\times$, and any compact open K is contained in such a subgroup.

In the particular case $K = \prod_v \mathfrak{o}_v$, we obtain the quotient $k_\infty/\mathfrak{o}$, which is compact, hence $[G]$ is compact. □

13.1.4. The multiplicative group.

Proposition 13.1.5. *Let $G = \mathbb{G}_m$. The product of absolute values defines a homomorphism $[G] \to \mathbb{R}_+^\times$, whose kernel is compact. If $K \subset \mathbb{A}_f^\times$ is the maximal compact subgroup (the product of local units), then the $k_\infty^\times$-orbits on $[G]/K$ are canonically parametrized (under the natural homomorphism from ideles to fractional ideals, sending a uniformizer at a finite place to the corresponding ideal) by the class group of k, and each is isomorphic to $k_\infty^\times/\mathfrak{o}^\times$. The orbits of the identity component $(k_\infty^\times)^0$ are parametrized by the narrow class group of k.*

More generally, if $K = \prod_{v \notin S} \mathfrak{o}_v^\times \prod_{v \in S}(1 + \mathfrak{p}_v^{r_v})$, where S is a finite set of finite primes, the $k_\infty^\times$-orbits on $[G]/K$ are canonically parametrized by the ray class group of modulus $\mathfrak{m} = \prod_{v \in S} \mathfrak{p}_v^{r_v}$, and the $(k_\infty^\times)^0$-orbits by the corresponding narrow ray class group.

Proof. For an idele $a = (a_v)_v$, let $\lambda(a)$ be the number of $\alpha \in k$ with $|\alpha|_v \le |a_v|$ for all v. Then, there are constants c_1, c_2, depending only on k, such that

$$c_1 \le \frac{\lambda(a)}{|a|} \le c_2$$

for any a. Indeed, by restriction of scalars for $\mathbb{G}_a$, the problem reduces to the base field $\mathbb{Q}$ or $\mathbb{F}_q(t)$, where it is straightforward.

In particular, for $|a| > c_1^{-1}$, there will be an $\alpha \in k^\times$ with $|\alpha a_v|_v \ge 1$ for all v. On the other hand, $|\alpha a_v|_v = \frac{\prod_w |\alpha a_w|_w}{\prod_{w \ne v} |\alpha a_w|_w} \le |a|$.

Let $r = |a| > c_1^{-1}$, and let $[G]^r$ be the set of idele classes of norm r. We conclude that $[G]^r$ is contained in the image of the set

$$U = \prod_v k_v^{[1,r]},$$

where $k_v^{[1,r]}$ denotes the elements of k_v of absolute value in the interval $[1, r]$. For all but a finite number of v's, this is the same as $k_v^{[1,1]} = \mathfrak{o}_v^\times$, since the valuation is discrete and the residual degree is $> r$ (for almost all v).

Thus, the set U is compact in $G(\mathbb{A})$, and so is $[G]^r$. But $[G]^r$ is a torsor for $[G]^1$, therefore $[G]^1$ is compact.

The rest are left to the reader. □

Notice that this implies the Dirichlet unit theorem:

Proposition 13.1.6. *If F is a number field, with r_1 real places and r_2 complex places, then the unit group $\mathfrak{o}^\times$ is a finitely generated abelian group of rank r_1+r_2-1.*

Proof. If $K = \prod_{v<\infty} \mathfrak{o}_v^\times$, then $[\mathbb{G}_m] = k_\infty^\times/\mathfrak{o}^\times$, and the logarithm of the Archimedean absolute values define a surjection $k_\infty^\times \to \mathbb{R}^{r_1+r_2}$ with compact kernel. This kernel, intersected with the discrete subgroup $\mathfrak{o}^\times$, is finite (the torsion subgroup of $\mathfrak{o}^\times$), while the image of $\mathfrak{o}^\times$ will be a discrete subgroup, and by Proposition 13.1.5 it will be cocompact inside the kernel of $\mathbb{R}^{r_1+r_2} \xrightarrow{\sum} \mathbb{R}$. Therefore, it is a finitely generated abelian group of rank $r_1 + r_2 - 1$. □

13.1.7. Tori.

Proposition 13.1.8. *Let T be a torus over k. Let $X_k^*(T)$ be the k-character group of T, and $\mathfrak{a} = \mathrm{Hom}(X_k^*(T), \mathbb{R})$. The map $X_k^*(T) \times [T] \ni (\chi, t) \mapsto \log(|\chi(t)|)$ defines a homomorphism $\log_T : [T] \to \mathfrak{a}$ with compact kernel and cokernel. In particular, $[T]$ is compact if and only if T is anisotropic.*

Proof. The only nontrivial statement is that the kernel of $\log_T$ is compact. Let $[T]^1$ be this kernel (and similarly for any torus).

First, reduce to the case when T is anisotropic: if T_1 is the kernel of all morphisms to $\mathbb{G}_m$, the quotient T/T_1 is isomorphic to $\mathbb{G}_m^r$ for some r, and we have a map $[T]^1 \to [T/T_1]^1$ with kernel $[T_1]$. The case of $\mathbb{G}_m$ has already been treated in Proposition 13.1.5, so we are reduced to the case $T = T_1$, i.e., T is anisotropic.

Assume this to be the case. By Lemma 7.1.4, there is an induced torus S, together with a surjection $S \twoheadrightarrow T$. If S_1 denotes the common kernel of all characters of S, since T is anisotropic, we have a surjection $S_1 \twoheadrightarrow T$. By Proposition 13.1.5, $[S_1]$ is compact. On the other hand, the image of the map $[S_1] \to [T]$ will have finite index modulo any compact open subgroup of $T(\mathbb{A})$ (exercise!). Therefore, $[T]$ is compact. $\square$

Remark 13.1.9. Generalizing the Dirichlet unit theorem 13.1.6, for a torus T over $\mathbb{Q}$, the group $T(\mathbb{Z})$ is a finitely generated abelian group of rank equal to $\mathrm{splr}_{\mathbb{R}}(T) - \mathrm{splr}_{\mathbb{Q}}(T)$, where splr denotes the split rank (the rank of the character group) of the torus over the indicated field.

13.2. Parabolic automorphic spaces

When studying the geometry and harmonic analysis of the space $[G]$, a very important role is played by certain related $G(\mathbb{A})$-homogeneous spaces, that we will call *parabolic automorphic spaces*, or *boundary degenerations*. The last term is not standard, but is borrowed from [**SV17**], where it was used in a local setting, and it is a useful concept that unifies the ideas of harmonic analysis globally and locally.

Let $\mathcal{P}$ denote a (conjugacy) class of parabolics in G; it can be understood as a homogeneous space of G, endowed with the tautological sub-group scheme $\mathbb{P}$ of the constant scheme $\mathcal{P} \times G$, where the fiber of $\mathbb{P}$ over a parabolic P is P itself. It admits a canonical quotient $\mathbb{L}$, where the fiber over P is the Levi quotient of P, and a further canonical quotient $\mathbb{L}^{ab}$, where the fiber is the abelianization of the Levi quotient. Notice that the action of P is trivial on its fiber in $\mathbb{L}^{ab}$, but not on its fiber in $\mathbb{L}$, unless $\mathcal{P} = \mathcal{B}$, the class of Borel subgroups, where the group scheme $\mathbb{L}$ is the constant "universal Cartan" group scheme (Definition 7.5.5), $\mathbb{L} = \mathbf{A}^G \times \mathcal{B}$.

Definition 13.2.1. Let G be a reductive group over a field k, let $\mathcal{P}$ denote a (conjugacy) class of parabolics in G, and let $\mathbb{L}$ be the group scheme of Levi quotients over $\mathcal{P}$. A *pre-flag variety* [can someone suggest a better term?] associated to $\mathcal{P}$ is a G-equivariant $\mathbb{L}$-torsor R over $\mathcal{P}$, where any $P \in \mathcal{P}$ acts on its fiber R_P through its Levi quotient $P \to \mathbb{L}_P$. A *degenerate pre-flag variety* is a G-equivariant $\mathbb{L}^{ab}$-torsor R over $\mathcal{P}$, where any $P \in \mathcal{P}$ acts on its fiber R_P through its abelianized Levi quotient $P \to \mathbb{L}_P^{ab}$.

Remark 13.2.2. In this section, we will always take a (degenerate) pre-flag variety to have points over the field of definition, unless otherwise stated. In this case is isomorphic to $U_P\backslash G$ (resp. $[P,P]\backslash G$) — but the definition is formulated in a way to avoid having a chosen base point. For example, if $\mathcal{P} = \{G\}$, then the pre-flag variety is simply a G-torsor.

Clearly, the isomorphism class of the pre-flag variety (which has a point) depends only on the conjugacy class $\mathcal{P}$, but we will often abuse language, pick a parabolic $P \in \mathcal{P}$, and say that $U_P\backslash G$ is "the" pre-flag variety associated to P — but without a fixed point, unless otherwise stated.

Definition 13.2.3. Let G be a reductive group over a global field k, and $\mathcal{P}$ a class of parabolics. Fix a pre-flag variety Y associated to $\mathcal{P}$. The *parabolic automorphic space* or *boundary degeneration* $[G]_{\mathcal{P}}$ of the automorphic space $[G]$ associated to these data is the set of pairs $(y \in M)$, where M is a $G(\mathbb{A})$-translate of $Y(k)$ in $Y(\mathbb{A})$, and $y \in M$, modulo the action of $\mathrm{Aut}^G(Y)(k)$.

Equivalently, fixing a base point $P \in \mathcal{P}(k)$ with unipotent radical U and Levi quotient L, the boundary degeneration $[G]_{\mathcal{P}}$, which by abuse of notation will also be denoted by $[G]_P$, is the space $L(k)U(\mathbb{A})\backslash G(\mathbb{A})$.

Remark 13.2.4. For a different choice of parabolic $P' = L'U' \in \mathcal{P}(k)$, there is a canonical isomorphism $L(k)U(\mathbb{A})\backslash G(\mathbb{A}) \simeq L'(k)U'(\mathbb{A})\backslash G(\mathbb{A})$, induced by translation by an element of $G(k)$, which is unique modulo left $P(k)$-translation. Hence, in this case, there is no ambiguity in saying that $[G]_{\mathcal{P}}$ is "the" space $L(k)U(\mathbb{A})\backslash G(\mathbb{A})$.

Remark 13.2.5. Here is an alternate, and more straightforward construction of the boundary degeneration: For every class $\mathcal{P}$ of parabolics over k, the Levi quotients L of any two elements of $\mathcal{P}$ are isomorphic, canonically up to $L(k)$-conjugacy. (As in the case of the universal Cartan: the parabolics are conjugate by an element of $G(k)$ unique up to multiplication by $P(k)$.)

Choosing such a parabolic with Levi quotient L, we have

$$[G]_P = [L] \times^{P(\mathbb{A})} G(\mathbb{A}). \tag{13.2.5.1}$$

One easily checks that for any two parabolics, any element of $G(k)$ conjugating one to another defines the same isomorphism between the corresponding spaces defined by (13.2.5.1).

Finally, another way to define the same space is the following: Consider $\mathcal{P}$ as an algebraic variety, and consider the space

$$Z = \mathcal{P}(k) \times^{G(k)} G(\mathbb{A}).$$

If we choose a $P \in \mathcal{P}(k)$, this is isomorphic to $P(k)\backslash G(\mathbb{A})$. Now, the inertia group scheme of $\mathcal{P}$ has fiber P over the point P, and its unipotent radical is a group scheme $\mathbf{U} \to \mathcal{P}$. If we divide the space Z, which lives over $\mathcal{P}(\mathbb{A})$, by the action of $\mathbf{U}(\mathbb{A})$, we obtain the space $[G]_P$.

The importance of boundary degenerations lies in the fact that, as we will see, they model the space $[G]$ "at infinity", while having a larger group of symmetries:

Lemma 13.2.6. *Let $[G]_P$ be a boundary degeneration of the automorphic space, and let Z be the center of the Levi quotient of P. The $G(\mathbb{A})$-automorphism group of $[G]_P$ is identified with $[Z]$, through its action descending from the action on the pre-flag variety.*

Proof. The $G(\mathbb{A})$-automorphism group is the quotient of the normalizer of $H := L(k)U(\mathbb{A})$ by H. The closure of the projection of H to any place v of k is the parabolic $P(k_v)$, and since P is self-normalizing, an adele of G normalizing H must lie in $P(\mathbb{A})$. Then it acts on $L(\mathbb{A})$ by conjugation, and in order to normalize $L(k)$ it has to belong to the center of $L(\mathbb{A})$. $\square$

We will use the action of this abelian group in order to construct (partial) compactifications of the boundary degeneration (and, later, of the automorphic space). There are several slight variants of how to do it, but they all follow the same idea: Let $H \subset [Z]$ (with notation as in Lemma 13.2.6) be a subgroup, and let $\bar{H}$ be a partial compactification of H (or a partial compactification of a H-torsor). Then, we can form the space

$$\bar{H} \times^H [G]_P,$$

which is a partial compactification of $[G]_P$. Here, $A \times^H B$ denotes the *topological* quotient of the product $A \times B$ by the action of H, i.e., by the equivalence relations $(ah, b) \sim (a, hb)$, $h \in H$.

In practice, H will be the points of a torus and, $\bar{H}$ will arise from some toric variety. For what follows, if T is any torus over a field F, we denote by the corresponding gothic lowercase letter the vector space $\mathfrak{t} := \mathrm{Hom}(\mathbb{G}_m, T) \otimes \mathbb{R}$. If F is a valued field (or ring), we have a well-defined logarithmic map

$$\log : T(F) \to \mathfrak{t} \tag{13.2.6.1}$$

given by $\langle \log(t), \chi\rangle = \log|\chi(t)|$ for any $\chi \in \mathrm{Hom}(T, \mathbb{G}_m)$. The same definition can be given, globally, replacing $T(F)$ by $[T]$, and using the adelic absolute value.

Recall that a normal affine embedding Y of a torus T over a field k is given by a *strictly convex, rational polyhedral cone* $C \subset \mathfrak{t}$. The faces of this cone are in bijection with T-orbits on Y, in such a way that cocharacters λ in the relative interior of a face are those for which $\lim_{t\to 0} \lambda(t)$ belongs to the corresponding orbit. (A "face", here, is the intersection with the kernel of a linear functional χ such that $\chi|_C \geq 0$; this way, $\{0\}$ is a face.) The bijection is closure-reversing, e.g., $\{0\}$ corresponds to the open orbit T, and the relative interior of C corresponds to the unique closed orbit. More general normal embeddings of T are described by *fans* in $\mathfrak{t}$, i.e., collections of such cones closed under the operation of passing to a face of a cone and with disjoint relative interiors.

Returning to our group G, let $\mathbf{A}^G$ be its universal Cartan (Definition 7.5.5), and let $A \subset \mathbf{A}^G$ be its maximal split subtorus. We denote by $\mathfrak{a}^- \subset \mathfrak{a}$ the antidominant cone, and by $\mathfrak{a}^-_{ss} \subset \mathfrak{a}^-$ its intersection with the weight space of the associated semisimple group. Faces of $\mathfrak{a}^-$ (or $\mathfrak{a}^-_{ss}$) correspond to conjugacy classes of parabolic subgroups defined over k, and the span of the face $\mathfrak{a}^-_P$ (resp. $\mathfrak{a}^-_{ss,P}$) associated to P will be denoted by $\mathfrak{a}_P$ (resp. $\mathfrak{a}^-_{ss,P}$). The center Z of a Levi quotient as in Lemma 13.2.6 is canonically a subgroup of $\mathbf{A}^G$, and the cocharacters in $\mathfrak{a}_P$ span the maximal split subtorus A_P of Z.

Now, the face $\mathfrak{a}^-_{ss,P} \subset \mathfrak{a}_P$ defines an affine embedding $A_P \hookrightarrow \overline{A_P}$. We can define a corresponding partial compactification $\overline{[G]_P}$ of $[G]_P$, by either of the construction in the following definition:

Definition 13.2.7. A *standard embedding* of the space $[G]_P$ is either of the following spaces:

- $\overline{A_P(k_\infty)^0} \times^{A_P(k_\infty)^0} [G]_P$, where $\overline{A_P(k_\infty)^0}$ is the closure of the identity component of $A_P(k_\infty)$ in $\overline{A_P}(k_\infty)$. This makes sense only if k is a number field. When $k = \mathbb{Q}$ (which we can always assume, by restriction of scalars), it leads to the so-called *reductive Borel–Serre compactification* [**Zuc83, BJ05**].

- $\overline{A_P}(k_\infty) \times^{A_P(k_\infty)} [G]_P$. Recall that, in the function field case, we just pick a place (or a place of the base field) that we call infinity.
- $\overline{A_P}(k)A_P(\mathbb{A}) \times^{A_P(\mathbb{A})} [G]_P$. Here, $\overline{A_P}(k)A_P(\mathbb{A})$ is the subset of "primitive elements" in $\overline{A_P}(\mathbb{A})$. Notice that $A_P(\mathbb{A})$ is not open in the space $\overline{A_P}(\mathbb{A})$, but it is open in the subset of primitive elements (with the induced topology).

Lemma 13.2.8. *Assume that $k = \mathbb{Q}$, and let $K \subset G(\mathbb{A}_f)$ be a compact open subgroup. The space $\overline{A_P(\mathbb{R})^0} \times^{A_P(\mathbb{R})^0} [G]_P/K$ has the structure of a manifold with corners, so that the quotient $\overline{A_P(\mathbb{R})^0} \times [G]_P/K \to \overline{A_P(\mathbb{R})^0} \times^{A_P(\mathbb{R})^0} [G]_P/K$ is an $A_P(\mathbb{R})^0$-torsor in manifolds with corners.*

Proof. The space $[G]_P/K$, under the $G(\mathbb{R})$-action, is a union of homogeneous manifolds, and the space $\overline{A_P(\mathbb{R})^0}$ is a manifold with corners (isomorphic to a product of copies of $\mathbb{R}_{>0}$ and $\mathbb{R}_{\ge 0}$) . The stabilizer for the action of $A_P(\mathbb{R})^0$ on it is a compact subgroup, hence trivial, and therefore the quotient of $\overline{A_P(\mathbb{R})^0} \times [G]_P/K$ by the action of $A_P(\mathbb{R})^0$ inherits the structure of a manifold with corners. ☐

There is some coarser topology than the manifold topology that these spaces have, in the number field case, namely some sort of "semialgebraic" topology. This plays a role in harmonic analysis; we will return to it when appropriate.

Finally, we introduce the notion of the cusp:

Definition 13.2.9. Consider a standard embedding $[G]_P \subset \overline{[G]_P}$, as in Definition 13.2.7. The *P-cusp* in $\overline{[G]_P}$ is the closed $G(\mathbb{A})$-orbit. A *neighborhood of the P-cusp* in $[G]_P$ is the intersection of $[G]_P$ with a neighborhood of the P-cusp in $\overline{[G]_P}$. When P is the class of minimal parabolics, the P-cusp will simply be called "the cusp".

Notice that the distinction "P-cusp", as opposed to "cusp", is important: e.g., when $P = G$, the partial compactification $\overline{[G]}$ above is trivial, so the G-cusp is $[G]$ itself, but there will be a "cusp", which, to define, we first need to discuss reduction theory, and a full compactification.

Lemma 13.2.10. *All standard embeddings of Definition 13.2.7 give rise to the same neighborhoods of the P-cusp.*

Proof. This relies on the "baby case" of the partial compactification $\mathbb{G}_m \hookrightarrow \mathbb{G}_a$. We leave the rest to the reader.

Under the natural map

$$k_\infty \times^{k_\infty^\times} [\mathbb{G}_m] \to \mathbb{A} \times^{\mathbb{A}^\times} [\mathbb{G}_m] = \mathbb{A}/k^\times,$$

a basis of neighborhoods of the "cusp" (represented by $0 \in \mathbb{A}$) on the right hand side maps to a basis of neighborhoods of the "cusp" (represented by $0 \in k_\infty$) on the left hand side.

The map is continuous, so it is enough to show that any element in a basis of neighborhoods $(V_r)_r$ on the left contains the preimage of a neighborhood on the right. Since the cusp is invariant under $[\mathbb{G}_m]$, and in particular under the compact subgoup $K = [\mathbb{G}_m]^1$, we can take the basis on the left to be invariant under K, and then it is seen that the sets $V_r = \{x \in [\mathbb{G}_m], |x| < r\}$ with $r \to 0^+$ form such a basis. If we choose an adele b with $|b| < r$, and let V' be the union of $k^\times$-translates of the set $\{a \in \mathbb{A}, \forall v : |a_v| \le |b_v|\}$, then V' is open, and its preimage belongs to V_r. This proves the claim.

□

Remark 13.2.11. Let us explicate neighborhoods of the cusp in the case where G is split, hence the minimal parabolic is a Borel subgroup. (A similar description will be valid for the minimal parabolic P in every case, once we show — Theorem 13.5.3 — that $[L']$ is compact, where L' is the derived subgroup of the Levi quotient of P.)

Choose a Borel subgroup $B = AN$, and a maximal compact subgroup K of $G(\mathbb{A})$ which satisfies the Iwasawa decomposition $G(\mathbb{A}) = B(\mathbb{A})K$. Then, for *any* identification $\bar{Y} = N\backslash G$ over k, a basis of neighborhoods of the cusp in $[G]_B$ is given by the neighborhoods $V_\epsilon := N(\mathbb{A})[A]^{\le\epsilon}K$ (as $\epsilon \to 0^+$), where $[A]^{\le\epsilon}$ is the set of elements a with $|e^\alpha(a)| \ge \epsilon^{-1}$ for all positive roots α.

We reformulate the definition of the P-cusp, when the last choice of standard embedding in Definition 13.2.7 is made: Let Y be the pre-flag variety $U\backslash G$. The action of A_P by G-automorphisms allows us to define a (so-called *toroidal*) partial compactification $\bar{Y} \supset Y$, as $\bar{Y} = \overline{A_P} \times^{A_P} Y$, where $\overline{A_P}$ is the affine embedding defined by the wall of the (semisimple) antidominant chamber, as above. In particular, $\bar{Y}$ contains a closed G-orbit Y_0, which we will call *the P-cusp in $\bar{Y}$*. Now, consider an $\mathrm{Aut}^G(Y)(k)$-stable neighborhood V of $Y_0(\mathbb{A})$ in $\bar{Y}(\mathbb{A})$ — recall that $\mathrm{Aut}^G(Y)$ is identified with the Levi quotient of P, once we fix a point whose stabilizer is the unipotent radical of P. Notice also that $Y(\mathbb{A})$ is not open in $\bar{Y}(\mathbb{A})$ — but it doesn't matter! All that matters is the subset $\bar{Y}(k)G(\mathbb{A})$ of "primitive" elements, where $Y(\mathbb{A})$ is open.

Now recall from Definition 13.2.3 that $[G]_P$ is defined as the set of pairs (M, y) modulo $\mathrm{Aut}^G(Y)(k)$, where M is a $G(\mathbb{A})$-translate of $Y(k)$, and $y \in M$. Let $\tilde{V} \subset [G]_P$ be the subset given by the condition $y \in V$. Then, the neighborhoods of the P-cusp in $[G]_P$ are precisely the sets of the form $\tilde{V}$, where V is an $\mathrm{Aut}^G(Y)(k)$-invariant neighborhood of the P-cusp in $\bar{Y}$, and $\tilde{V}$ is obtained from V as above.

We will now modify this description to define a degenerate version of the P-cusp, that leads to a coarser collection of "neighborhoods of infinity". Let Y_{deg} be the degenerate pre-flag variety $[P, P]\backslash G$. Again, the action of A_P by G-automorphisms allows us to define a partial compactification $\bar{Y}_{deg} \supset Y_{deg}$, as $\bar{Y}_{deg} = \overline{A_P} \times^{A_P} Y_{deg}$, where $\overline{A_P}$ is the affine embedding defined by the wall of the (semisimple) antidominant chamber, as above. In particular, $\bar{Y}_{deg}$ contains a closed G-orbit $Y_{deg,0}$, which we will call *the degenerate P-cusp in $\bar{Y}_{deg}$*. Now, consider an $L^{ab}(k)$-stable neighborhood V_{deg} of $Y_{deg,0}(\mathbb{A})$ in $\bar{Y}_{deg}(\mathbb{A})$, where L^{ab} is the abelianization of $L = \mathrm{Aut}^G(Y)$.

Using again Definition 13.2.3 for $[G]_P$ as the set of pairs (M, y) modulo $\mathrm{Aut}^G(Y)(k)$, we let $\tilde{V} \subset [G]_P$ be the subset given by the condition $\bar{y} \in V_{deg}$, where $\bar{y}$ is the image of y under $U\backslash G \to [P, P]\backslash G$. In other words, these are neighborhoods of the cusp obtained from neighborhoods $V \subset U\backslash G(\mathbb{A})$, as before, except that V should be stable under the commutator of the Levi.

Definition 13.2.12. A *neighborhood of the degenerate P-cusp* in $[G]_P$ is a set of the form $\tilde{V}$ as above, where V is a neighborhood of the degenerate P-cusp in $\bar{Y}_{deg}(\mathbb{A}) = \overline{[P, P]\backslash G}(\mathbb{A})$.

Hence, a neighborhood of the degenerate P-cusp in $[G]_P$ is a neighborhood of the P-cusp, but a neighborhood of the P-cusp is a neighborhood of the degenerate P-cusp only when it is the preimage of a set under the map

$$L(k)U(\mathbb{A}) \to G(\mathbb{A}) \to L^{ab}(k)[P,P](\mathbb{A})\backslash G(\mathbb{A}).$$

Of course, the two notions coincide when P is in the class of Borel subgroups.

13.3. Adelic heights, and the case of $\mathbf{SL}_2$

Definition 13.3.1. An *adelic height* on a vector space V over k is a function of the form $\|x\| = \prod_v \|x_v\|_v$ on $V(\mathbb{A})$, where:

- $\| \bullet \|_v$ is a norm on $V(k_v)$, i.e., a subadditive, $\mathbb{R}_{\ge 0}$ valued function that is zero only at 0 and satisfies $\|ax\|_v = |a|_v \cdot \|x\|_v$ for every $a \in k_v$, $x \in V(k_v)$;
- there is a basis of V over k such that for almost every non-Archimedean place, $\|x\|_v$ is the maximum of the absolute values of the coordinates of x in that basis.

Obviously, the factors of the product are almost all equal to 1 if $x \in (V \smallsetminus \{0\})(\mathbb{A})$, but the height extends continuously by zero to the entire space $V(\mathbb{A})$ (though we will never use that).

Lemma 13.3.2. *Adelic height functions (Definition 13.3.1) on a vector space V have the following properties:*

(1) *For any two height functions $\| \bullet \|'$, $\| \bullet \|''$, the quotient $\frac{\|\bullet\|'}{\|\bullet\|''}$ is bounded in $\mathbb{R}^\times_{>0}$.*
(2) *For all $a \in \mathbb{A}^\times$, $x \in V(\mathbb{A})$, we have $\|ax\| = |a| \cdot \|x\|$; in particular, $\| \bullet \|$ is invariant under $k^\times$-multiplication.*
(3) *The restriction of $\|\bullet\|$ to the quotient space $k^\times\backslash V(k)\cdot GL_V(\mathbb{A}) = k^\times\backslash(V^*(\mathbb{A})\cup \{0\})$, where V^* denotes the complement of zero, defines a basis of neighborhoods of zero.*
(4) *For every $g \in GL_V(\mathbb{A})$ and any $c > 0$, there is only a finite number of classes $[y] \in k^\times\backslash V(k)$ such that $\|y\| < c$; in particular, the set $\|V^*(k)g\|$ has a minimum.*

Proof. Left to the reader. $\square$

Let us now discuss the case of $G = \mathrm{SL}_2$, with its standard representation V. Notice that $V^* = V \smallsetminus \{0\}$ can be identified with the pre-flag variety Y for the class of Borel subgroups of G (Definition 13.2.1), and V is simply its affine closure, i.e., $V = \mathrm{Spec} k[Y]$. In this case, the cusp can also be defined with the help of the affine embedding V, instead of the toroidal embedding $\bar{Y}$. That is, instead of choosing a $k^\times$-invariant neighborhood of the cusp U in $\bar{Y}(\mathbb{A})$, one can choose a $k^\times$-invariant neighborhood U of zero in $V(\mathbb{A})$, and pull it back to $B(k)\backslash G(\mathbb{A})$ to define a neighborhood of the cusp there. By Lemma 13.3.2, a basis of such neighborhoods of zero is determined by height functions.

Remark 13.3.3. We caution the reader that, in higher rank, the affine embedding $\bar{Y}^{\mathrm{aff}} = \overline{N\backslash G}^{\mathrm{aff}} = \mathrm{Spec} k[N\backslash G]$ (where G is assumed split semisimple, and N is a maximal unipotent subgroup) is *not* the appropriate space to define the cusp, i.e., not every neighborhood of the cusp in $[G]_B$ is the pullback of an $A(k)$-invariant neighborhood of "zero" (=the unique G-fixed point) in $Y^{\mathrm{aff}}(\mathbb{A})$.

In terms of toric varieties, this is because the toroidal embedding of Y used in the definition of the P-cusp corresponds to the antidominant Weyl chamber, while the affine closure Y^{aff} corresponds to the *negative root cone*, in the sense that it admits a blowup $\tilde{Y} \to \bar{Y}^{\mathrm{aff}}$, with $\tilde{Y} = \tilde{A} \times^A Y$, and $\tilde{A}$ is the affine embedding of $\tilde{A}$ corresponding to the negative root cone. Thus, $A(k)$-invariant neighborhoods of zero in $Y^{\mathrm{aff}}(\mathbb{A})$ are *much larger* than neighborhoods of the cusp, in higher rank, and the latter cannot be defined using adelic heights on an affine embedding of Y.

Now, we formulate the main theorem of reduction theory for $G = \mathrm{SL}_2$:

Theorem 13.3.4. *Let $G = SL_2$, and consider the maps*

$$\begin{tikzcd} & B(k)\backslash G(\mathbb{A}) \arrow[dl, "\pi_G"'] \arrow[dr, "\pi_B"] & \\ {[G]} & & {[G]_B.} \end{tikzcd} \tag{13.3.4.1}$$

Fix an adelic height function $\|\bullet\|$ (Definition 13.3.1) on the space V of the standard representation of G, and for every $\epsilon > 0$, let U_ϵ be the preimage in $[G]_B$ of the set of points $y \in Y(\mathbb{A})$ (where $Y = V^* = V \smallsetminus \{0\}$) with $\|y\| < \epsilon$, and $\tilde{U}_\epsilon$ its preimage in $B(k)\backslash G(\mathbb{A})$.

Then, for ϵ sufficiently small, the map $\pi_G|_{\tilde{U}_\epsilon}$ is injective, while for ϵ sufficiently large, $\pi_G|_{\tilde{U}_\epsilon}$ is surjective. In particular, for every ϵ, the complement of $\pi_G(\tilde{U}_\epsilon)$ is relatively compact in $[G]$.

Proof. Injectivity: It suffices to show that, if ϵ is sufficiently small, for any $g \in G(\mathbb{A})$ the set $\{y \in Y(k)g | \|y\| < \epsilon\}$ contains at most one $k^\times$-orbit.

Hence, fix $[g] \in [G]$, and let us also fix the point $y_0 = (0,1) \in Y$, so that its image in the flag variety is stabilized by the upper triangular Borel subgroup B. Let us fix a good maximal compact subgroup K that satisfies the Iwasawa decomposition $G(\mathbb{A}) = B(\mathbb{A})K$, and we may without loss of generality assume that the adelic norm $\| \bullet \|$ is K-invariant. Assume that $y \in Y(k)g$ has height $< \epsilon$. We may choose a representative $g \in G(\mathbb{A})$ for $[g]$ such that $y = y_0 g$. Writing $g = bk$ according to the Iwasawa decomposition, we have $b = \begin{pmatrix} a & r \\ & a^{-1} \end{pmatrix}$, and $\|y_0 g\| = \|y_0 b\| = \|a^{-1} y_0\| = |a|^{-1}\|y_0\|$, so $|a| > \epsilon^{-1}\|y_0\|$.

Our goal is to show that, if ϵ is sufficiently small (independently of $[g]$), any other element $z \in Y(k)g$ with height $< \epsilon$ is a $k^\times$-multiple of y. If not, we have $z = z_0 g$ with $z_0 = (\kappa, \lambda) \in k^2$ with $\kappa \neq 0$. Up to the $k^\times$-action, we may assume that $\kappa = 1$. But $\|z_0 g\| = \|z_0 b\|$, and $\|z_0 b\| = \|(a, r + a^{-1}\lambda)\|$. It is now clear from the definition of heights that if ϵ is sufficiently small, so that $|a|$ is sufficiently large, the last expression is $> \epsilon$.

(For a more geometric, and conceptual, version of the same argument, see the proof of the general case in Theorem 13.4.2.)

Surjectivity: Vice versa, it suffices to show that if ϵ is large enough, then for every $[g] \in [G]$ the set $\{y \in Y(k)g | \|y\| > \epsilon\}$ is nonempty. Given $[g]$, we may choose a representative $g \in G(\mathbb{A})$ so that $\|y_0 g\|$ is minimal in $\|Y(k)g\|$. (The minimum exists, by Lemma 13.3.2.) Write $g = bk$ by the Iwasawa decomposition, where $b = \begin{pmatrix} a & r \\ & a^{-1} \end{pmatrix}$. The minimality of $\|y_0 g\|$ implies that $\|(a, r + a^{-1}\lambda)\| \geq \|y_0 g\| =$

$|a|^{-1}\|y_0\|$ for all $\lambda \in k$, hence $|a|^2 \geq \frac{\|y_0\|}{\|(1,a^{-1}r+a^{-2}\lambda)\|}$. The denominator is bounded away from zero, giving a lower bound for the absolute value of the positive root $|e^\alpha(b)| = |a|^2$, and an upper bound for $\|y_0 g\| = |a|^{-1}\|y_0\|$. Hence, if ϵ is larger than this upper bound, the class of $[g]$ is represented in $\tilde{U}_\epsilon$.

□

13.4. Reduction theory and compactifications in the split case

Reduction theory is the description of the space $[G]$ "at infinity" in terms of its boundary degenerations. In this section, we will prove the main result of reduction theory in the split case. We will use it to define a certain compactification of $[G]$, which will be used in the next section to address the general case.

Before we generalize from SL_2 to the general case, we need a purely algebro-geometric result on closure of horocycles on pre-flag varieties. The result is analogous to the following statement in the case of SL_2: Let $N \subset \mathrm{SL}_2$ be the stabilizer of a nonzero vector on the two-dimensional plane, let ℓ be the line spanned by that vector, and let ℓ' be a different, affine line parallel to ℓ; notice that ℓ' is an N-orbit. Let $v_n \in \ell'$ be a sequence of points such that the lines that they span approach ℓ in the projective space; the statement is that $v_n \to \infty$. This fact is obvious in this case, or for any unipotent orbit on a quasiaffine space, since by a result of Rosenlicht [**Ros61**] orbits of unipotent groups on affine spaces are closed. Here, however, we need finer information for the quasiaffine space $Y = N\backslash G$ in higher rank: Roughly speaking, we need to know that if a point is far from the cusp, so is its entire N-orbit. Since the cusp cannot be described in terms of affine embeddings in higher rank (see Remark 13.3.3), we need to consider more general compactifications, and Rosenlicht's theorem will not be enough.

Let $Y = N\backslash G$ be "the" pre-flag variety for G. In order to keep track of possible limits in various directions, let us consider a full compactification $\bar{A}^F$ of the torus A, described by a fan F whose support is the entire space $\mathfrak{a}$. Let $\bar{Y}^F = \bar{A}^F \times^A Y$ — it is a proper variety over the flag variety $\mathcal{B}$.

Choose a Borel subgroup $B \in \mathcal{B}(k)$, and a section of the canonical map $B \to A$, and denote its image by T; hence, T is a maximal torus in B, identified with A through the quotient map. We use $\bar{T}$, $\bar{T}^F$ etc. for T, as for A. For every $B' \in \mathcal{B}$ and any strictly antidominant cocharacter λ into T, we have $\lim_{x\to 0} B'^{\lambda(x)} = B^w$, for some w in the Weyl group of W — this is the Bruhat decomposition!

The following proposition is very essential in what follows; although purely of algebrogeometric nature, it will enventually translate to information about the "heights" of points on N-orbits on the preflag variety. This information is useful not only for reduction theory, but also for the study of Radon transforms and intertwining operators.

Proposition 13.4.1. *Let $Y \to \mathcal{B}$ be the preflag variety of a reductive group G, $y_0, y' \in Y$ be two points lying over Borel subgroups B, B', let $T \subset B$ be a maximal torus, and assume that $y' \in y_0\tilde{w}N$, where $N \subset B$ is the unipotent radical, and $\tilde{w}$ is an element in the normalizer of T.*

(1) *Let $\bar{Y}^F = \bar{A}^F \times^A Y \to \mathcal{B}$ be the fiberwise compactification of Y determined by the fan of all Weyl chambers. If $\lambda : \mathbb{G}_m \to T$ is a strictly antidominant cocharacter with respect to B, then $\lim_{x\to 0} y'\lambda(x) \in \bar{Y}^F$ is the A-fixed*

point in the fiber over $B^w \in \mathcal{B}$ which corresponds to the right-w translate of the antidominant cone in $\mathfrak{a}$.

(2) *Let $\bar{A}^{R(w^{-1})^+}$ be the embedding of A corresponding to the cone $R(w^{-1})^+$ spanned by the positive coroots $\check\gamma$ such that $w^{-1}\check\gamma < 0$, and let $\bar{Y}^{R(w^{-1})^+}$ be the corresponding embedding of Y, so that G-orbits on $\bar{Y}^{R(w^{-1})^+}$ (or, equivalently, A-orbits in each fiber over the flag variety) correspond to faces of the cone $R(w^{-1})^+$. Then, the closure of $y_0\tilde{w}N$ in $\bar{Y}^{R(w^{-1})^+}$ is proper, and the points in the boundary of $y_0\tilde{w}N$ lie in the complement of the open G-orbit.*

Proof. The first statement is not needed for what follows, and is left to the reader.

Let us prove the second statement: First, fix any compacification $\bar{Y}^F = \bar{A}^F \times^A Y$, so that the closure of $y_0\tilde{w}N$ in $\bar{Y}^F$ is proper. This closure corresponds to the closure of the set $A\cdot(1, y_0\tilde{w}N)$ in $\bar{A}^F \times Y$, where A acts by $a\cdot(x,y) = (a^{-1}x, ay)$.

Let $Y_w \subset Y$ be the preimage of the closed Schubert cell $S_w := \overline{B\backslash BwN}$. We write $\mathring{S}_w$, $\mathring{Y}_w$ for the corresponding open Schubert cell and its preimage, and identify the ("geometric") quotient of $\mathring{Y}_w$ by N with A, equivariantly under the right action of $B = A/N$, by fixing the base point $\overline{y_0\tilde{w}}$=the image of $y_0\tilde{w}$. Notice that this is *not* the equivariant identification with respect to the canonical action of A on Y. The complement of $\mathring{S}_w$ in S_w is a union of Schubert divisors $S_{w'}$ with $w' = ww_\gamma$ for some (not necessarily simple) root γ and length $\ell(w') = \ell(w) - 1$. The coordinate ring $k[Y_w]^N$ is generated by those characters of A which, considered as functions on $\mathring{Y}_w$, have ≥ 0 valuations on the corresponding divisors $Y_{w'}$. We claim that these are the characters which are ≥ 0 on the cone $R(w)^+$ spanned by the positive coroots $\check\gamma$ such that $w\check\gamma < 0$. Indeed, consider a reduced decomposition $w = w_{\alpha_1}w_{\alpha_2}\cdots w_{\alpha_n}$ into simple reflections, and work on the corresponding Bott–Samelson resolution of S_w: $\tilde{S}_w = B\backslash P_{\alpha_1}\times^B P_{\alpha_2}\times^B\cdots\times^B P_{\alpha_n}$, and the corresponding resolution $\tilde{Y}_w$ of Y_w (divide by N on the left instead of by B). Each codimension-one divisor $S_{w'}$ is the image of the subset $\tilde{S}_{w'} \subset \tilde{S}_w$ obtained by omitting a factor P_{α_i}. Is is easy to see that the divisor $N\backslash B$ induces the valuation $\langle\check\alpha_i, \chi\rangle$ on B-eigenfunctions with character χ on $N\backslash P_\alpha$, and therefore $\tilde{Y}_{w'}$ induces the valuation $\langle\check\gamma,\chi\rangle$ on B-eigenfunctions on $\tilde{Y}_w$, where $\check\gamma = w^{-1}\check\alpha_i$. These are precisely the positive coroots such that $w\check\gamma < 0$. Thus, the regular characters are those which are ≥ 0 on $R(w)^+$.

Thus, $Y_w /\!\!/ N \simeq \bar{A}^{R(w)^+}$, the embedding of A corresponding to the cone $R(w)^+$, equivariantly under the right action of $A = B/N$. If instead we use the ("left") action of A on Y, we identify this space with the embedding $\bar{A}^{w\cdot R(w)^+}$. Passing to the corresponding quotient of $\bar{A}^F \times Y_w$, the closure of $A\cdot(1, y_0\tilde{w}N)$ maps to the closure of $A \hookrightarrow \bar{A}^F \times \bar{A}^{wR(w)^+}$, where the embedding is the antidiagonal one. Notice that $wR(w)^+ = -R(w^{-1})^+$. If we now assume that $\bar{A}^F$ contains $\bar{A}^{R(w^{-1})^+}$ as an open subset, we see that the closure of $A \hookrightarrow \bar{A}^F \times \bar{A}^{wR_w^+}$ maps into $\bar{A}^{R(w^{-1})^+}$ under the projection to the first factor. This proves the properness of the closure of $y_0\tilde{w}N$ in $\bar{Y}^{R(w^{-1})^+}$. Moreover, the proof shows that a point in the boundary of this set, hence lying over a non-open orbit of $Y_w /\!\!/ A$, will project to the complement of the open A-orbit in $A^{R(w^{-1})^+}$. ∎

Fix a class $\mathcal{P}$ of parabolics, and a parabolic $P \in \mathcal{P}(k)$. (Nothing will depend on P, but it is notationally convenient.) Consider the homogeneous space $\mathcal{P}(k)\times^{G(k)} G(\mathbb{A}) = P(k)\backslash G(\mathbb{A})$, already encountered in Remark 13.2.5. It admits a pair of quotient maps:

(13.4.1.1)
$$\begin{tikzcd} & P(k)\backslash G(\mathbb{A}) \arrow[dl, "\pi_G"'] \arrow[dr, "\pi_P"] & \\ {[G]} & & {[G]_P,} \end{tikzcd}$$

and notice that π_G is a local homeomorphism, while π_P has compact fibers.

We will call "neighborhood of the (degenerate) P-cusp" in $P(k)\backslash G(\mathbb{A})$ the preimage of any neighborhood of the (degenerate) P-cusp in $[G]_P$. Generally, for a neighborhood V of the P-cusp in $[G]_P$, we will use the notation $\tilde{V}$ for its preimage in $P(k)\backslash G(\mathbb{A})$. Recall that the P-cusp is a $G(\mathbb{A})$-orbit in a certain partial compactification $\overline{[G]_P}$ of $[G]_P$, see Definition 13.2.9. A "scaling" of such a neighborhood will be the neighborhood that we obtain from it by the action of an element $a \in A_P(\mathbb{A})$ on $[G]_P$.

Theorem 13.4.2. *Let G be a split reductive connected group over k, and $P \subset G$ a (class of) parabolic(s).*

- **(I):** *Every compact subset of the P-cusp in $\overline{[G]_P}$ has a neighborhood V such that $\pi_G|_{\tilde{V}}$ is injective, where $\tilde{V}$ is the preimage of V in $P(k)\backslash G(\mathbb{A})$.*[1]
- **(S):** *Every $[A_G]$-invariant neighborhood of the degenerate P-cusp in $[G]_P$ (where A_G is the maximal split torus in the center of G) can be scaled to a neighborhood V such that $\pi_G|_{\tilde{V}}$ is surjective, where $\tilde{V}$ is as above.*

Proof. There are two approaches to proving this theorem: One, due to Borel and Harish-Chandra [**BHC62**], is to prove it first for GL_n over $\mathbb{Q}$, and then deduce it for a general reductive group via an embedding $G \to \mathrm{GL}_n$ (where G, by restriction of scalars, can be considered as a group over $\mathbb{Q}$, at least in the number field case. Here and in the next section, we will present the second, following Godement [**God95**] and Springer [**Spr94**], but heavily reformulated. [We caution that there are serious gaps in both Godement and Springer, especially at the point that corresponds to Proposition 13.4.1.]

Having addressed the case of split tori in Proposition 13.1.5, we will now assume that G is semisimple; the combination of the two to address the general case is "easy", and left to the reader.

The proof of (I) relies on Proposition 13.4.1. Fix the base point $y_0 \in Y(k)$, and let B be the stabilizer of its A-orbit. Fix a maximal compact subgroup satisfying the Iwasawa decomposition $G(\mathbb{A}) = B(\mathbb{A})K$. We use this, together with the logarithmic maps $B(\mathbb{A}) \to \mathfrak{a}$, defined as in (13.2.6.1), to define a "height" function on $[G]_B = A(k)\backslash y_0 G(\mathbb{A})$:
$$h(g) = \log(b) \in \mathfrak{a}.$$
This is similar to the height functions of Definition 13.3.1, except that it does not use an affine embedding of Y; rather, the sets $V_\epsilon = \{[g] \in [G]_B | \langle \alpha, h(g) \rangle > -\log \epsilon \text{ for all positive roots } \alpha\}$ form a basis of neighborhoods for the cusp. (See Remarks 13.2.11 and 13.3.3.)

We claim that the statement of Proposition 13.4.1 about the closure of $y_0 \tilde{w} N$ implies the following:

[1] Obviously, "every compact subset" is superfluous, but we include it to stress that there is no uniformity in the choice of neighborhood.

- There is a compact $C_0 \subset \mathfrak{a}$ such that

$$h(y_0\tilde{w}N(\mathbb{A})) \subset C_0 - R(w^{-1})^+. \tag{13.4.2.1}$$

[We leave it to the reader to check this corollary, for now. We remark only that the embedding of $N\backslash G$ which corresponds to the cone $R(w^{-1})^+$ is the one where, for a cocharacter $\lambda \in R(w^{-1})^+$ into the torus A, and a point $y \in N\backslash G$, the limit of $\lambda(t)\cdot y$ exists *as* $t \to 0$; the logarithm of such an element $\lambda(t)$, for t close to zero, belongs to the *negative* of the cone $R(w^{-1})^+$, hence the negative sign above.]

Returning to the proof of (I), consider the surjective maps

$$B(k)\backslash G(\mathbb{A}) \xrightarrow{\pi_P^B} P(k)\backslash G(\mathbb{A}) \xrightarrow{\pi_G} [G],$$

for any parabolic $P \supset B$. Fix a compact $\Omega \subset [G]_P$, and consider the subsets $\Omega_\epsilon = [A_P]_\epsilon \cdot \Omega$, where $[A_P]_\epsilon$ is the set of elements a with $\langle \alpha, \log(a)\rangle > -\log\epsilon$ for every simple root α in the unipotent radical of P; recall that $\log(a) \in \mathfrak{a}_P$, for $a \in A_P(\mathbb{A})$. These subsets form a basis of neighborhoods of the image of Ω in the P-cusp as $\epsilon \to 0$ (which, by abuse of notation, we will denote by $A_P\backslash\Omega$). In turn, Ω is contained in the image Ω' of a compact subset of $B(k)\backslash G(\mathbb{A})$ under $B(k)\backslash G(\mathbb{A}) \to P(k)\backslash G(\mathbb{A}) \to [G]_P$. Thus, we see

- There is a countable basis of neighborhoods in $[G]_P$ of the subset $A_P\backslash\Omega$ of the P-cusp which, for any $\lambda_0 \in \mathfrak{a}_P$, is eventually contained in the image of a set of the form $h^{-1}(C + \lambda_0 + \mathfrak{a}_P^+) \subset B(k)\backslash G(\mathbb{A})$ under the map π_P^B above, where C is a fixed compact subset depending on Ω.

Thus, to prove (I), it suffices to show that, if λ_0 is sufficiently dominant (i.e., sufficiently deep in the relative interior of $\mathfrak{a}_P^+$), any two elements $g, g' \in B(k)\backslash G(\mathbb{A})$ with $h(g), h(g') \in C + \lambda_0 + \mathfrak{a}_P^+$ and the same image in $[G]$, have the same image in $P(k)\backslash G(\mathbb{A})$. In other words, we need to show that if $y = y_0\tilde{w}\nu$ for $\tilde{w} \in \mathcal{N}(T)(k)$ and $\nu \in N(k)$, and $yg = y_0g'$ for two elements as above, then $y \in y_0L(k)$.

Now, on one hand, $h(y_0g) = h(g)$ and $h(yg) = h(y_0g') = h(g')$ belong to $C+\lambda_0+\mathfrak{a}_P^+$, by assumption. On the other, writing $g = ntk$ according to the Iwasawa decomposition, we have $h(y_0g) = h(t)$ and $h(yg) = h(y_0\tilde{w}\nu ntk) \in h(y_0{}^wt\tilde{w}N(\mathbb{A}))$. By (13.4.2.1), which translates in the obvious way under the action of A, we obtain $h(y_0{}^wt\tilde{w}N(\mathbb{A})) \subset C_0 + h({}^wt) - R(w^{-1})^+ \subset C_0 + wC + w\lambda_0 + w\mathfrak{a}_P^+ + wR(w)^+$. Thus, it suffices to show that the intersection

$$(C + \lambda_0 + \mathfrak{a}_P^+) \cap (C_0 + wC + w\lambda_0 + w\mathfrak{a}_P^+ + wR(w)^+) \tag{13.4.2.2}$$

is empty for sufficiently large λ_0 (i.e., sufficiently deep in the interior of $\mathfrak{a}_P^+$), unless w preserves $\mathfrak{a}_P^+$, that is, unless $\tilde{w} \subset P(k)$.

Write $\mathcal{C}$ for the cone $-R(w^{-1})^+ = wR(w)^+$. As a first hint for the proof, notice that, because $\lambda_0 \in \mathfrak{a}_P^+$ is dominant, the difference $w\lambda_0 - \lambda_0$ lies in $\mathcal{C}$. If this difference is nonzero, i.e., if w does not preserve λ_0 (equivalently, does not preserve the face $\mathfrak{a}_P^+$), then by a simple scaling argument the difference can be made to avoid any compact set, by taking λ_0 large enough, so that $(C+\lambda_0)\cap(C_0+wC+w\lambda_0+\mathcal{C}) = \emptyset$.

To show the stronger statement that, for such w, the intersection (13.4.2.2) is empty, it suffices to prove that the intersection

$$\mathfrak{a}^+ \cap (w\mathfrak{a}^+ + \mathcal{C})$$

does not meet the relative interior of $\mathfrak{a}^+$. (In fact, the argument will show that the intersection consists precisely of those faces of $\mathfrak{a}^+$ which are fixed by w, i.e., are orthogonal to $\mathcal{C}$.) To see this, let $R^\pm$ denote the positive and negative root cones, and notice that $\mathcal{C}$ is equal to $wR^+ \cap R^-$. Let $\mathcal{C}^\vee$ denote the dual cone of all elements of $\mathfrak{a}$ that are ≥ 0 on $\mathcal{C}$ (under one, equivalently any, W-invariant positive definite quadratic form). Then $w\mathfrak{a}^+ \subset \mathcal{C}^\vee$, while $\mathfrak{a}^+ \subset -\mathcal{C}^\vee$. Therefore, $\mathfrak{a}^+ \cap (w\mathfrak{a}^+ + \mathcal{C}) \subset -\mathcal{C}^\vee \cap (\mathcal{C} + \mathcal{C}^\vee) = \mathcal{C}^\perp$.

This proves (I).

Now we come to the proof of (S). It is enough to prove it for the minimal parabolic (i.e., the Borel). The cusp of $\overline{[G]_B}$ is isomorphic to $\mathcal{B}(\mathbb{A})$, hence compact. Fix an ordering $\alpha_1, \alpha_2, \ldots, \alpha_r$ of the simple roots in the root system of G, and let P_i be the parabolic which contains α_j with $j \leq i$ in its unipotent radical, and α_j with $j > i$ in its Levi. Hence, $P_0 = G$ and $P_r = B$. For each i, fix a (right) representation V_i and a vector $v_i \in V_i(k)$ such that v_i is an eigenvector for P_i, but no larger parabolic. Fix adelic heights on the V_i's, and consider the set

$$\Omega = \{g \in B(k)\backslash G(\mathbb{A}) | \forall i \geq 1, \forall \gamma \in P_{i-1}(k), \|v_i \gamma g\| \geq \|v_i g\|\}.$$

Then, by the properties of heights (Lemma 13.3.2), $G(k)\Omega = G(\mathbb{A})$.

We may fix again a good maximal compact subgroup K with Iwasawa decomposition $G(\mathbb{A}) = B(\mathbb{A})K$, and assume that V is K-invariant. Writing an element of $[G]_B$ as $g = bk$, accordingly, and defining a "height" function (nothing to do with the adelic height functions defined previously, but this is a standard name) $h(g) = \log(b) \in \mathfrak{a}$, it is enough to show that $h(\Omega)$ lies in a translate of $\mathfrak{a}^+$. Such a translate is defined by equations $\langle \lambda, \alpha_i \rangle \geq T_i$ for some scalars T_i and any $i = 1, \ldots, r$, i.e., we need to show that $\langle h(\Omega), \alpha_i \rangle$ is bounded below, for all i.

By induction on r, we may assume this to be the case for $i \geq 2$. But then, to show it for α_1, we may replace G by the group G' =the Levi quotient of the parabolic Q generated by P and the root spaces proportional to α_1. Indeed, the condition $\|v_1 \gamma g\| \geq \|v_1 g\|$ holds a fortiori for $\gamma \in Q(k)$, and Q acts on v_1 through its Levi quotient. Thus, we are reduced to the case of G being of *semisimple rank one*, which we now assume. We may keep assuming that G is semisimple, since the center evidently plays no role. But then we are in the case of $G = \mathrm{SL}_2$, covered in Theorem 13.3.4, or PGL_2, which is similar. Thus, the theorem is proven. $\square$

We will now use Theorem 13.4.2 to construct a compactification of $[G]$. Set-theoretically, the compactification is

$$\overline{[G]}^{RBS} = \bigcup_P A_P \backslash [G]_P,$$

where P ranges over all conjugacy classes of parabolics, so $A_P\backslash[G]_P$ is the P-cusp. Here, we are being a bit ambiguous about the meaning of A_P: it depends on which of the "standard embeddings" of $[G]_P$ (Definition 13.2.7) one chooses; corresponding to the choices in this definition, one can take $A_P = A_P(k_\infty)^0$, $A_P(k_\infty)$, or $[A_P]$. The last one this seems the most natural choice, in general, but, when $k = \mathbb{Q}$, and we take $A_P = A_P(k_\infty)^0$ (or, more generally, when k is a number field, and we replace k_∞ by the diagonal embedding of $\mathbb{R}$ in k_∞), we obtain the *reductive Borel–Serre compactification*, which is a manifold with corners (after we mod out by a sufficiently small compact open subgroup of $G(\mathbb{A}_f)$) — this is why we use the notation $\overline{[G]}^{RBS}$ in general. Thus, we will be using A_P, and leave it to the reader to choose their favorite version, except where we need to use a specific one.

We will define, more generally, an embedding $[G]^F$ for every fan F of rational, strictly convex polyhedral cones supported in $\mathfrak{a}^-$, following [**Sak16**]. Set-theoretically, it will be the union of $G(\mathbb{A})$-orbits Z_C, for all cones $C \in F$. The RBS compactification will be the one corresponding to $F =$ the fan consisting of the faces of $\mathfrak{a}_{ss}^-$.

Assume, first, that F is a fan supported entirely on the face of the anti-dominant cone corresponding to a parabolic P with Levi quotient L, whose split center A_P is canonically a subtorus of A. Then F defines a toric embedding $A_P \hookrightarrow \overline{A_P}^F$, and we set:

$$[L]^F := \overline{A_P}^F \times^{A_P} [L], \tag{13.4.2.3}$$

$$[G]_P^F := [L]^F \times^{P(\mathbb{A})} G(\mathbb{A}). \tag{13.4.2.4}$$

Its $G(\mathbb{A})$-orbits are in natural bijection with cones in the fan, as is the case with A_P-orbits in $\overline{A_P}^F$. If $C \in F$ does not belong to the face corresponding to any larger parabolic, we let Z_C be the orbit corresponding to a cone C. If C belongs to the face corresponding to a larger parabolic Q, we will denote its orbit by Z_C^P, because Z_C will be defined by the analogous construction for Q.

For a general fan F, the $G(\mathbb{A})$-space $[G]_P^F$ will be defined by the formula (13.4.2.4), once $[L]^F$ is defined. To define $[L]^F$, we may assume that $L = G$, and that the spaces $[L]^F$, $[G]_P^F$ have been defined for all proper parabolics P.

We first consider the restriction F_G of F to $\mathfrak{a}_G$ (i.e. the sub-fan consisting of all cones which are contained in the central cocharacter space of G). By (13.4.2.3), it defines an embedding $[G]^{F_G}$ of $[G]$. Now, all the strata Z_C have been defined: If C belongs to $\mathfrak{a}_G$, then $Z_C \subset [G]^{F_G}$. If not, then Z_C has been defined is a stratum of $[G]_P^F$, for $P =$ the maximal parabolic such that $C \subset \mathfrak{a}_P$, and, by the inductive construction, it is a stratum of $[G]_Q^F$, for all $Q \supset P$ other than G. There remains to explain how to glue those onto $[G]^{F_G}$.

It suffices to consider a maximal parabolic P. First, we lift the embedding $[G]_P^F$ to an embedding of the space $\overline{A_G}^{F_G} \times^{A_G} P(k)\backslash G(\mathbb{A})$ (recall that F_G is the restriction of F to $\mathfrak{a}_G$), by considering the universal $G(\mathbb{A})$-space X fitting in a diagram of continuous, $G(\mathbb{A})$-equivariant maps:

$$\begin{tikzcd}
\overline{A_G}^{F_G} \times^{A_G} P(k)\backslash G(\mathbb{A}) \arrow[r] \arrow[d] & X \arrow[d] \\
\overline{A_G}^{F_G} \times^{A_G} [G]_P \arrow[r] & {[G]_P^F}
\end{tikzcd},$$

where the first horizontal map is supposed to be an isomorphism over $\overline{A_G}^{F_G} \times^{A_G} [G]_P$. Explicity, the universal such X (to be denoted $\overline{P(k)\backslash G(\mathbb{A})}^F$) is the union of $\overline{A_G}^{F_G} \times^{A_G} P(k)\backslash G(\mathbb{A})$ with all $G(\mathbb{A})$-orbits $Z_C \subset [G]_P^F$ with $C \in F$ not belonging to $\mathfrak{a}_G$. (We stress again that, for $C \subset \mathfrak{a}_G$, the corresponding orbit $Z_C^P \subset [G]_P^F$ is different from the orbit $Z_C \subset [G]^{F_G}$, which will be part of the final compactification $[G]^F$.)

Now, consider the pair of quotient maps (13.4.1.1). By Theorem 13.4.2, there is an $[A_G]$-stable neighborhood of the P-cusp where the map π_G is an isomorphism. Thus, we can glue $\overline{P(k)\backslash G(\mathbb{A})}^F$ to $[G]$ over such a neighborhood. In fact, the map

π_G extends to a morphism

$$\overline{A_G}^{F_G} \times^{A_G} P(k)\backslash G(\mathbb{A}) \to [G]^{F_G},$$

which is an isomorphism on an $[A_G]$-invariant neighborhood of the P-cusp. All orbits $Z_C \subset \overline{P(k)\backslash G(\mathbb{A})}^F$ with C not belonging to $\mathfrak{a}_G$ have a neighborhood which is contained in such a neighborhood of the P-cusp, *therefore we can use such a neighborhood to glue Z_C to $[G]^{F_G}$.*

Finally, if $Z_C \subset [G]^F_P$ and $Z_C \subset [G]^F_Q$ for two different maximal parabolics, there is a smaller parabolic $R \subset P \cap Q$ such that $Z_C \subset [G]^F_R$, and a neighborhood of Z_C in $[G]^F_R$ is contained in a neighborhood of the R-cusp whose preimage in $R(k)\backslash G(\mathbb{A})$ injects into both $P(k)\backslash G(\mathbb{A})$ and $Q(k)\backslash G(\mathbb{A})$, and into $[G]$. Thus, the way that Z_C is glued to $[G]$ does not depend on the choice of maximal parabolic P or Q.

Definition 13.4.3. The (partial) compactification $[G]^F$ described above, where F is a fan supported on $\mathfrak{a}^-$, will be called the *equivariant toroidal compactification* attached to F. If $F = \mathfrak{a}^-_{ss}$, it will be denoted by $[G]^{RBS}$, for *reductive Borel–Serre compactification.* (It is a partial compactification if $\mathfrak{a}_G$ is nontrivial.)

Remark 13.4.4. The equivariant toroidal compactifications are not related to the "toroidal compactifications" of locally symmetric spaces, defined by Ash–Mumford–Rapoport–Tai in [**AMRT75**]. However, the name is appropriate as they are have the local structure of toric varieties.

13.5. Reduction theory in the general case

We will now use the split case to deduce the general theorem of reduction theory. The main tool will be the compactification $[G]^{RBS}$ of Definition 13.4.3, and the fact that "it is defined over k", in the following sense:

Proposition 13.5.1. *Let G be a connected reductive group over k, and let $k \hookrightarrow l$ be a Galois extension, with Galois group Γ, such that G splits over l. Then, the action of Γ on $[G_l]$ extends continuously to the RBS compactification $[G_l]^{RBS}$.*

Proof. The basic ingredient of the proof is that *the boundary degeneration $[G]_B$ is "defined" over k*, in the sense that $[G_l]_B$ admits a canonical action of Γ. The reason is that the flag variety $\mathcal{B}$ of G is defined over k, even if G has no Borel subgroups over k. Recall that $B(l)\backslash G(\mathbb{A}_l) = \mathcal{B}(l) \times^{G(l)} G(\mathbb{A}_l)$, so it has a natural action of Γ. This action descends to $[G_l]_B$ (= the quotient of $\mathcal{B}(l) \times^{G(l)} G(\mathbb{A}_l)$ by the group bundle of unipotent radicals over $\mathcal{B}(\mathbb{A}_l)$).

The same holds for the partial flag varieties of the form $P\backslash G$, except that one has to group them into *associate classes*: Two conjugacy classes of parabolics are called *associate* if they have conjugate Levi subgroups. An associate class $\mathcal{A}$ of parabolics is a variety defined over k, such that $\mathcal{A}_l$ splits into a disjoint union of conjugacy classes $\mathcal{P}_i$ of parabolics. Thus, we have a well-defined action of Γ on $\sqcup_i \mathcal{P}_i(l)$, and from this we obtain a well-defined action of Γ on the union $\sqcup_i P_i(l)\backslash G(l)$, or the union of boundary degenerations $\sqcup_i [G_l]_{P_i}$.

Using these facts, it is not hard to see that the Γ-action extends to $[G_l]^{RBS}$; we leave the details to the reader. $\square$

If, now, G is a reductive group over k, and $k \hookrightarrow l$ is a Galois extension, with Galois group Γ, such that G splits over l, we have an embedding $[G_k] \hookrightarrow [G_l]^\Gamma \hookrightarrow ([G_l]^{RBS})^\Gamma$.

Proposition 13.5.2. *If $\mathcal{P}$ is a conjugacy class of parabolics in G_l, and $[G_k]$ has an accumulation point in the $\mathcal{P}$-cusp of $[G_l]^{RBS}$, then $\mathcal{P}$ has an element P defined over k. Moreover, there is a neighborhood V of the P-cusp of $[G_l]^{RBS}$ such that $\pi_G(\tilde{V}) \cap [G_k] = \pi_G(\tilde{V} \cap P(k)\backslash G(\mathbb{A}))$, where π_G is the map of (13.4.1.1).*

Proof. The statement is local over the P-cusp, so it is enough to replace "neighborhood of the cusp" by "neighborhood of a compact subset of the cusp". Let V to be any Γ-stable neighborhood in $[G_l]_P$ of a compact subset of accumulation points of $[G_k]$ in the P-cusp, satisfying the conclusion of the injectivity statement (I) of Theorem 13.4.2. If $[g] \in [G_k]$ is a point in $\pi_G(\tilde{V})$ by the injectivity property, and the stability of V, g under Γ, the preimage $\tilde{g}$ of g in $\tilde{V}$ is also Γ-stable. But

$$\tilde{g} \in P(l)g \in \mathcal{P}(l) \times^{G(k)} G(\mathbb{A}_k) \subset \mathcal{P}(l) \times^{G(l)} G(\mathbb{A}_l) = P(l)\backslash G(\mathbb{A}_l),$$

hence the Galois action on the subset $\mathcal{P}(l) \times^{G(k)} G(\mathbb{A}_k)$ mapping to $[G_k]$ comes entirely from its action on $\mathcal{P}(l)$ (more precisely, on $\mathcal{A}(l)$, where $\mathcal{A}$ is the union of associate classes to $\mathcal{P}$). Thus, the existence of a Γ-fixed point on $P(l)\backslash G(\mathbb{A}_l)$, mapping to $[G_k]$, implies that $\mathcal{P}(k) \neq \emptyset$, and $g \in [G_k]_P$. $\square$

Theorem 13.5.3. *Let G be a reductive connected group over k, and $P \subset G$ a (class of) parabolic(s).*

- **(C):** *$[G]$ is compact iff G is anisotropic (i.e., does not contain any split torus).*
- **(I):** *Every compact subset of the P-cusp in $\overline{[G]_P}$ has a neighborhood V such that $\pi_G|_{\tilde{V}}$ is injective, where $\tilde{V}$ is the preimage of V in $P(k)\backslash G(\mathbb{A})$.*
- **(S):** *Every $[A_G]$-invariant neighborhood of the degenerate P-cusp in $[G]_P$ (where A_G is the maximal split torus in the center of G) can be scaled to a neighborhood V such that $\pi_G|_{\tilde{V}}$ is surjective, where $\tilde{V}$ is as above.*

Proof. Again, having addressed the case of tori in Proposition 13.1.8, we will now assume that G is semisimple; the combination of the two to address the general case is left to the reader.

We start with (C): Let $k \hookrightarrow l$ be a Galois extension, with Galois group Γ, such that G splits over l. Then, we have a closed embedding $[G_k] \hookrightarrow [G_l]^\Gamma$. If $[G_k]$ is not compact, then it has a Γ-stable accumulation point in some boundary component (some P-cusp) of $[G_l]^{RBS}$. By Proposition 13.5.2, it must have a proper parabolic, which contradicts the assumption that G is anisotropic over k.

For (I), there is nothing to prove, since by embedding again $[G_k] \hookrightarrow [G_l]$, the statement follows from the corresponding statement for split groups, contained in Theorem 13.4.2.

Now we pass to (S). The same argument as in the split case reduces us to the case of split semisimple rank one. Thus, we assume that G has a unique class of proper parabolics P over k. Since P is minimal, by the statement (C) applied to its Levi quotient L, its cusp will be compact, and therefore the neighborhoods of the P-cusp coincide with the neighborhoods of the degenerate P-cusp. If, again, l is a splitting field as above, and V is any small neighborhood of the P-cusp in $[G_l]_P$ satisfying the conclusion of Proposition 13.5.2, the proposition shows that, on one hand, $[G_k] \smallsetminus \pi_G(\tilde V)$ is relatively compact, and on the other $P(k)\backslash G(\mathbb{A}) \cap \tilde V$ surjects onto $[G_k] \cap \pi_G(\tilde V)$. Thus, there is an $A_P(\mathbb{A})$-scaling V' of $V \cap [G_k]_P$ such that $\pi_G(\tilde V') = [G_k]$. $\square$

Now we draw some corollaries from Theorem 13.5.3: the finiteness of class numbers, and a more classical formulation, saying that $[G]$ can be covered by domains of a particular form, called *Siegel sets*.

Definition 13.5.4. A *fundamental domain* for the action of a discrete subgroup Γ on a locally compact group G is an open subset D of G such that no two points of D are in the same Γ-orbit, and such that $G = \cup\gamma \in \Gamma\gamma\bar D$.

A *fundamental set* Ω for $\Gamma\backslash G$ is a subset of G such that $\Gamma \cdot \Omega = G$ and the set $\{\gamma \in \Gamma | \gamma\Omega \cap \Omega \neq \emptyset\}$ is finite.

Definition 13.5.5. Let $P \subset G$ be a minimal parabolic subgroup, and fix a maximal compact subgroup $K \subset G(\mathbb{A})$ satisfying the Iwasawa decomposition $G(\mathbb{A}) = P(\mathbb{A})K$. A *Siegel set* is a subset of $G(\mathbb{A})$ of the form: $\Omega A_\epsilon K$, where:

- Ω is a compact subset of $P(\mathbb{A})$;
- $A_\epsilon \subset A_P(k_\infty)$ (or, in the number field case, $A_\epsilon \subset A_P(\mathbb{R})^0$, where $\mathbb{R} \hookrightarrow k_\infty$ is the diagonal embedding) is the subset of those elements t satisfying $|e^\alpha(t)| > \epsilon > 0$ for all positive roots α. [2]

Then, a corollary of Theorem 13.5.3 is:

Theorem 13.5.6. (1) *For every compact open subgroup $J \subset G(\mathbb{A}_f)$, the number of $G(k_\infty)$-orbits on $[G]/J$ is finite.*
(2) *There exists a Siegel set which is a fundamental set for $[G]$.*

Proof. Let $P \subset G$ be a minimal parabolic, and $K \subset G(\mathbb{A})$ a maximal compact subgroup satisfying the Iwasawa decomposition $G(\mathbb{A}) = P(\mathbb{A})K$. Using the Iwasawa decomposition, the question is reduced to the question of $P(k_\infty)$-orbits on $[P]/J_P$, where J_P is an open compact subgroup of $P(\mathbb{A}_f)$. If $P \to L \to L^{ab}$ are the Levi quotient of P and its abelianization, a torus, the problem is easily reduced from P to L by means of Proposition 13.1.3, from L to L^{ab} by means of the compactness of $[L']$, where L' is the derived subgroup of L (Statement (C) of Theorem 13.5.3), and the case of L^{ab} is Proposition 13.1.8.

For the second statement, first choose a compact subset $\Omega \subset P_0(\mathbb{A})$, such that $U(k)\Omega \supset U(\mathbb{A})\Omega$ (possible by Proposition 13.1.3), and $U(\mathbb{A})A_P(k_\infty)\Omega = G(\mathbb{A})$ (or, respectively, $U(\mathbb{A})A_P(\mathbb{R})\Omega = G(\mathbb{A})$ — this is possible by the compactness of $[L']$, and the finiteness of the class number for tori, Proposition 13.1.8, applied to L^{ab}.

The surjectivity, now, of a subset of the form $\Omega A_\epsilon K$ onto $[G]$ follows from Statement (S) of Theorem 13.5.3. $\square$

Remark 13.5.7. By Theorem 13.5.6, the $G(k_\infty)$-space $[G]/J$ is a finite disjoint union

$$\bigsqcup_i \Gamma_i\backslash G(k_\infty),$$

where $\Gamma_i = g_i^{-1}G(k)g_i \cap JG(k_\infty)$, for g_i ranging over a set of representatives of the $G(k_\infty)$-orbits, is a discrete *congruence subgroup*.

[2] More generally, one can specify different values of ϵ for different roots.

13.6. Weak and strong approximation.

Definition 13.6.1. We say that a (geometrically integral) variety X over k satisfies *weak approximation* if:

For every finite set of places S, $X(k)$ is dense in $X_S = \prod_{v \in S} X(k_v)$.

Equivalently, if:

$X(k)$ is dense in $\prod_v X(k_v)$

the product taken over all places.

We say that X has the property of weak approximation away from a finite set of places Σ if this property holds with the product taken over all places outside of Σ. For instance, if $\Sigma = \infty$ and an integral model (i.e. the structure of an $\mathfrak{o}$-scheme, where $\mathfrak{o}$ is the ring of integers in k) is given, then weak approximation outside of Σ means that for every finite set of finite places S, every integer N and every set of points $(x_v \in X(k_v))_{v \in S}$ we can find $x \in X(k)$ such that $x \equiv x_v \mod \mathfrak{p}_v^N$.

We have:

Theorem 13.6.2 (Kneser, Platonov)**.** *Let G be semisimple simply connected or adjoint. Then G satisfies weak approximation.*

Proof. See [**PR94**, Theorem 7.8]. □

There are many more examples of groups which satisfy weak approximation, for instance GL_n. (Proof: $\mathrm{GL}_{n,S}$ is open in $\mathrm{Mat}_{n,S}$ and carries the induced topology, so since Mat_n satisfies weak approximation, so does GL_n.) In fact, any split reductive group, being a rational variety (by the Bruhat decomposition), satisfies weak approximation. However, weak approximation can fail already for tori:

Example 13.6.3. Let $L = \mathbb{Q}(\sqrt{-1}, \sqrt{2})$, and let T be the kernel of the norm map $L^\times \to \mathbb{Q}^\times$, considered as an algebraic torus over $\mathbb{Q}$. Then, the closure of $T(\mathbb{Q})$ has index 2 in $T(\mathbb{Q}_2)$; see [**PR94**, p.423].

Definition 13.6.4. We say that a variety X satisfies *strong approximation* away from a finite set of places Σ if:

$X(k)$ is dense in $X^\Sigma = X(\mathbb{A}^\Sigma)$.

Sometimes if $\Sigma = \infty$ we say that X satisfies strong approximation without mentioning Σ. Hence, strong approximation (away from ∞) is a strengthening of the statement "class number = 1". Notice that the above condition is much stronger than being dense in $\prod_v (k_v)$, because the topology on the adeles is finer than the induced topology from $\prod_v (k_v)$. For instance, if $G = \mathrm{GL}_n$ and $\Sigma = \infty$ then the property reads: For every set S of finite places and for all $(x_v \in k_v)_{v \in S}$ *there exist S-integers* in $k^\times$ (i.e. elements of $k^\times \cap \prod_{v \in S \cup \infty} k_v^\times \prod_{v \notin S \cup \infty} \mathfrak{o}_v^\times$) which approximate $(x_v)_{v \in S}$.

A slightly weaker version of the following theorem was proven by Kneser:

Theorem 13.6.5 (Platonov, Prasad)**.** *Let G be an algebraic group over a global field k, and let Σ be a finite set of places. Then, G satisfies strong approximation outside of Σ if and only if G is connected and simply connected (in particular, semisimple), and there is no simple component $G_1 \subset G$ over k such that $G_1(k_\Sigma)$ is compact.*

Proof. See [**PR94**, §7.4] for number fields, and for the "only if" statement. Prasad proved the function field case in [**Pra77**]. Let us only outline an elementary proof in the case of SL_n, for any nonempty set of places Σ: In this case, for any place v, the group $\mathrm{SL}_n(k_v)$ is generated by the elementary subgroups $I + k_v E_{ij}$ (where E_{ij} is the matrix with 1 in the (i,j)-th entry, and zero otherwise), and by the case of the additive group, Proposition 13.1.3, $I + kE_{ij}$ is dense in $I + \mathbb{A}^\Sigma E_{ij}$. Therefore, $\mathrm{SL}_n(k)$ is dense in $\mathrm{SL}_n(\mathbb{A}^\Sigma)$. □

Notice that this statement implies, in particular, that $\mathrm{SL}_n(\mathbb{Z})$ is dense in $\mathrm{SL}_n(\widehat{\mathbb{Z}})$, i.e. the map: $\mathrm{SL}_n(\mathbb{Z}) \to \mathrm{SL}_n(\mathbb{Z}/n)$ is surjective for every n. Such a result is certainly not true for the multiplicative group, for instance: $\mathbb{Z}^\times$ does not surject onto $(\mathbb{Z}/5)^\times$.

Example 13.6.6. Here is an example for the failure of strong approximation: Assume that G is a linear algebraic group, and Σ is a finite, nonempty set of places such that $G(k_\Sigma)$ is compact. Since $G(k)$ is discrete in $G(\mathbb{A}) = G(\mathbb{A}^\Sigma) \times G(k_\Sigma)$, and $G(k_\Sigma)$ is compact, it follows that $G(k)$ is discrete in $G(\mathbb{A}^\Sigma)$, and therefore it is not dense.

13.7. The class number of a reductive group

Definition 13.7.1. Let G be a linear algebraic group over the ring of integers $\mathfrak{o}$ of a number field k, or over the integers of a function field away from a non-empty set of places that we will denote as k_∞. The cardinality of the set $G(k)\backslash G(\mathbb{A}_f)/\prod_{v\neq\infty} G(\mathfrak{o}_v)$ is the *class number* of G.

The class number is always finite, by Theorem 13.5.6. Here, we will use strong approximation (definitely a deeper fact than reduction theory) in order to get a more precise calculation of the class number in several cases.

Proposition 13.7.2. *Assume that the derived group G' of G is simply connected, and does not contain any factor G'_1 over k with $G'_1(k_\infty)$ compact. Let G^{ab} be the abelianization of G, $K \subset G(\mathbb{A}_f)$ a compact open subgroup, and K^{ab} the image of K in G^{ab}.*

Then, the double quotients $G(k)\backslash G(\mathbb{A}_f)/K$ and $G^{ab}(k)\backslash G^{ab}(\mathbb{A}_f)/K^{ab}$ are in bijection, under the natural map $G(\mathbb{A}_f) \to G^{ab}(\mathbb{A}_f)$.

In particular, the class numbers of G and G^{ab} coincide.

Proof. By Theorem 9.3.2, the Galois cohomology $H^1(k_v, G')$ is trivial for any finite place v, therefore the map $G(\mathbb{A}) \to G^{ab}(\mathbb{A})$ is surjective.

Similarly, by Theorem 9.4.3, the Galois cohomology $H^1(k, G')$ injects into $H^1(k_\infty, G) := \prod_{v|\infty} H^1(k_v, G)$ (actually, surjects, by Proposition 9.4.4, but we won't use that).

Hence, we have exact sequences $G(k) \to G^{ab}(k) \to H^1(k_\infty, G')$ and $G(\mathbb{A}) \to G^{ab}(\mathbb{A}) \to H^1(k_\infty, G')$, compatible with the embeddings $G(k) \to G(\mathbb{A})$ and $G^{ab}(k) \to G^{ab}(\mathbb{A})$, and, in particular, since $G^{ab}(\mathbb{A}_f)$ lies in the kernel of the map to $H^1(k_\infty, G')$, the images of $G(k)$ and $G^{ab}(k)$ in $G^{ab}(\mathbb{A}_f)$ are equal.

Thus, the fiber [...]

Therefore, the quotients $G(k)\backslash G(\mathbb{A}_f)/K$ and $G^{ab}(k)\backslash G^{ab}(\mathbb{A}_f)/K^{ab}$ are in bijection. □

Notice that this gives a group structure to the double quotient $G(k)\backslash G(\mathbb{A}_f)/K$ and, in particular, we can talk about the class group of G. More generally,

Proposition 13.7.3. [**PR94**, Proposition 8.8]

Proof. □

Definition 13.7.4. If G satisfies the conditions of Proposition 13.7.3, the double quotient $G(k)\backslash G(\mathbb{A}_f)/\prod_{v\neq\infty} G(\mathfrak{o}_v)$, with the group structure inherited from this proposition, is the *class group* of G.

For more general reductive groups, we can analyze their class number/group with the help of *z-extensions*:

Definition 13.7.5. A *z-extension* of a reductive group G is a short exact sequence of algebraic (necessarily reductive) groups

$$1 \to T \to \tilde{G} \to G \to 1,$$

where T is an induced torus (Definition 7.1.3), and the derived group of G' is simply connected.

Proposition 13.7.6. *Any reductive group over a field k admits a z-extension.*

Proof. □

[To be continued: describe the class group of G in terms of the class group of $\tilde{G}^{ab}$, where $\tilde{G}$ is a z-extension.]

13.8. Tamagawa numbers

13.8.1. Motivating example I. Consider the group $\mathrm{SL}_2/\mathbb{Q}$, on which we have an invariant algebraic differential $\omega = dxdydz/x$, where we are realizing SL_2 as matrices of the form $\begin{pmatrix} x & y \\ z & \frac{1+yz}{x} \end{pmatrix}$. As such we obtain an invariant measure $\mu_\infty = |\omega|$ on the Lie group $\mathrm{SL}_2(\mathbb{R})$. Since the quotient $\mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{SL}_2(\mathbb{R})$ is of finite volume, we may try to consider the measure induced by μ_∞ on it, and compute its mass. And, it turns out we get the number $\zeta(2) = \pi^2/6$.

On the other hand we may consider the p-adic analogues: consider the group $\mathrm{SL}_2(\mathbb{Q}_p)$, and we form $\mu_p = |\omega|_p = |dxdydz/x|_p$ (in the following we drop the subscript p for simplicity), which we understand as a real-valued measure by setting $dx(\mathbb{Z}_p) = 1$. Now, we compute the mass of $\mathrm{SL}_2(\mathbb{Z}_p)$ under μ_p. Consider the subgroup $1 + p\mathrm{Mat}_2(\mathbb{Q}_p)$: it has mass $1/p^3$ by our measure since it coincides with $(1+p\mathbb{Z}_p)\times(p\mathbb{Z}_p)^2$ in coordinates (x, y, z). Also, this subgroup is open compact, with number of coset representatives equal to $|\mathrm{SL}_2(\mathbb{F}_p)| = p(p^2-1)$. So we conclude that $\mu_p(\mathrm{SL}_2(\mathbb{Z}_p)) = 1 - p^{-2}$, seemingly miraculous to coincide with the reverse of the Euler factor of $\zeta(2)$ at p! Hence, we see that $\mu_\infty(\mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{SL}_2(\mathbb{R}))\cdot\prod_p \mu_p(\mathrm{SL}_2(\mathbb{Z}_p)) = 1$.

A nice way to (partly) reformulate our computation is that, when we use the measure $\prod_v \mu_v$ on $\mathrm{SL}_2(\mathbb{A})$, the mass of the compact quotient $\mathrm{SL}_2(\mathbb{Q})\backslash\mathrm{SL}_2(\mathbb{A}) \simeq \mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{SL}_2(\mathbb{R}) \times \prod_p \mathrm{SL}_2(\mathbb{Z}_p)$ is 1. This example is in fact almost general enough to the classical definition of the Tamagawa measure and the attached Tamagawa number, being the generalization of $\prod_v \mu_v$ and the mass of $G(\mathbb{Q})\backslash G(\mathbb{A})$ when G is semisimple.

13.8.2. Motivating example II. Now consider the group $\mathrm{SO}_2/\mathbb{Q}$, or more precisely the special orthogonal group for the rational quadratic form $Q(x,y) = x^2+y^2$. We realize SO_2 as matrices of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ where $a^2+b^2=1$, and one can check that the algebraic differential $\omega = da/b$ is invariant.

Now we can imitate the previous example: for real points, $\mathrm{SO}_2(\mathbb{R})$ is already compact, and the mass is

$$\int_{(a,b)\in S^1} \left|\frac{da}{b}\right|_\infty = 2\int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} = 2\pi.$$

Next for a prime $p \neq 2$, we consider the group $\mathrm{SO}_2(\mathbb{Q}_p)$ and $\mu_p = |\omega|_p = |da/b|_p$, with the group being identified as $\{(a,b) \in \mathbb{Q}_p^2 : a^2+b^2 = 1\}$. We use again the trick of reduction (which is available as the quadratic form is integral): the map $\mathrm{SO}_2(\mathbb{Z}_p) \to \mathrm{SO}_2(\mathbb{F}_p)$ is surjective, as when $p \neq 2$ the derivatives of Q doesn't vanish and we may employ a Hensel-lemma type of argument. Furthermore, the kernel of the reduction map is same as $\mathrm{SO}_2(\mathbb{Z}_p) \cap 1 + p\mathrm{Mat}(\mathbb{Z}_p)$, which is parametrized by $b \in p\mathbb{Z}_p$, and hence on it $|da/b|_p = |-db/a|_p = |db|_p$, and hence the mass of the kernel is equal to $1/p$. Next we count $|\mathrm{SO}_2(\mathbb{F}_p)|$. When $\sqrt{-1} \in \mathbb{F}_p$ we see the group is the same as $\{(a,b) \in \mathbb{F}_p : (a+ib)(a-ib) = 1\}$, which is in bijection with $\mathbb{F}_p^\times$ by the map $(a,b) \mapsto a+ib$. So the number is $p-1$. When $\sqrt{-1} \notin \mathbb{F}_p$ denote $q = p^2$. We then have an embedding

$$\mathrm{SO}_2(\mathbb{F}_p) \to \mathbb{F}_q^\times \qquad (a,b) \longmapsto a+ib,$$

the image being identified with $u \in \mathbb{F}_q^\times$ having norm 1. Now again using $p \neq 2$ we can show the norm map $\mathbb{F}_q^\times \to \mathbb{F}_p^\times$ is surjective, and hence $|\mathrm{SO}_2(\mathbb{F}_p)| = \frac{p^2-1}{p-1} = p+1$. So in summary, when $p \neq 2$ the mass of $\mathrm{SO}_2(\mathbb{Z}_p)$ is $1-\chi(p)/p$, where $\chi(p)$ is the Legendre symbol $(\frac{-1}{p})$.

At $p=2$, one can carry out a small computation to verify that the mass is $1 = 1/2 \cdot 2$ again using reduction (albeit not surjective). Thus we did all the local computations.

Finally to put everything together, we see that, while $\prod_{p\neq 2}(1-\chi(p)/p)$ is not absolutely convergent, we may "equate" it to the reverse of $1-1/3+1/5-1/7+\cdots = \pi/4$ by the formula of Leibniz. And by multiplying with the mass at ∞ we get 8. To convert this to our reformulation as before, it asserts that, if we define the *Tamagawa measure* $\mu = \frac{1}{L(1,\chi)}\mu_\infty \cdot \prod_p L_p(1,\chi)\mu_p$ to ensure the convergence, then it becomes a measure on $\mathrm{SO}_2(\mathbb{A})$, and the mass of $\mathrm{SO}_2(\mathbb{Q})\backslash\mathrm{SO}_2(\mathbb{A}) \simeq \mathrm{SO}_2(\mathbb{R})/\mu_4 \times \prod_p \mathrm{SO}_2(\mathbb{Z}_p)$ is 2.

13.8.3. Volume forms and measures. To a top algebraic differential on a linear algebraic group we want to attach a measure on its points over a local field.

Definition 13.8.4. The *standard Haar measure* dx on $\mathbb{A}$ is the measure that assigns volume 1 to the quotient $\mathbb{A}/k$. The *standard multiplicative Haar measure* on $\mathbb{A}^\times$ is the measure $\frac{dx}{|x|}$.

Remark 13.8.5. The standard Haar measure on $\mathbb{A}$ can be factorized as $\prod_v dx_v$, where dx_v is a Haar measure on k_v. Although there is no "canonical" Haar measure on k_v, there is a standard choice that one can make, which at every non-Archimedean completion with discriminant D_v over the base field ($\mathbb{Q}_p$ or $\mathbb{F}_p(t)$)

assigns mass $|D_v|^{-\frac{1}{2}}$ to the ring of integers $\mathfrak{o}_v$, at real places is the usual Lebesgue measure, and at complex places is twice the usual Lebesgue measure.

Definition 13.8.6. Let F be a local field, endowed with a Haar measure dx, and let X be a smooth algebraic variety of dimension n over F. Let ω be a volume form on X. The *absolute value of the volume form* ω is the measure $|\omega|$ which, in any open chart $(U,(x_i)_i)$, that is, an open subset (in the F-topology) $U \subset X(F)$ endowed with a set of algebraic coordinates $x_1,\dots,x_n$, if ω is written as $f(\underline{x})dx_1\wedge\cdots\wedge dx_n$, then $|\omega||_U$ is equal to the measure $|f(\underline{x})|dx_1\cdots dx_n$.

By "a set of algebraic coordinates" we mean a set of algebraic functions $x_1,\dots x_n$, which are regular at all points of U, and such that the volume form $\omega' = dx_1\wedge\cdots\wedge dx_n$ is nowhere vanishing on U; hence, the quotient $\frac{\omega}{\omega'}$ is a rational function that is regular at all points of U. The definition, of course, presupposes that the measure $|f(\underline{x})|dx_1\cdots dx_n$ does not depend on the coordinates chosen, which is easily checked. See also Weil's [**Wei82**].

When X is a smooth scheme over the ring of integers $\mathfrak{o}$ of a non-Archimedean field, with "standard" choices of measures, this measure is just counting points over the residue field:

Lemma 13.8.7. *Suppose that X is a smooth scheme of dimension n over the ring of integers $\mathfrak{o}$ of a non-Archimedean field F, with residue field $\mathbb{F}_q$. If ω is a volume form on X that is defined over $\mathfrak{o}$ and residually non-vanishing, then*

$$\int_{X(\mathfrak{o})} |\omega| = q^{-n}\#X(\mathbb{F}_q),$$

when $|\omega|$ is defined with respect to the measure on F that assigns mass 1 to $\mathfrak{o}$.

Proof. By smoothness (Hensel's lemma), the map $X(\mathfrak{o}) \to X(\mathbb{F})$ is surjective, so we need to show that the preimage of every point in $X(\mathbb{F}_q)$ has mass equal to q^{-n}.

By smoothness, locally on X, there is an embedding $X \hookrightarrow A^r_{\mathfrak{o}}$, where $A^r_{\mathfrak{o}}$ denotes affine r-space over $\mathfrak{o}$ (whose coordinates we will denote by x_i), and a set of $r-n$ equations $f_{n+1},\dots,f_r$, such that $df_{n+1}\wedge\cdots\wedge f_r$ is nowhere vanishing, such that X is the zero locus of the f_i's. By Hensel's lemma, the map $X(\mathfrak{o}) \to X(\mathbb{F})$ is surjective, and for any point $x = (x_i)_i \in X(\mathfrak{o})$, the points with coordinates $(x_1+\mathfrak{p}, x_2+\mathfrak{p},\dots,x_n+\mathfrak{p})$ lift uniquely to $X(\mathfrak{o})$. This defines a bijection from $\mathfrak{p}^n$ to the residual neighborhood of x; moreover, ω can be written as a nowhere vanishing, integral multiple of $dx_1\wedge\cdots\wedge dx_n$, hence $|\omega| = dx_1\cdots dx_n$. Therefore, the volume of this neighborhood is $\int_{\mathfrak{p}^n} dx_1\cdots dx_n = q^{-n}$. □

Now, we discuss volume forms on algebraic groups.

Lemma 13.8.8. *If G is an algebraic group over a field k, then G has a unique, up to scaling, nonzero right-invariant volume form ω_G^R, and a unique, up to scalar, nonzero left-invariant volume form ω_G^L. There is a character $\partial_G : G \to \mathbb{G}_m$ such that, if ω_G^R is a right-invariant volume form on G, then $\omega_G^R(ag) = \partial_G(a)\cdot\omega_G^R(g)$, or equivalently, ∂_G is the quotient $\frac{\omega_G^R}{\omega_G^L}$ between a right- and a(n appropriately scaled) left-invariant volume form.*

If G is reductive, unipotent, or projective (i.e., an abelian variety), then $\partial_G = 1$, i.e., left- and right-invariant volume forms coincide.

Proof. Take a nonzero vector from $\wedge^{\dim(G)}\mathfrak{g}^*$ at the identity, and translate it by right- or left-multiplication. As such we obtain a left- (resp. right-)invariant nonvanishing global section ω_G^L (resp. ω_G^R) trivializing the sheaf of volume forms: $\wedge^{\dim(G)}\Omega_{G/k} = \mathcal{O}_G \cdot \omega_G^L = \mathcal{O}_G \cdot \omega_G^R$. Immediate from this:

- The quotient ω_G^L/ω_G^R is an invertible global function.
- The set of left-(right-) invariant differentials is a one dimensional k-subspace.

The action of G by left translations on the one-dimensional space of right-invariant sections of $\wedge^{\dim(G)}\Omega_{G/k}$ defines the algebraic character $\partial_G : G \to \mathbb{G}_m$. It is easily checked that ∂_G is the character of the left coadjoint representation of G on the one-dimensional space $\bigwedge^{\mathrm{top}} \mathfrak{g}^*$.

Unipotent algebraic groups do not have nontrivial algebraic characters, by the Jordan decomposition. For an abelian variety there is no nontrivial character since it only has constant global functions. In the reductive case, because roots come in opposite pairs, the determinant of the coadjoint representation is trivial on semisimple elements, hence (by density of semisimple elements) on the entire group. □

Definition 13.8.9. A left- or right- invariant volume form on an algebraic group G is called a (left- or right-) *Haar volume form.* The character ∂_G of Lemma 13.8.8 (or, abusing the notation when G is over F, its absolute value on $G(F)$, $\delta_G(g) = |\partial_G(g)|$) is called the *modular character* of G.[3] The group $G(F)$ is *unimodular* if $\delta_G = 1$.

13.8.10. Definition of Tamagawa measure. An interesting feature of adelic groups is that they come with *canonical* measures. The idea of the definition is as follows: Given an algebraic group G over a global field k, a (left- or right-)invariant volume form ω gives rise to a measure $\mu_v = |\omega|_v$ at every place, and we will set $\mu = \prod_v \mu_v$ on $G(\mathbb{A})$, the Tamagawa measure. If the definition makes sense, it will be independent of the choice of ω, because any other invariant volume form is of the form $a\omega$ with $a \in k^\times$, and $|a\omega| = |a| \cdot |\omega| = |\omega|$, by the product formula. Still, this definition needs some caution, because of the Euler product; we need to make sure that $\prod_v \mu_v$ is finite on compact subsets of $G(\mathbb{A})$. This amounts to making sense of the partial Euler product $\prod_{v\notin S} G(\mathfrak{o}_v)$, where S is a finite number of places, including the Archimedean ones and we are fixing a model for G over the S-integers $\mathfrak{o}_S$. Taking S large enough, we may assume that G is a reductive group scheme over $\mathfrak{o}_S$.

If G is a reductive group scheme over $\mathfrak{o}_v$, the volume of $G(\mathfrak{o}_v)$ is expressed in terms of the *motive of* G,[4] see [**Gro97**]:

Definition 13.8.11. Let G be a connected reductive group defined over a field k, with absolute Galois group Γ. Let T be the universal Cartan of G (Definition 7.5.5), with absolute Weyl group W and absolute cocharacter group $X_{*,k^s}(T)$. Set $\mathfrak{c} = \mathfrak{t} /\!\!/ W$, where $\mathfrak{t} = X_{*,k^s}(T) \otimes \mathbb{Q}$, where T is a maximal torus, and notice that it contains a distinguished point $0 \in \mathfrak{c}$, the image of $0 \in \mathfrak{t}$.

We define $M_G^\vee(1) =$ the tangent space $V = T_0\mathfrak{c}$ as a vector space, considered as a graded $\mathbb{Q}$-vector space with the grading descending from the $\mathbb{G}_{m,\mathbb{Q}}$-action on

[3] The convention in Bourbaki is the opposite one: left over right; however, right over left is the standard definition of the modular character in more recent literature.

[4] For everything that follows, the reader can interpret "motive" as "ℓ-adic Galois representation", for the decomposition group at a non-Archimedean place of residual degree $q = p^r$, $p \neq \ell$. The Tate motive $\mathbb{Z}(1)$ corresponds to the module $\lim_{\leftarrow n} \mu_{\ell^n}$ on which the Frobenius morphism acts as multiplication by q

$\mathfrak{t}^*$, and as a Γ-module by $M_G^\vee(1) = \bigoplus_d V_d(d)$, i.e., the twist of the natural Γ-action on V by the d-th power of the Tate motive on the d-th graded piece. We define the *motive of* G to be the motive M_G, defined as the notation suggests, i.e., $M_G = (M_G^\vee(1))^\vee(1) = \bigoplus_d V_d^\vee(1-d)$.

Proposition 13.8.12. *Let G be a reductive group scheme over the valuation ring $\mathfrak{o}$ of a local non-Archimedean field F with residue field $\mathbb{F}_q$, and let ω be an invariant volume form on G, defined over $\mathfrak{o}$ and residually non-vanishing. Then,*

$$\int_{G(\mathfrak{o})} |\omega| = |G(\mathbb{F}_q)|/q^{\dim(G)} = L(M_G^\vee(1))^{-1} = \det(I - \mathcal{F}|M^\vee(1)), \tag{13.8.12.1}$$

where $\mathcal{F}$ is the geometric Frobenius morphism (acting by q^{-1} on $\mathbb{Q}(1)$).

Note that, under the assumption that G is reductive over $\mathfrak{o}$, the action of Γ on $M^\vee(1)$ is unramified.

Proof. The equality of the integral with $|G(\mathbb{F}_q)|/q^{\dim(G)}$ is Lemma 13.8.7, and the number of points over the finite field is a result of Steinberg [**Ste68**, p79]. [This is an important result with number-theoretic consequences; its proof should be added.] $\square$

Proposition 13.8.13. *Let G be a connected, reductive group over the global field k, and let ω be a nonzero right-invariant volume form on G over k. Fix a finite set S of places, including the Archimedean ones, a reductive model for G over the S-integers $\mathfrak{o}_S$, such that the form ω is integral and residually nonvanishing, and a factorization $dx = \prod_v dx_v$ of the standard Haar measures on $\mathbb{A}$ (Definition 13.8.4) such that $dx_v(\mathfrak{o}_v) = 1$ for $v \notin S$. Then:*

(1) *The partial L-function*
$$L^S(M^\vee(1), s) = \prod_{v \notin S} \det(I - q_v^{-s}\mathcal{F}_v|M^\vee(1))^{-1}$$
admits a meromorphic continuation to $s = 0$, with pole of order equal to the k-character group of G.

(2) *The Haar measure μ_{Tam} on $G(\mathbb{A})$, which assigns to an open neighborhood $U = \prod_v U_v$ of the identity, with $U_v = G(\mathfrak{o}_v)$ for $v \notin S$ the value*
$$\mu_{Tam}(U) = \frac{\prod_{v\in S} |\omega|_v(U_v)}{L^S(M^\vee(1))^*},$$
where $L^S(M^\vee(1))^*$ is the leading coefficient of the Laurent expansion of $L^S(M^\vee(1), s)$ at $s = 0$, is independent of the volume form ω, the set of places S, or the integral model of G over $\mathfrak{o}_S$.

Proof. By the definition of $M^\vee(1)$, the L-function $L(M^\vee(1), s)$ is a product of Artin L-functions evaluated at the points $s + d$, where d varies over the fundamental invariants of the group; thus, $s = 0$ corresponds to a point in the domain of convergence of the Euler product, except for the factor with $d = 1$, which corresponds to the linear invariants $\mathfrak{z} \hookrightarrow \mathfrak{c}$, where $\mathfrak{z} \subset \mathfrak{t}$ is spanned by the cocharacters into the center of G. The order of pole is equal to the multiplicity of the trivial representation in $\mathfrak{z}$, which is equal to the order of the k-character group of G.

The second statement follows from Proposition 13.8.12 and the product formula: adding more places to the set S will multiply the numerator and the denominator by the same factor; this depends only on the reductivity of the integral model away from S; and, any other invariant volume form is equal to $c\cdot\omega$ for some $c\in k^\times$; for it to be also integral and residually nonvanishing over $\mathfrak{o}_S$, we must have $c\in\mathfrak{o}_S^\times$, and then the product formula shows that $\prod_{v\in S}|c\omega|_v=\prod_{v\in S}|\omega|_v$. □

Definition 13.8.14. The *Tamagawa measure* on $G(\mathbb{A})$ (or on $[G]$) is the measure μ_{Tam} of Proposition 13.8.13 (resp., its descent to $[G]$).

The *Tamagawa number* of G is the number

$$\tau(G)=\mathrm{AvgVol}([G],\mu_{Tam})=\lim_{c\to\infty}\frac{\mu_{Tam}(\log^{-1}(cB))}{\mathrm{Vol}(cB)}, \tag{13.8.14.1}$$

where $\log:[G]\to\mathfrak{a}_G=\mathrm{Hom}(G,\mathbb{G}_m)^\vee\otimes\mathbb{R}$ is the logarithmic map defined as in (13.2.6.1), B is the unit ball of any norm on the real vector space $\mathfrak{a}_G$, and $\mathrm{Vol}(cB)$ is taken with respect to the measure on $\mathfrak{a}_G$ which assigns covolume 1 to the lattice $\mathrm{Hom}(G,\mathbb{G}_m)^\vee$.

Remark 13.8.15. This definition is usually expressed in terms of the volume of $[G]^1=\{g\in[G]|\forall\chi\in\mathrm{Hom}(G,\mathbb{G}_m),|\chi(g)|=1\}$, which one endows with the Haar measure μ'_{Tam} such that the measure μ_{Tam} on $[G]$ factorizes as

$$\int_{[G]}\mu_{Tam}=\int_{\mathfrak{a}_G}(\int_{[G]^1}\mu'_{Tam})dx,$$

with dx the same measure on $\mathfrak{a}_G$ as in the definition, see [**Oes84**]. In the case of the function field of a curve over a finite field with q elements, the image of $[G]$ in $\mathfrak{a}_G$ is not the entire space, but just the lattice $\log q\cdot\mathrm{Hom}(G,\mathbb{G}_m)^\vee$, and the above integral over $\mathfrak{a}_G$ should be replaced with a sum over that lattice, multiplied by $\log q$. The definition with "average volume" unifies the number field and function field cases, and provides a more conceptual way to understand the regularization procedure, whereby the infinite $L^S(M^\vee(1),0)$ is replaced by its leading coefficient $L^S(M^\vee(1))^*$: formally, we are computing the volume of *the entire space* $[G]$ with respect to a measure defined using the infinite quantity $L^S(M^\vee(1),0)$. The "infinities" of the space $[G]$ and this quantity are "of the same order", and cancel each other.

13.8.16. The Tamagawa number conjecture.

Theorem 13.8.17 (Conjecture of Weil, theorem of Langlands-Lai-Kottwitz)**.** *For any simply connected semisimple linear algebraic group G over a global field, $\tau(G)=1$.*

Proof. The proof of Weil conjecture over number fields, to name the most prominent contributions, was started by Langlands using Eisenstein series, generalized by King Fai Lai, and completed by Kottwitz in [**Kot88**]. Strictly speaking, the proof was not fully finished by Kottwitz's Annals paper, as it relies on the the Hasse principle due to works of Kneser and Harder, who proved it only for groups without E_8 factors. Its complete resolution only came after Chernousov proved the Hasse principle for E_8 in his 1989 paper. Over function fields, a completely different proof was given by Gaitsgory and Lurie [**GL**]. □

As a *corollary*, we state the following.

Theorem 13.8.18 (Ono-Sansuc)**.** *For any semisimple linear algebraic group G over a global field E, let $\tilde{G}\to G$ be its simply connected cover over the same field* *and M the kernel. Then*

$$\tau(G) = \frac{|X^*(M)^{\Gamma_E}|}{|Sha(X^*(M))|} = \frac{|\mathrm{Pic}(G)|}{|Sha(G)|},$$

where for a Γ_E-module A (where $\Gamma_E = Gal(\bar{E}/E)$ for a separable closure $\bar{E}$), $Sha(A)$ denotes the obstruction group $\ker(H^1(E, A) \to \prod_{v:\ places\ of\ E} H^1(E_v, A))$*, and $Sha(G) = Sha(G(\bar{E}))$.*

Proof. That the first equality is a corollary of the Tamagawa number conjecture is proved by Ono in [**Ono65**]. The second equality is due to Sansuc in [**San81**]. In Kottwitz's final paper [**Kot88**] the formula takes a slightly different form, and its relation to the formula of Sansuc is explained in [**Kot84**]. Of course, we should also be able to specialize to 13.8.17. And this follows from Sansuc's formula and

- $\mathrm{Pic}(G) = 1$ if G is simply connected [**Mil17**, Cor. 18.24].
- The Hasse principle mentioned above, due to Kneser, Harder, Chernousov.

□

13.8.19. Application: The Smith–Minkowski–Siegel mass formula for quadratic forms.

Definition 13.8.20. Let R be a commutative ring. A *quadratic module* over R is an R-module Λ equipped with a *quadratic form*, i.e., a map $q : \Lambda \to R$ such that

(1) The map $\Lambda \times \Lambda \to R$ given by $(\lambda, \lambda') \mapsto q(\lambda+\lambda') - q(\lambda) - q(\lambda')$ is R-bilinear.
(2) $q(r\lambda) = r^2 q(\lambda)$ for any $\lambda \in \Lambda$ and $r \in R$.

We will consider quadratic modules as a groupoid, that is, a morphism of quadratic modules (Λ_1, q_1), (Λ_2, q_2) is an isomorphism of R-modules $T : \Lambda_1 \xrightarrow{\sim} \Lambda_2$ with $T^* q_2 = q_1$. The automorphism group of of (Λ, q) is its *orthogonal group*, denoted by $\mathrm{O}_q(R)$.

Remark 13.8.21. The base change $\Lambda \otimes_R R'$ of a quadratic module (Λ, q) to an R-algebra R' is a quadratic module. When R is finitely generated, the functor $R' \to \mathrm{O}_q(R')$ is represented by a closed subgroup scheme of GL_Λ over R.

We will be writing $q_1 \overset{R'}{\sim} q_2$ to denote that two quadratic forms are isomorphic over an R-algebra R'.

From now on we work, without extra mentioning, with quadratic forms on $\mathbb{Z}$-lattices, i.e., finitely generated free $\mathbb{Z}$-modules. The lattice will sometimes be implicit in the notation.

Definition 13.8.22. The *genus* of a quadratic form q over $\mathbb{Z}$ is the class of all quadratic forms q' that $q \overset{\mathbb{Z}_p}{\sim} q'$ for all primes p and $q \overset{\mathbb{R}}{\sim} q'$.

We will consider the genus as a groupoid, as well, with respect to $\mathbb{Z}$-equivalence.

Theorem 13.8.23 (Hasse principle)**.** *Let q and q' be two quadratic forms over $\mathbb{Q}$, then q is equivalent to q' over $\mathbb{Q}$ if and only if q and q' are equivalent over $\mathbb{Q}_p$ for all primes p and equivalent over $\mathbb{R}$.*

Proof. See [**Ser73**, p.44]. Note that this is not a direct corollary of the general Hasse principle of Theorem 9.4.3, since □

Remark 13.8.24. Suppose q and q' are in the same genus. Notice that since q and q' are equivalent over $\mathbb{Z}_p$, then they are also equivalent over $\mathbb{Q}_p$ through extension of scalars. Then by Hasse principle, they are also equivalent over $\mathbb{Q}$.

Proposition 13.8.25. *Fix an integral quadratic form (Λ, q) and let $G = O_q$ be its orthogonal group. For any (Λ', q') in the genus of q, there exists, by definition, $\alpha \in Isom_{\hat{\mathbb{Z}}\times\mathbb{R}}(\Lambda, \Lambda')$ such that $q = \alpha^* q$. By the Hasse principle, there also exists $\beta \in Isom_{\mathbb{Q}}(\Lambda, \Lambda')$ such that $q = \beta^* q'$. The composition $\gamma = \beta^{-1} \circ \alpha$ is then an element of $G(\mathbb{A})$.*

The coset of γ in $G(\mathbb{Q})\backslash G(\mathbb{A})/G(\hat{\mathbb{Z}} \times \mathbb{R})$ does not depend on choices, and the map $q' \mapsto [\beta^{-1} \circ \alpha]$ induces an equivalence of groupoids[5]

(13.8.25.1)
$$\{\textit{genus of } q\} \longleftrightarrow G(\mathbb{Q})\backslash G(\mathbb{A})/G(\hat{\mathbb{Z}} \times \mathbb{R}).$$

Proof. Since $q = \alpha^* q'$ and $q = \beta^* q'$, we get $q = \gamma^* q$, so $\gamma \in G(\mathbb{A})$. Any other isomorphism between q' and q over $\hat{\mathbb{Z}}\times\mathbb{R}$, it is given by $\alpha\circ\sigma$ for some $\sigma \in G(\hat{\mathbb{Z}}\times\mathbb{R})$. Similarly, any isomorphism between q' and q over $\mathbb{Q}$ is given by $\beta \circ \sigma'$ for some $\sigma' \in G(\mathbb{Q})$. So, the class of $\beta^{-1}\alpha$ in $Y := G(\mathbb{Q})\backslash G(\mathbb{A})/G(\hat{\mathbb{Z}} \times \mathbb{R})$ is well-defined, defining a map m from the genus of q to the set Y.

Let us prove that the map m is a bijection of equivalence classes. For injectivity, suppose first that $m(\Lambda', q')$ is the coset of $1 \in G(\mathbb{A})$. This means that, by choosing α and β appropriately, we can ensure that $\alpha = \beta \in \mathrm{Isom}_{\hat{\mathbb{Z}}\times\mathbb{R}}(\Lambda, \Lambda') \cap \mathrm{Isom}_{\mathbb{Q}}(\Lambda, \Lambda') = \mathrm{Isom}_{\mathbb{Z}}(\Lambda, \Lambda')$, hence $q' \overset{\mathbb{Z}}{\sim} q$.

The general injectivity statement reduces to this case, by noticing that an element β as above gives rise to an isomorphism $\beta^* : G' \xrightarrow{\sim} G$ over $\mathbb{Q}$, where $G' = \mathrm{O}_{q'}$, hence to a bijection (actually, equivalence of groupoids) between the corresponding double coset spaces Y and Y'. Hence, for any two forms in the genus of q with $m(\Lambda', q') = m(\Lambda'', q'')$, we can replace q by q' and apply the argument for $m(\Lambda'', q'') = 1$.

For surjectivity, any $\gamma \in G(\mathbb{A})$ defines a lattice $\Lambda_\gamma = \Lambda \otimes \mathbb{Q} \cap \gamma(\Lambda \otimes (\hat{\mathbb{Z}} \times \mathbb{R})) \subset \Lambda \otimes \mathbb{A}$. Note that $q|_{\Lambda_\gamma}$ takes values in $\mathbb{Q} \cap \hat{\mathbb{Z}} \times \mathbb{R}$, i.e., (Λ_γ, q) is an integral quadratic lattice. Moreover, picking $\beta \in \mathrm{GL}_\Lambda(\mathbb{Q})$ such that $\beta(\Lambda_\gamma) = \Lambda$, and setting $\alpha = \beta\gamma \in \mathrm{GL}_\Lambda(\hat{\mathbb{Z}} \times \mathbb{R})$, we see that (Λ_γ, q) corresponds to the class $[\gamma] \in Y$ under the above map — hence, the map m is surjective.

Finally, the automorphism group $G(\mathbb{Z})$ of (Λ, q) is the automorphism group of 1 in the double coset space Y (viewed now as a groupoid). By the same argument as above, using an element β to replace G by G', the same applies to any form (Λ', q') in the genus of q, i.e., $G'(\mathbb{Z})$ is isomorphic (canonically up to inner automorphisms) to the stabilizer of a representative in the class $m(\Lambda', q')$. □

We now restrict our attention to (positive or negative) definite integral quadratic forms, so that the automorphism group $G(\mathbb{Z})$ is finite (being a discrete subgroup in the compact Lie group $G(\mathbb{R})$.

Definition 13.8.26. Given a definite quadratic form q, the *mass* of the genus of q, denoted as $m(q)$, is the weighted count of isomorphism classes of forms in the

[5]In other words, a bijection between equivalence classes, and an identification of automorphism groups (up to inner automorphism).

genus, considered as a stack, i.e.,

$$m(q) = \sum_{q' \in X_q} \frac{1}{|\mathrm{Aut} q'|} = \sum_{q' \in X_q} \frac{1}{|\mathrm{O}_{q'}(\mathbb{Z})|},$$

where X_q denotes the set of isomorphism classes of elements in the genus of q.

The next proposition relates the mass of a genus to the Tamagawa number $\tau(\mathrm{SO}_q)$ of the special orthogonal group.

Remark 13.8.27. For rings where 2 is a unit, or in odd dimensions, the special orthogonal group is defined as the kernel of the determinant in O_q. In even dimensions and 2 non-invertible, in order to obtain the correct definition (that is, the reductive group scheme corresponding to the appropriate root datum), one needs to define SO_q as the kernel of the *Dickson morphism* $\mathrm{O}_q \to \mathbb{Z}/2$, which in this case is surjective, even in characteristic 2. The difference between the correct and the "naive" definition does not play any role in what follows, because the $\mathbb{Z}_2$-points of both are the same, but we will use the fact that $[\mathrm{O}_q(\mathbb{Z}_2) : \mathrm{SO}_q(\mathbb{Z}_2)] = 2$ in every case.

Proposition 13.8.28. *Let q be a definite integral quadratic form in n variables, $n \geq 2$. The mass of the genus of q is equal to*

$$m(q) = 2^{-1} \frac{\tau(SO_q)}{\mu_{Tam}(SO_q(\hat{\mathbb{Z}} \times \mathbb{R}))}.$$

Proof. Let q' be any element in X_q, and let $\gamma \in Y := G(\mathbb{Q}) \backslash G(\mathbb{A}) / G(\hat{\mathbb{Z}} \times \mathbb{R})$, where $G = \mathrm{O}_q$, be the element corresponding to q' under the bijection in Proposition 13.8.25. Then Proposition 13.8.25 implies that $\mathrm{O}_{q'}(\mathbb{Z}) = G(\hat{\mathbb{Z}} \times \mathbb{R}) \cap \gamma^{-1} G(\mathbb{Q}) \gamma$ and thus we get

$$m(q) = \sum_{\gamma \in Y} \frac{1}{|G(\hat{\mathbb{Z}} \times \mathbb{R}) \cap \gamma^{-1} G(\mathbb{Q}) \gamma|}.$$

Let μ be any Haar measure on $G(\mathbb{A})$, then μ also induces a measure on $G(\mathbb{Q}) \backslash G(\mathbb{A})$, which is invariant under the action of $G(\hat{\mathbb{Z}} \times \mathbb{R})$. Thus $G(\mathbb{Q}) \backslash G(\mathbb{A})$ can be written as a union $\bigcup_{\gamma \in Y} O_\gamma$ where O_γ is the orbit of γ under the action of $G(\hat{\mathbb{Z}} \times \mathbb{R})$. Since each O_γ can be identified with $(G(\hat{\mathbb{Z}} \times \mathbb{R}) \cap \gamma^{-1} G(\mathbb{Q}) \gamma) \backslash G(\hat{\mathbb{Z}} \times \mathbb{R})$, we get

$$\sum_{\gamma \in Y} \frac{1}{|G(\hat{\mathbb{Z}} \times \mathbb{R}) \cap \gamma^{-1} G(\mathbb{Q}) \gamma|} = \sum_{\gamma \in Y} \frac{\mu(O_\gamma)}{\mu(G(\hat{\mathbb{Z}} \times \mathbb{R}))} = \frac{\mu(G(\mathbb{Q}) \backslash G(\mathbb{A}))}{\mu(G(\hat{\mathbb{Z}} \times \mathbb{R}))}. \tag{13.8.28.1}$$

To compute (13.8.28.1), let us first consider the analogue of it by substituting the group scheme G by its subscheme SO_q and compute

$$\frac{\mu'(\mathrm{SO}_q(\mathbb{Q}) \backslash \mathrm{SO}_q(\mathbb{A}))}{\mu'(\mathrm{SO}_q(\hat{\mathbb{Z}} \times \mathbb{R}))} \tag{13.8.28.2}$$

for μ' a Haar measure. Moreover, if we take $\mu' = \mu_{\mathrm{Tam}}$ to be the Tamagawa measure, then the numerator becomes the Tamagawa number of SO_q, which we denote by $\tau(\mathrm{SO}_q)$. Now compute (13.8.28.1) by comparing it with (13.8.28.2). Let $U \subset \mathbb{A}^\times$ be the subgroup containing such elements $u \in \mathbb{A}^\times$ that $u^2 = 1$, we then obtain an exact sequence

$$1 \to \mathrm{SO}_q(\mathbb{A}) \to G(\mathbb{A}) \to U \to 1$$

where the map from $G(\mathbb{A})$ to U is the determinant map. Let μ_{Tam} be the Haar measure on $\mathrm{SO}_q(\mathbb{A})$ and μ'' be a Haar measure on U, then we can construct a Haar measure μ on $G(\mathbb{A})$ using μ_{Tam} and μ'' by setting

$$\mu(W) = \int_{u \in U} \mu_{\mathrm{Tam}}(\mathrm{SO}_q(\mathbb{A}) \cap \overline{u}^{-1} W) d\mu''$$

for $W \subset G(\mathbb{A})$ and $\overline{u} \in G(\mathbb{A})$ any element lying over u. Then since μ is compatible with the other two Haar measures μ_{Tam} and μ'', we get

$$\frac{\mu(G(\mathbb{Q})\backslash G(\mathbb{A}))}{\mu_{\mathrm{Tam}}(\mathrm{SO}_q(\mathbb{Q})\backslash \mathrm{SO}_q(\mathbb{A}))} = \mu''(\{\pm 1\}\backslash U) = \frac{\mu''(U)}{2}.$$

Similarly, let $V \subset U$ be the image of $\det|_{G(\hat{\mathbb{Z}} \times \mathbb{R})}$, then

$$\frac{\mu(G(\hat{\mathbb{Z}} \times \mathbb{R}))}{\mu_{\mathrm{Tam}}(\mathrm{SO}_q(\hat{\mathbb{Z}} \times \mathbb{R}))} = \mu''(V) = \frac{\mu''(U)}{|U/V|}.$$

Thus we get

$$\frac{\mu(G(\mathbb{Q})\backslash G(\mathbb{A}))}{\mu(G(\hat{\mathbb{Z}}) \times \mathbb{R})} = \frac{|U/V| \mu_{\mathrm{Tam}}(\mathrm{SO}_q(\mathbb{Q})\backslash \mathrm{SO}_q(\mathbb{A}))}{2\mu_{\mathrm{Tam}}(\mathrm{SO}_q(\hat{\mathbb{Z}} \times \mathbb{R}))} = \frac{|U/V| \tau(\mathrm{SO}_q)}{2\mu_{\mathrm{Tam}}(\mathrm{SO}_q(\hat{\mathbb{Z}} \times \mathbb{R}))}$$

Now, we have $|U/V| = 2^k$, where k is the number of primes p such that $\mathrm{SO}_q(\mathbb{Z}_p) = \mathrm{O}_q(\mathbb{Z}_p)$. Thus, we get

$$m(q) = \frac{\mu(G(\mathbb{Q})\backslash G(\mathbb{A}))}{\mu(G(\hat{\mathbb{Z}} \times \mathbb{R}))} = 2^{k-1} \frac{\tau(\mathrm{SO}_q)}{\mu_{\mathrm{Tam}}(\mathrm{SO}_q(\hat{\mathbb{Z}} \times \mathbb{R}))}.$$

Up to this point, the argument would apply, with obvious modifications, to any connected or disconnected reductive group G. Finally, we claim that for every prime p we have $[\mathrm{O}_q(\mathbb{Z}_2) : \mathrm{SO}_q(\mathbb{Z}_2)] = 2$. This is immediate for $p \neq 3$ by the existence of an orthogonal basis for every quadratic lattice [**O'M71**, §92]. For $p = 2$, see [**O'M71**, §93], and also Remark 13.8.27. □

Remark 13.8.29. The Smith–Minkowski–Siegel mass formula says that $\tau(\mathrm{SO}_q) = 2$, or, equivalently:

$$m(q) = \frac{1}{\mu_{\mathrm{Tam}}(\mathrm{SO}_q(\hat{\mathbb{Z}} \times \mathbb{R}))}. \tag{13.8.29.1}$$

Ultimately, this is a special case of Theorem 13.8.18.

[To add: calculation of p-adic densities.]

13.9. Incarnations of the automorphic space: moduli of principal G-bundles

Let G be a (smooth) affine algebraic group over a field $\mathbb{F}$. By a *(principal) G-bundle* over an $\mathbb{F}$-scheme C we will mean a G-torsor $P \to C$ in the fpqc or étale topology, that is, a G-equivariant sheaf (or, equivalently by fpqc descent, a G-equivariant scheme) over C, such that there is a finitely presented, quasicompact cover, or an étale cover $\tilde{C} \to C$, with a G-equivariant isomorphism

$$P \times_C \tilde{C} \simeq G \times \tilde{C}.$$

That the two topologies give rise to the same objects follows from the smoothness of G (and hence of any G-torsor), and the fact that smooth morphisms $X \to Y$

admit sections étale-locally. It is not, in general, true that an étale G-torsor is trivial Zariski-locally.

However, this is often true when C is a curve. Let C be a one-dimensional, reduced and irreducible scheme of finite type over $\mathbb{F}$. Let $k = \mathbb{F}(C)$ be the function field of the curve, $\mathbb{A}$ its ring of adeles, defined as the restricted tensor product, over all closed points x of the curve, of the fields k_x, and $\mathcal{O} := \prod_x \mathfrak{o}_x \subset \mathbb{A}$, the product of the associated stalks of rings.

The set of isomorphism classes of G-bundles over C will be denoted by $\mathrm{Bun}_G(\mathbb{F})$; it is the set of $\mathbb{F}$-isomorphism classes of a moduli stack Bun_G, which we will not define.

Proposition 13.9.1. *Assume that $H^1(k, G) = 1$ and, for every finite extension $\mathbb{F}'/\mathbb{F}$, $H^1(\mathbb{F}', G) = 1$. Then, there is a canonical isomorphism of* groupoids of sets *$[G]/G(\mathcal{O}) \leftrightarrow Bun_G(\mathbb{F})$, such that, if a G-bundle P corresponds to the class of $(g_v)_v$, which can be assumed to be equal to 1 for every $v \in U$, for some open dense $U \subset C$, then there are trivializations $t_U : P|_U \xrightarrow{\sim} G \times U$ and $t_v : P|_{Spec\mathfrak{o}_v} \xrightarrow{\sim} G \times Spec\mathfrak{o}_v$ for $v \notin U$, such that $t_U \circ t_v^{-1}|_{Speck_v} = g_v$.*

"Groupoid of sets" means (small) categories where all morphisms are isomorphisms; such a category is equivalent to a set, where every point is equipped with a group of automorphisms.

Proof. Citing [**Mat**, 112593]—look there for more details and links to sources:

Since $H^1(k, G) = 1$, any G-torsor is trivial over the generic point of C, hence over a nonempty upen $U \subset C$. Fix such an open set and an isomorphism $P|_U \simeq G \times U$, i.e. a section $\rho : U \to P$.

On the other hand, for every $v \in C$, consider the restriction of P to the formal neighborhood $D_v = \mathrm{Spec}\mathfrak{o}_v$ at v. The residue field $\mathbb{F}(v)$ is a finite extension $\mathbb{F}'$ of $\mathbb{F}$, and since $H^1(\mathbb{F}', G) = 1$, the restriction to P over the special fiber is trivial, i.e., admits a section $\mathrm{Spec}\mathbb{F}' \to P$. Since P is formally smooth, this extends to a section $\sigma_v : D_v \to P$, i.e., P is trivial over D_v; fix such a section σ_v, for every $v \notin U$.

Then, on the intersection $U \cap D_v = \mathrm{Spec}k_v$, we have the restrictions of two sections ρ and σ_v, and thus there is a $g_v \in G(k_v)$ such that $g_v \cdot \sigma_v = \rho$. Set $g_v = 1$ for $v \in U$; the collection $(g_v)_v$ gives an element of $G(\mathbb{A})$, which depends on choices made. All different choices for ρ (possibly for different U) are of the form $G(k)\rho$, and for σ_v of the form $G(\mathfrak{o}_v)\sigma_v$. Thus, independently of choices, the G-bundle P gives rise to an element of $G(k)\backslash G(\mathbb{A})/G(\mathcal{O})$.

Vice versa, descent for the fpqc cover $\mathrm{Spec}(k) \sqcup \mathrm{Spec}\mathfrak{o}_v \to \mathrm{Spec}\mathfrak{o}_{(v)}$ (where $\mathfrak{o}_{(v)} \subset k$ is the local ring at v) shows that the category of G-torsors over $\mathrm{Spec}(\mathfrak{o}_{(v)})$ is equivalent to the category of data (P_v, τ_v), where P_v is a G-torsor over $\mathfrak{o}_v$, and τ_v is a descent datum from its generic fiber to the trivial bundle over k — equivalently, a section of P_v over $\mathrm{Spec}(k_v)$.

If two G-bundles P_1, P_2 give rise to the same class in $[G]/G(\mathcal{O})$, and we represent them as the trivial bundle over an open set U (which we can assume to be the same for both) and data $(P_{i,v}, \tau_{i,v})$ for $v \in S := C \smallsetminus U$, the fact that the classes in $[G]/G(\mathcal{O})$ are the same means that there are isomorphisms $r_v : P_{1,v} \xrightarrow{\sim} P_{2,v}$ such that $r_v \circ \tau_{1,v} = \tau_{2,v}$. By the equivalence of categories, these data give rise to an isomorphism $r : P_1 \xrightarrow{\sim} P_2$. A similar argument shows that an automorphism r of a P-bundle presented as above is the same as an element $r_U \in G(\mathfrak{o}_S)$ (=a section of G over U) and sections $r_v \in G(\mathfrak{o}_v)$ such that $r_v = g_v^{-1} r g_v$, i.e.,

$r \in G(k) \cap (g_v)_v G(\mathcal{O})(g_v)_v^{-1}$, which is the stabilizer of the corresponding point in $[G]/G(\mathcal{O})$.

Finally, any class of $G(k)\backslash G(\mathbb{A})/G(\mathcal{O})$ can be represented by an element $(g_v)_v$ which is equal to 1 at the points of an open subset $U \subset C$, and then we can form an associated G-bundle by *modification* of the trivial vector bundle over the points of $C \smallsetminus U$, that is, by glueing $G \times U$ to $G \times \mathrm{Spec}(\mathfrak{o}_v)$ according to g_v over $\mathrm{Spec}(k_v)$. $\square$

Remark 13.9.2. The condition $H^1(k, G) = 1$ is satisfied if $\mathbb{F}$ is algebraically closed and G is connected reductive, by results of Tsen, and Springer $(+\epsilon)$; it is satisfied when $\mathbb{F}$ is finite and G is semisimple, simply connected, by Harder's proof of the Hasse principle 9.4.3. The condition $H^1(\mathbb{F}', G)$ holds, trivially, when $\mathbb{F} = \mathbb{C}$, and by Lang's theorem 9.1.1 when $\mathbb{F}$ is finite and G is connected.

On the other hand, the μ_n-torsor $\mathbb{G}_m \xrightarrow{n} \mathbb{G}_m$ is not locally trivial in the Zariski topology, for any $n \geq 2$. Moreover, over higher-dimensional bases (e.g., surfaces), even for simply connected semisimple groups, there are étale torsors which are not locally trivial in the Zariski topology.

Remark 13.9.3. Note that the G-automorphism group of a G-bundle makes sense as a group scheme G' over the curve, and is an inner form of G (compare with 8.3.4). The automorphisms appearing in Proposition 13.9.1 are simply the *global sections* of G'.

It is interesting to discuss the isomorphism of Proposition 13.9.1 in the case of $G = \mathrm{GL}_n$. In this case the category of G-bundles is equivalent to the category $\mathrm{Vect}_n(C)$ of rank n vector bundles via the functor $P \mapsto E = V \times^G P$, where V is the standard representation of GL_n, with inverse $E \mapsto$ the frame bundle of E. In the rest of this subsection, we prove that points in a neighborhood of cusps corresponds to unstable vector bundles through this identification. [we ignore the automorphisms, at least for now.] In order to understand the asymptotic behavior of corresponding vector bundles, we would like to translate the situation to the boundary degeneration of $[G]$. For simplicity, we will restrict to the case $n = 2$. For general n, while everything except Proposition 13.9.8 holds mutatis mutandis, for the final result we will need to consider P-cusps for all conjugate class of parabolic subgroups, not only the Borel subgroup. First note the following:

Lemma 13.9.4. *Let $\mathbb{F} = \mathbb{F}_q$, and let H be one of GL_2, its chosen Borel subgroup B, its unipotent radical $\mathbb{G}_a$, or its universal Cartan group scheme $\mathbf{A} \cong (\mathbb{G}_m)^2$. Then the conditions $H^1(k, H) = 1$ and $H^1(\mathbb{F}', H) = 1$ (for $\mathbb{F}'$ any finite extension of $\mathbb{F}$) in Proposition 13.9.1 are satisfied. In particular, there are canonical isomorphisms $[H]/H(\mathcal{O}) \leftrightarrow \mathrm{Bun}_H(\mathbb{F})$.*

Proof. Remark 13.9.2 applies directly to $H^1(\mathbb{F}', \mathrm{GL}_2) = 1$. When $H = \mathbb{G}_a$ or $\mathbb{G}_m$, by Hilbert's theorem 90 we have $H^1(k, H) = 1$. The cohomology long exact sequence (of pointed sets) for $1 \to \mathbb{G}_a \to B \to \mathbf{A} \to 1$ tells us that all the first cohomology sets in question are trivial. $\square$

Since B is the stabilizer of a complete flag $V_\bullet = (0 = V_0 \subset V_1 \subset V_2 = V)$, the category of B-bundles is equivalent to the category $\mathrm{Flag}_2(C)$ of rank 2 complete flag bundles via $P \mapsto E_\bullet = V_\bullet \times^B P$. Similarly, since $\mathbf{A} \cong \mathbb{G}_m \times \mathbb{G}_m$, the category of $\mathbf{A}$-bundles is equivalent to the category $\mathrm{Vect}_1(C) \times \mathrm{Vect}_1(C)$. Note that the set of isomorphism classes of $\mathrm{Vect}_1(C)$ is a group with respect to the tensor product, which is denoted by $\mathrm{Pic}(C)$. The following lemma relates these bundle interpretations of double-cosets:

Lemma 13.9.5. *Through the isomorphisms of Proposition 13.9.1,*

(1) *the map* $f : [B]/B(\mathcal{O}) \to [G]/G(\mathcal{O})$ *induced by the inclusion* $B \hookrightarrow G$ *corresponds to the map* $\mathrm{fgt} : \mathrm{Flag}_2(C) \simeq \mathrm{Bun}_B(\mathbb{F}) \to \mathrm{Bun}_G(\mathbb{F}) \simeq \mathrm{Vect}_2(C)$ *which sends a flag bundle* $E_\bullet$ *to its top filtration* $E = E_2$.
(2) *the map* $g : [B]/B(\mathcal{O}) \to [\mathbf{A}]/\mathbf{A}(\mathcal{O})$ *induced by the quotient* $B \to \mathbf{A}$ *corresponds to the map* $\mathrm{gr} : \mathrm{Flag}_2(C) \simeq \mathrm{Bun}_B(\mathbb{F}) \to \mathrm{Bun}_{\mathbf{A}}(\mathbb{F}) \simeq \mathrm{Pic}(C) \times \mathrm{Pic}(C)$ *which sends a flag bundle* $E_\bullet = (0 \subset E_1 \subset E)$ *to its associated graded bundle* $\mathrm{gr}_\bullet(E_\bullet) = (E_1, E/E_1)$.

Proof. The isomorphism of Proposition 13.9.1 is given by seeing a double-coset as gluing data for trivial bundles on a fixed covering. Therefore f corresponds to the map $G \times^B (-) : \mathrm{Bun}_B(C) \to \mathrm{Bun}_G(C)$. Passing to vector bundles, $V \times^G G \times^B P \cong V \times^B P$ is the top filtration of the flag bundle $V_\bullet \times^B P$. Similarly, the map $B \to \mathbf{A}$ induces, on flag bundles, the map $E_\bullet = V_\bullet \times^B P \mapsto (\mathrm{gr}_\bullet V_\bullet) \times^B P \cong \mathrm{gr}_\bullet E_\bullet$. ☐

For $\mathbb{G}_m = \mathrm{GL}_1$, we can write down the isomorphism more explicitly as follows: Let $\mathrm{Div}(C)$ denote the divisor group of C, i.e., the free abelian group generated by the closed points of C. The normalized discrete valuation $\mathrm{val}_x : k_x^\times \to \mathbb{Z}$ with the kernel $\mathfrak{o}_x^\times$ induces the map $\mathrm{val} : \mathbb{A}^\times \to \mathrm{Div}(C)$ between their restricted products. Dividing by the kernel we get a canonical isomorphism $\mathbb{A}^\times/\mathcal{O}^\times \xrightarrow{\cong} \mathrm{Div}(C)$. The elements in the image of $k^\times \subset \mathbb{A}^\times$ in $\mathrm{Div}(C)$ are called principal divisors, and we have an exact sequence $k^\times \to \mathrm{Div}(C) \twoheadrightarrow \mathrm{Pic}(C) \to 0$. So the valuation gives the isomorphism $[\mathrm{GL}_1]/\mathrm{GL}_1(\mathcal{O}) \cong \mathrm{Pic}(C)$.

Now we introduce the notion of slope-stability for vector bundles over a curve. See [**LP97**, Chapter 5] for more details.

Definition 13.9.6. (1) For any closed point $x \in C$ we define $\deg(x) = 1$, and linearly extend it to the *degree* homomorphism $\deg : \mathrm{Div}(C) \to \mathbb{Z}$. By the product formula of valuations the degree of a principal divisor is 0, so it factors through $\mathrm{Pic}(C) \to \mathbb{Z}$, which we also denote by deg.
(2) More generally, we define the *degree of a vector bundle* E as the degree of its determinant line bundle: $\deg E = \deg\big(\bigwedge^{\mathrm{rk}E} E\big)$.
(3) For a nonzero vector bundle E, we call the rational number $\mu(E) = \frac{\deg(E)}{\mathrm{rk}(E)}$ the *slope* of E.
(4) A vector bundle E is said to be *semistable* if for any nonzero subbundle $F \subset E$, we have $\mu(F) \leq \mu(E)$, and *unstable* otherwise.

Remark 13.9.7. (1) If we have a short exact sequence $0 \to E' \to E \to E'' \to 0$, then the canonical isomorphism $\bigwedge^{\mathrm{rk}E} E \cong \big(\bigwedge^{\mathrm{rk}E'} E'\big) \otimes \big(\bigwedge^{\mathrm{rk}E''} E''\big)$ exhibits that $\deg E = \deg E' + \deg E''$.
(2) The slope can be defined more generally for a coherent sheaf F on C by decomposing it into a direct sum of a vector bundle and a sheaf with finite support. The definition of semi-stability remains equivalent if we replace 'nonzero subbundle' by 'nonzero coherent subsheaf.'
(3) For any subbundle $F \subset E$, consider the point $p_F = (\mathrm{rk}(F), \deg(F)) \in \mathbb{Z}^2$ on a plane. Then the slope $\mu(F)$ is literally the slope of the segment $0p_F$. By the additivity of rk and deg for short exact sequences, we see

that the slope of the segment $p_F p_E$ is the slope $\mu(E/F)$ of the quotient. Considering the convex hull of $\{p_F \mid F \subset E\}$ and taking the upper edges $0p_{F_1}$, $p_{F_1}p_{F_2}, \dots,$ $p_{F_{k-1}}p_{F_k}$, it can be proved that there exists a unique filtration $0 \subset F_1 \subset F_2 \subset \cdots \subset F_k = E$ by subbundles that satisfies the following:
(a) $\mathrm{gr}_i = F_i/F_{i-1}$ is semistable,
(b) $\mu(\mathrm{gr}_1) > \mu(\mathrm{gr}_2) > \cdots > \mu(\mathrm{gr}_k)$.
This filtration is called the *Harder-Narasimhan filtration.*

(4) Let μ be a fixed rational number and consider the full subcategory $\mathcal{C}(\mu)$ of the abelian category Coh_C of coherent sheaves spanned by semistable vector bundles of slope μ. Then $\mathcal{C}(\mu)$ is closed under kernel, cokernel, and extensions. In particular, in contrast with the category of all vector bundles, the category $\mathcal{C}(\mu)$ is abelian.

Now back to our context, note that the composition

$$\mathbb{A}^\times \twoheadrightarrow [\mathrm{GL}_1]/\mathrm{GL}_1(\mathcal{O}) \xrightarrow{\mathrm{val}} \mathrm{Pic}(C) \xrightarrow{\deg} \mathbb{Z}$$

is given by $-\log$ of the adelic absolute value function.

Proposition 13.9.8. *The points in* $[\mathrm{GL}_2]$ *whose image in* $[\mathrm{GL}_2]/\mathrm{GL}_2(\mathcal{O})$ *corresponds to an unstable vector bundles forms a neighborhood of the (B-)cusp.*

Proof. The rank 2 vector bundle E is unstable iff there exists a complete flag $0 \subset F \subset E$ such that $\mu(F) > \mu(E/F)$, or equivalently, $\deg(F) > \deg(E/F)$. Considering $\mathrm{Vect}_2(C) \xleftarrow{\mathrm{fgt}} \mathrm{Flag}_2(C) \xrightarrow{\mathrm{gr}} \mathrm{Pic}(C)^{\oplus 2} \xrightarrow{\deg^{\oplus 2}} \mathbb{Z}^{\oplus 2}$, the unstable loci of $\mathrm{Vect}_2(C)$ is $\mathrm{fgt}\big((\deg^{\oplus 2} \circ \mathrm{gr})^{-1}\{(a,b) \in \mathbb{Z}^{\oplus 2} \mid a > b\}\big)$. Interpreting through Lemma 13.9.5, this diagram is isomorphic to $[G]/G(\mathcal{O}) \xleftarrow{f} [B]/B(\mathcal{O}) \xrightarrow{g} [\mathbf{A}]/\mathbf{A}(\mathcal{O}) \xrightarrow{-\log|-|^{\oplus 2}} \mathbb{Z}^{\oplus 2}$. Therefore the preimage U of $\{(a,b) \in \mathbb{Z}^{\oplus 2} \mid a > b\}$ in $[B]/B(\mathcal{O})$ is the set of points represented by an upper-triangular matrix (a_{ij}) such that $-\log|a_{11}| > -\log|a_{22}|$. Now recall from Theorem 13.4.2 that the map $B(k)\backslash G(\mathbb{A}) \xrightarrow{\pi_G} [G]$ is injective in a neighborhood of the cusp, and the sets $V_\epsilon = \{(a_{ij}) \mid -\log|a_{11}/a_{22}| > -\log\epsilon\}$ (in the same basis representation) forms a basis of neighborhoods of the cusp in $B(k)\backslash G(\mathbb{A})$. By Iwasawa decomposition $G(\mathbb{A}) = B(\mathbb{A})K$, where $K = G(\mathcal{O})$ in our case, the map f factors into $[B]/B(\mathcal{O}) \xrightarrow{\cong} B(k)\backslash G(\mathbb{A})/G(\mathcal{O}) \xrightarrow{\pi_G} [G]/G(\mathcal{O})$. Now we conclude by observing that the above set of unstable locus U is the image of V_1. $\square$

13.10. Incarnations of the automorphic space: locally symmetric spaces and Shimura varieties

13.10.1. Locally symmetric spaces.

Definition 13.10.2. A *Riemannian symmetric space* is a pair (M, g), where M is a (connected)[6] manifold, and g is a Riemannian metric on M, with the property that for every $x \in M$ there is an isometry $s_x : M \to M$, having x as an isolated fixed point, and acting by -1 on the tangent space of x.

Lemma 13.10.3. *If* (M, g) *is a connected Riemannian symmetric space, then it is geodesically complete, the Lie group* G *of isometries of* M *acts transitively on* M*, and the stabilizer of any point* $x \in M$ *is a compact subgroup.*

[6] "Connected" is not always part of the definition, but we will include it, for convenience.

Notice that this implies that the identity component G^0 of the Lie group already acts transitively.

Proof. [**Mil05**, Lemma 1.5 and Proposition 1.11]. □

Let G be a real reductive algebraic group. By Theorem 8.6.6, the group $G(\mathbb{R})$ admits a unique, up to conjugacy, Cartan involution. Let X be the set of its Cartan involutions, considered as a homogeneous manifold under the action of $G(\mathbb{R})$: $X \simeq G(\mathbb{R})/K$, where K is the fixed-point subgroup of a Cartan involution θ.

Proposition 13.10.4. *Assume that G is real semisimple. The space X of Cartan involutions of G admits a canonical $G(\mathbb{R})$-invariant Riemannian metric g, described as follows: Let $\theta \in X$ be a Cartan involution, with corresponding Cartan decomposition (Definition 8.6.1), $\mathfrak{g}(\mathbb{R}) = \mathfrak{k} \oplus \mathfrak{p}$, identifying the tangent space $T_\theta X \simeq \mathfrak{p}$. Then, the restriction of g to $T_\theta X$ is the Killing form of $\mathfrak{g}$, restricted to $\mathfrak{p}$. This Riemannian structure has negative sectional curvature.*

Proof. See [**Par09**, §5.4]. □

Definition 13.10.5. The *symmetric space of a real reductive group* $G(\mathbb{R})$ is the set X of its Cartan involutions.

The symmetric space of $G(\mathbb{R})$ can be considered as a Riemannian symmetric space under any $G(\mathbb{R})$-invariant Riemannian metric. Such a metric is the combination of a multiple of the Killing form of Proposition 13.10.4 on each of the simple factors, and the Euclidean metric on the symmetric space of its center.

Definition 13.10.6. A *Riemannian locally symmetric space* is a Riemannian manifold whose universal cover is a symmetric space.

Remark 13.10.7. Let G be a reductive algebraic group over a number field k, $[G]$ its automorphic space, K_f an open compact subgroup of the finite adeles, and K_∞ a maximal compact subgroup of $G(k_\infty)$ (unique up to conjugation, by Theorem 8.6.6).

By Remark 13.5.7, the quotient $[G]/K_f K_\infty$ can be written as a finite disjoint union

$$\bigsqcup_i \Gamma_i \backslash X,$$

where X is the symmetric space of $G(k_\infty)$, and the Γ_i's are congruence subgroups. The subgroups Γ_i act with finite stabilizers on X (because they are discrete, and stabilizers of points on X are compact), and taking K_f small enough, they act properly discontinuously, so that $\Gamma_i \backslash X$ has the natural structure of a manifold with universal cover X. Thus, for K_f small enough, $[G]/K_f K_\infty$ has a natural structure of locally symmetric space.

13.10.8. Hermitian locally symmetric spaces. There are instances where the space G_∞/K_∞ of the previous subsection has more structure, that of a *Hermitian symmetric space.*

Definition 13.10.9. A *Hermitian symmetric space* is a pair (M, g), where M is a complex manifold, and g is a Hermitian metric on M, with the property that for every $m \in M$ there is an isometry $s_x : M \to M$, having m as an isolated fixed point, and acting by -1 on the tangent space of m.

A *Hermitian symmetric domain* is a Hermitian symmetric space which, as a complex manifold, is isomorphic to a bounded open subset of $\mathbb{C}^n$.

Remark 13.10.10. For a Hermitian symmetric domain realized as a bounded open $U \subset \mathbb{C}^n$, the metric on U is the *Bergman metric*, [**Mil05**, Theorem 1.3]. There is also another way to define Hermitian symmetric domains, as *Hermitian symmetric spaces of noncompact type*, [**Mil05**, §1].

Lemma 13.10.11. *If (M, g) is a hermitian symmetric domain, and G is the Lie group of its isometries as a Riemannian manifold, then G^0 acts by holomorphic automorphisms. In particular, by Lemma 13.10.3, M is homogeneous, with compact stabilizers, under the group of its Hermitian automorphisms.*

Proof. [**Mil05**, Proposition 1.6]. □

From now on, "automorphisms" of a Hermitian symmetric domain will mean "Hermitian automorphisms", i.e., holomorphic isometries. The Hermitian structure upgrades the ± 1-symmetry at each point to an $S^1 (= U_1)$-symmetry:

Lemma 13.10.12. *If (M, g) is a Hermitian symmetric domain with group G of holomorphic isometries, then for every $x \in M$ and every $z \in \mathbb{C}^1$ there is an automorphism $s_{x,z}$ of M, fixing x and acting by z on its tangent space.*

Proof. [**Mil05**, Theorem 1.9]. □

In particular, $s_{x,-1}$ is the automorphism s_x of the symmetric structure. Another way to phrase this lemma is that there is a homomorphism $U_1 \to G_x$, acting by the tautological character on the tangent space $\mathfrak{g}/\mathfrak{g}_x$ (which has a complex structure).

Lemma 13.10.13. *The image of the homomorphism $U_1 \to G_x$ lies in the center of G_x.*

Proof. The group G_x acts by holomorphic isometries on M, hence by complex linear automorphisms of the tangent space T_xM. Hence, its action on T_xM commutes with the action of U_1; but an element in the image of $U_1 \subset G_x^0$ is completely determined by its action on the tangent space. □

For our purposes, we want a realization of G in terms of algebraic groups.

Lemma 13.10.14. *Let (M, g) be a hermitian symmetric domain with automorphism (Lie) group A. Then, there is a unique connected, adjoint algebraic subgroup G of $GL(\mathfrak{a})$ over $\mathbb{R}$ such that $A^0 \xrightarrow{\sim} G(\mathbb{R})^0$ under the adjoint representation.*

Proof. [**Mil05**, Proposition 1.7]. □

Thus, we can identify M with the real points of an algebraic symmetric space G/G_x over $\mathbb{C}$. Every representation of a compact group is algebraic, so the homomorphism $s_x : U_1(\mathbb{R}) \to G_x(\mathbb{R})$ comes from an algebraic morphism: $U_1 \to G_x$. Base-changing to $\mathbb{C}$, we obtain a cocharacter $\mathbb{G}_m \to G_{x,\mathbb{C}}$, which acts on the complexification $\mathfrak{g}_{\mathbb{C}}/\mathfrak{g}_{x,\mathbb{C}}$ by the characters $z \mapsto z^{\pm 1}$. Since the image lies on the center of G_x (Lemma 13.10.13), it acts trivially on $\mathfrak{g}_{x,\mathbb{C}}$.

Definition 13.10.15. Let $\mathfrak{g}$ be a semisimple Lie algebra over an algebraically closed field, and $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra. A *minuscule coweight* is a coweight $\mu \in \mathfrak{h}$ such that for every root α we have $\langle \alpha, \mu \rangle \in \{-1, 0, 1\}$. Similarly, replacing roots by coroots one defines a *minuscule weight*.

Remark 13.10.16. Minuscule weights have the following representation-theoretic interpretation: An integral, dominant (with respect to some base of the root system — can always assume by Weyl translation) weight μ is minuscule if and only if the only weights appearing in the associated irreducible heighest weight module V_μ are Weyl group-translates of μ, and zero. This follows directly from the Weyl character formula.

Minuscule coweights are relatively rare, and so are Hermitian symmetric domains. In the Langlands program, certain general constructions over function fields (Drinfeld's shtukas) that depend on an irreducible representation/dominant weight for the dual group (= coweight for G), only have known (partial) analogs (Shimura varieties) over number fields when the weight is minuscule.

Theorem 13.10.17. *The map that assigns to pointed,* connected *Hermitian symmetric domain (M,g,x) (i.e., a connected Hermitian symmetric domain (M,g) and a point $x\in M$) the pair (G,s_x), where G is the real algebraic group such that $G(\mathbb{R})^0 = Aut(M,g)$ (Lemma 13.10.14) and $s_x : U_1 \to G$ is the S^1-symmetry fixing x (Lemma 13.10.12), is an equivalence between the groupoids of pointed connected Hermitian symmetric domains and pairs (G,s) consisting of a real adjoint algebraic group G and a homomorphism $s: U_1 \to G$ such that:*

- *s is minuscule, i.e., its complexification is a minuscule cocharacter;*
- *$Ad(s(-1))$ is a Cartan involution for G;*
- *no simple factor of G is compact; equivalently, $Ad(s(-1))$ does not project to 1 on any simple factor of G.*

The inverse functor assigns to such a pair (G,s) the set M of $G(\mathbb{R})^0$-conjugates of s, which has a unique structure of a Hermitian symmetric domain such that s, as a homomorphism, is the S^1-symmetry at s, as a point of M.

Proof. [**Mil05**, Theorem 1.21]. □

Definition 13.10.18. A *Hermitian locally symmetric space* is a Hermitian manifold whose universal cover is a Hermitian symmetric space.

Remark 13.10.19. Let G be a reductive group over a number field. By Remark 13.10.7, the quotient $[G]/K_f K_\infty$ of the automorphic space is naturally a locally symmetric space (for K_f small enough). If $G(k_\infty)$ admits a homomorphism $s: U_1 \to G(k_\infty)$ satisfying the conditions of Theorem 13.10.17, this quotient becomes a Hermitian locally symmetric space. *Shimura varieties* are (inverse) limits of such spaces, as $K_f \to 1$. Of course, one needs to justify the term "varieties", which is outside the current scope of these notes.

13.10.20. Variation of Hodge structures. In this subsection, we present a moduli description of Hermitian symmetric domains, following Milne [**Mil05**].

Definition 13.10.21. A (pure) *Hodge decomposition* of a real vector space V is a decomposition

$$V(\mathbb{C}) = \bigoplus_{p,q\in\mathbb{Z}} V^{p,q}$$

such that $V^{q,p}$ is the complex conjugate of $V_{p,q}$. A (pure) *Hodge structure* is a real vector space together with a Hodge decomposition. For each integer n, the subspace $\bigoplus_{p+q=n} V^{p,q}$ of $V(\mathbb{C})$ is stable under complex conjugation, and so it is defined over $\mathbb{R}$, i.e., there is a subspace V_n of V such that

$$V_n(\mathbb{C}) = \bigoplus_{p+q=n} V^{p,q}.$$

Then $V = \bigoplus_n V_n$ is called the *weight decomposition* of V. If $V = V_n$, then V is said to have weight n. Let V, W be two Hodge structures. A *morphism of Hodge structures* is a linear map $V \to W$, sending $V^{p,q}$ to $W^{p,q}$.

Remark 13.10.22. This definition comes from the fact that for a compact complex manifold admitting a Kähler metric, we can always factor its cohomology group as

$$H^n(X, \mathbb{C}) = \bigoplus_{p+q=n} H^q(X, \Omega_X^p).$$

Note that given a Hodge structure V, if we let $d(p,q) = \dim_{\mathbb{C}} V^{p,q}$, then $V^{p,q}$ naturally corresponds to a point in the Grassmannian $G_{d(p,q)}(V(\mathbb{C}))$. Hence V corresponds to a point in $\prod_{p,q:d(p,q)\neq 0} G_{d(p,q)}(V(\mathbb{C}))$.

There are two other ways to give a Hodge structure to a real vector space. Given a Hodge structure V of weight n, we can assign to $V(\mathbb{C})$ the *Hodge filtration*

$$F^\bullet : \cdots \supseteq F^p \supseteq F^{p+1} \supseteq \cdots,$$

where $F^p = \bigoplus_{r\geq p} V^{r,n-r}$. Then, the Hodge structure can be recovered as $V^{p,q} = F^p \cap \overline{F^q}$. The Hodge filtration satisfies the axioms $F^p \cap \overline{F^{n+1-p}} = 0$ and $F^p \oplus \overline{F^{n+1-p}} = V(\mathbb{C})$; vice versa, these axioms ensure that the filtration comes from a pure Hodge structure.

The Hodge filtration is, in some sense, a more natural object to consider, as it generalizes to a description of *mixed Hodge structures*, which can also be associated to singular or non-proper varieties.

In this way, a Hodge structure on V with $d(p,q) = \dim_{\mathbb{C}} V^{p,q}$ corresponds to a flag $F^\bullet$ of $V(\mathbb{C})$ satisfying $d_p = \dim_{\mathbb{C}} F^p = \sum_{r\geq p} d(r, n-r)$. Equivalently, it corresponds to a point in the flag variety $G_{\mathfrak{d}}(V(\mathbb{C}))$, where $\mathfrak{d} = (d_1, \cdots, d_p, \cdots)$.

Another way to give a Hodge structure is related to the represention of the *Deligne torus* $\mathbb{S}$, which is defined to be the restriction of scalars $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_{\mathrm{m}}$, as a linear algebraic group over $\mathbb{R}$.

Lemma 13.10.23. *For any $\mathbb{R}-$ algebra A, $\mathbb{S}(A) \cong \{(a,b) \in A \times A | a^2 + b^2 \neq 0\}$, with multiplication given by $(a,b)\cdot(a',b') = (aa' - bb', ab' + a'b)$.*

Proof. We can construct the following isomorphism,

$$\begin{aligned} f : \{(a,b) \in A \times A | a^2 + b^2 \neq 0\} &\to (A \otimes_{\mathbb{R}} \mathbb{C})^\times = \mathbb{S}(A) \\ (a,b) &\mapsto a \otimes 1 + b \otimes i \end{aligned}$$

The verification is obvious. □

Let's consider characters of $\mathbb{S}(\mathbb{C})$. Using the isomorphism $\{(a,b) \in \mathbb{C}\times\mathbb{C} | a^2 + b^2 \neq 0\} \ni (x,y) \mapsto (x+yi, x-yi) \in \mathbb{C}^\times \times \mathbb{C}^\times$, we may identify $\mathbb{S}(\mathbb{C})$ as $\mathbb{C}^\times \times \mathbb{C}^\times$, hence its character group is $\mathbb{Z}\times\mathbb{Z}$. For each pair $(a,b) \in \mathbb{Z}\times\mathbb{Z}$, let $\sigma_{a,b}$ be the character such that $\sigma_{a,b}(z,w) = z^a w^b$. Restricting $\sigma_{a,b}$ to $\mathbb{S}(\mathbb{R}) = \mathbb{C}^\times$ gives $\chi_{a,b}(z) = z^a \bar{z}^b$ for $z \in \mathbb{C}^\times$.

Now, given a Hodge structure on a real vector space V, we can construct a representation h of $\mathbb{S}(\mathbb{R})$ on $V(\mathbb{C})$ as follows: let $z \in \mathbb{S}(\mathbb{R})$ act on $V^{p,q}$ by the character $\chi_{-p,-q}$. Conversely, given a representation $h : \mathbb{S}(\mathbb{R}) \to GL(V)$, we can define $V^{p,q}$ to be the subspace of $V(\mathbb{C})$ with $\mathbb{S}(\mathbb{R})$ action given by $h(z)v = \chi_{-p,-q}(z)v = z^{-p}\bar{z}^{-q}v$. It is clear that $V^{p,q} = \overline{V^{q,p}}$. Therefore, we also use (V, h) to denote a Hodge structure.

The morphism between Hodge structures $f : (V, h_v) \to (W, h_w)$ can also be interpreted as a linear map satisfying $f(h_v(z)v) = h_w(z)f(v)$ for all $v \in V, z \in \mathbb{S}(\mathbb{R})$, or, in other words, a morphism between representations h_v and h_w.

We often want to consider the variation of Hodge structures. Let S be a connected complex manifold and V a real vector space. Given weight n, suppose that, for each $s \in S$, we have a Hodge structure on V, denoted by (V_s, h_s), with Hodge filtration given by $F_s^\bullet$.

Definition 13.10.24. (1) A *continuous family of Hodge structures* on a topological space S is a family $(h_s)_{s\in S}$ of Hodge structures on a fixed real vector space V, parametrized by the points of S, such that the dimension $d(p,q)$ of $V_s^{p,q}$ is locally constant in $s \in S$, and the map from S to the product of Grassmannians defined by

$$S \ni s \mapsto (V_s^{p,q})_{p,q:d(p,q)\neq 0} \in \prod_{p,q:d(p,q)\neq 0} G_{d(p,q)}(V(\mathbb{C}))$$

is continuous.

(2) A *holomorphic family of Hodge structures* on a connected complex manifold S is a continuous family of Hodge structures $(h_s)_{s\in S}$ on a real vector space V, such hat the map ϕ froms S to flag varieties defined by

$$S \ni s \mapsto F_s^\bullet \in G_{\mathfrak{d}}(V(\mathbb{C}))$$

is holomorphic, where $\mathfrak{d} = (d_1, \cdots, d_p, \cdots)$ and $d_p = \dim_{\mathbb{C}} F_s^p$. Note that continuity implies that d_p does not vary with $s \in S$.

Note that it would not make sense to require the spaces $V^{p,q}$ to vary holomorphically, since the operation of taking the conjugate of a subspace does not preserve this property.

Some differential geometry shows that $\mathrm{T}_{F_s^\bullet}(G_{\mathfrak{d}}(V(\mathbb{C})))$, the tangent space of $G_{\mathfrak{d}}(V(\mathbb{C}))$ at $F_s^\bullet$, is naturally realized as a subspace of $\bigoplus_p \mathrm{Hom}(F_s^p, V(\mathbb{C})/F_s^p)$. We will make a strong restriction and give the following definition:

Definition 13.10.25. A *variation of Hodge structures* is a holomorphic family of Hodge structures $(h_s)_{s\in S}$ such that the differential of the map $\phi : S \to G_{\mathfrak{d}}(V(\mathbb{C}))$ has its image in $\bigoplus_p \mathrm{Hom}(F_s^p, F_s^{p-1}/F_s^p)$. This condition is known as *Griffiths transversality.*

In order to give the classification theorem, we are going to define polarizations.

Definition 13.10.26. The *tensor product of Hodge structures* V and W of weights m and n is $V \otimes W$, a Hodge structure of weight $m+n$, satisfying

$$(V \otimes W)^{p,q} = \bigoplus_{r+r'=p s+s'=q} V^{r,s} \otimes W^{r',s'}.$$

In terms of representations of $\mathbb{S}(\mathbb{R})$, $(V, h_v) \otimes (W, h_w) = (V \otimes W, h_v \otimes h_w)$.

Definition 13.10.27. The *weight homomorphism* of $\mathbb{S}(\mathbb{R})$ is the homomorphism $w : \mathbb{R}^+ \to \mathbb{S}(\mathbb{R})$, defined by $w(t) = t^{-1}$.

We observe that a Hodge structure V has weight n if and only if its associated representation $h : \mathbb{S}(\mathbb{R}) \to GL(V(\mathbb{C}))$ satisfies $h(w(t))v = t^{-n}v$ for every $t \in \mathbb{R}^+, v \in V$.

One important example is the unique one-dimensional real vector space of weight $-2n$: $V = \mathbb{R}(n)$, whose underlying vector space is $\mathbb{R}$, and whose Hodge structure is given by $V(\mathbb{C}) = V^{-n,-n} = \mathbb{C}$, or equivalently, given by a representation $h : \mathbb{S}(\mathbb{R}) \to \mathbb{C}^\times$ satisfying $h(w(t)) = t^{2n}$ for all $t \in \mathbb{R}^+$.

It is time to define Hodge tensors.

Definition 13.10.28. A multilinear form $t : V^r \to \mathbb{R}$ on a Hodge structure V of weight n is called a *Hodge tensor* if the map

$$\underbrace{V \otimes V \otimes \cdots \otimes V}_{r \text{ copies}} \to \mathbb{R}(-nr/2)$$

it defines is a morphism of Hodge structures. In other words, t is a Hodge tensor if

$$t(h(z)v_1, \cdots, h(z)v_r) = (z\bar{z})^{-nr/2} t(v_1, \cdots, v_r)$$

for all $z \in \mathbb{C}^\times$ and $v_i \in V$.

We can now move on to the heart of our definitions:

Definition 13.10.29. A *polarization of a Hodge structure* (V, h) of weight n is a Hodge tensor $\psi : V \times V \to \mathbb{R}(-n)$, such that the map $\psi_{h(i)}$ defined by $V \times V \ni (v, w) \mapsto \psi(v, h(i)w) \in \mathbb{R}$ is symmetric and positive definite.

13.10.30. Hermitian symmetric domains as parameter spaces for Hodge structures. Now we explain and sketch a proof of Deligne's realization of hermitian symmetric domains as parameter spaces for Hodge structures. Let V be a real vector space and T be a family of tensors on V including a nondegenerate bilinear form t_0, and let $d : \mathbb{Z} \times \mathbb{Z} \to \mathbb{N}$ be a function such that

$$\begin{cases} d(p,q) = 0 \text{ for almost all p,q ;} \\ d(q,p) = d(p,q); \\ d(p,q) = 0 \text{ if } p + q \neq n. \end{cases}$$

Define $S(d, T)$ to be the set of Hodge structures h on V such that

(1) $\dim V_h^{p,q} = d(p,q)$ for all p, q;

(2) each $t \in T$ is a Hodge tensor for h;

(3) t_0 is a polarization for h.

By Remark 13.10.22, we can endow $S(d, T)$ with the subset topology of $\prod_{p,q:d(p,q)\neq 0} G_{d(p,q)}(V(\mathbb{C}))$.

Our main theorem is the following:

Theorem 13.10.31. *Let S^+ be a connected component of $S(d, T)$.*

(1) The space S^+ has a unique complex structure for which $(h_s)_{s\in S}$ is a holomorphic family of Hodge structures.

(2) With this complex structure, S^+ is a hermitian symmetric domain if $(h_s)_{s\in S}$ is a variation of Hodge structures.

(3) Every irreducible hermitian symmetric domain is of the form S^+ for a suitable choice of V, d, T.

Proof. (Sketch)

(1) Let G be the smallest algebraic subgroup of $GL(V)$ such that $h(\mathbb{S}) \subseteq G$ for all $h \in S^+$. Take any $h_0 \in S^+$, then for all $g \in G(\mathbb{R})^0$, $gh_0g^{-1} \in S^+$. It was

proven by Deligne [**Del79**] that the map $G(\mathbb{R})^0 \ni g \mapsto gh_0g^{-1} \in S^+$ is surjective. Let K be the subgroup of $G(\mathbb{R})^0$ fixing h_0, then $S^+ \cong G(\mathbb{R})^0/K$. And since K is closed, S^+ now admits a smooth manifold structure. Therefore we may consider its tangent space T at h_0, which is the quotient of Lie algebras $\mathfrak{g}/\mathfrak{k}$. It suffices to show that, under the embedding $S^+ \hookrightarrow G_{\mathfrak{d}}(V(\mathbb{C}))$ coming from the Hodge filtration (notation as in Definition 13.10.24), $\mathfrak{g}/\mathfrak{k}$ is a complex subspace of the tangent space of the Grassmannian.

Note that V has a Hodge structure h_0, which induces a Hodge structure on $W := V \otimes V^* = \mathrm{End}V = \mathfrak{gl}(V)$. What is more, W has weight 0, and its Hodge structure restricts to a Hodge structure on the subspace $\mathfrak{g}$ (which, in terms of the action of the Deligne torus $\mathbb{S}$, is simply the composition of $h_0 : \mathbb{S} \to G$ with the adjoint representation). It is a general fact that, if W is a Hodge structure of weight 0, setting $W^{00} = W^{0,0} \cap W$ we have $W/W^{00} = W(\mathbb{C})/F^0$. Indeed, $W^{00} = F^0 \cap \overline{F^0} \cap W =$ the kernel of the surjective map $W \to W(\mathbb{C})/F^0$.

Since K is the stabilizer of h_0, we have $\mathfrak{g}^{00} = \mathfrak{k}$, and therefore $T = \mathfrak{g}/\mathfrak{g}^{00} = \mathfrak{g}(\mathbb{C})/F^0_{\mathfrak{g}}$, a complex subspace of $W/W^{00} = W(\mathbb{C})/F^0_W$. The latter space is none other than the tangent space of $G_{\mathfrak{d}}(V(\mathbb{C}))$ at h_0. This shows that S^+ can be identified as a complex submanifold of $G_{\mathfrak{d}}(V(\mathbb{C}))$, hence admits a complex structure, as desired.

(2) Let G_{ad} be the adjoint group of G. We want to apply Theorem 13.10.17 to G_{ad}. For all $r \in \mathbb{R}^\times$, $h(r)$ acts as r^{-n} on V, therefore belongs to the center of $\mathrm{GL}(V)$. Therefore, we can define a homomorphism $s_0 : S^1 \ni z \mapsto h_0(\sqrt{z}) \in G_{\mathrm{ad}}$. Let $C = h_0(i)$. Consider the faithful representation $G \to GL(V)$. Since t_0 is a Hodge tensor for h, t_0 is invariant under $h(z)$, for any $h \in S^+$ and $z \in \mathbb{C}^\times$; therefore, t_0 is G-invariant. The form $(v, w) \mapsto t_0(v, Cw)$ is symmetric and positive definite, defining a Cartan involution $g \mapsto g^{-t}$ (transpose under this inner product) on $\mathrm{GL}(V)$, by Example 8.6.2, which leaves G stable; therefore, $\mathrm{Ad}C = \mathrm{Ad}(s_0(-1))$ is a Cartan involution of G. Griffiths transversality of h_0 ensures that s_0 is minuscule condition (a); condition (c) is clear. Hence, by Theorem 13.10.17, the set of all conjugates u of s_0 by $G_{\mathrm{ad}}(\mathbb{R})^0$ admits the structure of a hermitian symmetric domain. Since for each u, we can obtain an $h \in S^+$ as follows: $\mathbb{S}(\mathbb{R}) \ni z \mapsto u(z/\bar{z}) \in GL(V)$, this set can be identified with S^+.

(3) Let D be an irreducible symmetric domain. Let G be the connected adjoint group such that $G(\mathbb{R})^0$ is the identity component of the holomorphic automorphisms of D (Lemma 13.10.14). Choose a faithful representation $G \to GL(V)$, and let t_0 be a nondegenerate G-invariant bilinear form on V. We can find a set of tensors T containing t_0 such that G is the subgroup of $GL(V)$ fixing each $t \in T$. Fix a point $x \in D$, let $s_0 : U_1 \to G$ be the corresponding homomorphism (Lemma 13.10.12), and let h_0 be a Hodge structure on V obtained from s_0 using (2). Then all $t \in T$ are Hodge tensors for h_0 and t_0 is a polarization. Now we can check that D is naturally identified with the component of $S(d, T)^+$ containing this Hodge structure. $\square$

13.11. Other chapters

CHAPTER 14

Automorphic forms

14.1. Representations of adelic groups

Let G be a linear algebraic group over a global field k, with ring of adeles $\mathbb{A}$. For this chapter, a *representation* π of $G(\mathbb{A})$ will always be a smooth representation of the finite adeles $G(\mathbb{A}_f)$, and topologized as the strict direct limit

$$\lim_{\overrightarrow{K_f}} \pi^{K_f}$$

over a basis of open compact neighborhoods of the identity in K_f. Any property of representations of groups over local fields will be applied to representations of $G(\mathbb{A})$, by restricting it to the spaces of K^S-invariants, considered as a $G(k_S)$-representation, where S is a finite set of places including the Archimedean ones, and K^S is any compact open subgroup of the adeles away from S. For example, π is an SF-representation, or smooth representation of moderate growth, if π^{K^S} is such a representation of $G(k_S)$ (Definition 10.1.2), for any such K^S.

If k has Archimedean places, we will denote by $\mathfrak{g}$ the *complexified* Lie algebra $\mathfrak{g}(k_\infty) \otimes_{\mathbb{R}} \mathbb{C}$, by $U(\mathfrak{g})$ its universal enveloping algebra, and by $Z(\mathfrak{g})$ the (Harish-Chandra) center of $U(\mathfrak{g})$.

14.2. The space of automorphic forms

Definition 14.2.1. A continuous function f on the automorphic space $[G]$ is of *moderate growth* if on one, equivalently any, Siegel fundamental set $\Omega A_\epsilon K$ (in the language and notation of Definitions 13.5.4, 13.5.5), and a norm on the space $\mathfrak{a} = \mathrm{Hom}(\mathbb{G}_m, A) \otimes \mathbb{R}$, the function satisfies a bound of the form

$$|f(\omega a k)| \ll e^{s|\log(a)|}$$

for some $s > 0$, where $\log : A(\mathbb{A}) \to \mathfrak{a}$ is the logarithmic map of (13.2.6.1).

For fixed s and norm, the functions above form a Banach space with norm $\sup(|f(\omega a k)| e^{-s|\log(a)|})$. The space $C_{mg}([G])$ of moderate growth functions is the direct limit of these spaces.

A function is of *uniform moderate growth* if it belongs to the space $C_{mg}([G])^\infty$ of smooth vectors in this space. Equivalently, if it is fixed under an open compact subgroup K_f of the finite adeles $G(\mathbb{A}_f)$, and for every $D \in U(\mathfrak{g}(k_\infty))$, the function Df satisfies a bound as above, for the same s.

Remark 14.2.2. Equivalent definitions of moderate growth are the following:

First, fix a closed embedding of G into an affine space with coordinates $(x_1, \dots, x_n)$, and define $\|g\| = \max_i |x_i(g)|$. Then, a continuous function f on $[G]$ is of moderate growth iff $|f(g)| \ll \|g\|^s$ for some $s > 0$.

Explain why they are equivalent!

Another, more geometric, equivalent definition is the following: Consider any equivariant toroidal (full) compactification $\overline{[G]}$ (Definition 13.4.3). Recall that the comlement of $[G]$ is the union of P-cusps, as P ranges over all conjugacy classes of parabolics, and that the P-cusp has a neighborhood which maps, with compact fibers (see (13.4.1.1)), to a neighborhood of the closed orbit in a "standard embedding" (Definition 13.2.7) such as $\overline{A_P}(k_\infty)\times^{A_P(k_\infty)}[G]_P$. Now, fix a compact subset U of the P-cusp; it has a neighborhood of the form $U\times V$, where V is a neighborhood of the closed $A_P(k_\infty)$-orbit in $\overline{A_P}(k_\infty)$; use $\mathring{V}$ to denote its intersection with the open $A_P(k_\infty)$-orbit. Then, "moderate growth" means that for any cover of the boundary of $\overline{[G]}$ by such compact sets, the function is bounded on $U\times\mathring{V}$ by a multiple of ϵ^{-s}, where ϵ is an "algebraic distance function" from the closed orbit in V, that is, if the closed orbit is given by the vanishing of algebraic coordinates $x_1,\dots,x_n$, then $\epsilon\sim\max_i|x_i|$.

Proposition 14.2.3. *Let π be a Fréchet representation of moderate growth of $G(\mathbb{A})$. (See Section 14.1.) Any morphism $l:\pi\to C([G])$ factors through a continuous map to $C_{mg}([G])$. If it is a smooth Fréchet representation of moderate growth, it factors through a continuous map to $C_{mg}([G])^\infty$.*

Proof. We will use the first equivalent characterization of moderate growth of Remark 14.2.2: Since the map $l:\pi\to C([G])$ is continuous (the space on the right considered as a Fréchet space), for every K^S: compact open subgroup of $G(\mathbb{A}^S)$ there is a continuous seminorm q on π^{K^S} such that $|l(v)(1)|\le q(v)$. With π being an F-representation, we may assume that q is $G(k_S)$-continuous, and then by (2.6.2.1) we have that $|l(v)(g)|=|l(\pi(g)v)(1)|\le q(\pi(g)v)\le\|g\|^s q(v)$ for some $s>0$. Thus, the map factors continuously through $C_{mg}([G])$. Passing to smooth vectors, we get a continuous map $\pi^\infty\to C_{mg}([G])^\infty$. $\square$

If, in addition, π is admissible, elements in its image have the following properties:

Proposition 14.2.4. *Let π be an admissible smooth Fréchet representation of moderate growth of $G(\mathbb{A})$, and $l:\pi\to C^\infty([G])$ a morphism. Fix a maximal compact subgroup K_∞ of $G(k_\infty)$. Elements f in the image of π^{K_∞} have the following properties:*

1. *f is of uniform moderate growth;*
2. *f is K_f-finite, for every compact open subgroup of $G(\mathbb{A}_f)$;*
3. *f is K_∞-finite;*
4. *f is $Z(\mathfrak{g})$-finite, if k has Archimedean places. In the function field case, f is finite under the Bernstein center of $G(k_\infty)$, for some chosen place ∞.*

Conversely, every such function f generates an admissible SF-subrepresentation of $C_{mg}([G])^\infty$.

Proof. The first property is contained in Proposition 14.2.3, the second and third are obvious, and the fourth follows from the fact that $Z(\mathfrak{g})$ preserves the finite-dimensional K_fK_∞-isotypic space of f.

Vice versa, if f satisfies these properties, the statement to prove is admissibility. This follows from Harish-Chandra's theorem, which says that the space of functions as above with fixed K_fK_∞-type, and annihilated by a fixed ideal of finite codimension in $Z(\mathfrak{g})$, is finite-dimensional. [Not included yet in the notes.] $\square$

Definition 14.2.5. The space $\mathcal{A}([G])$ of *automorphic forms* on $[G]$ is the sum of all admissible subrepresentations of $C_{mg}([G])^\infty$ that are generated (in the sense of closure of the $G(\mathbb{A})$-translates) by their K_∞-finite vectors. An *automorphic representation* is any irreducible subquotient of the space of automorphic forms.

Remark 14.2.6. The definition of automorphic forms given above is not standard. Usually, all the conditions of Proposition 14.2.4 are imposed on automorphic forms, while we omitted K_∞-finiteness. The problem with the standard definition is that it depends on the choice of K_∞ (only up to translation by $G(k_\infty)$-though, since all K_∞ are conjugate), and it doesn't produce a representation of $G(k_\infty)$, but a $(\mathfrak{g}, K_\infty)$-module. On the other hand, the definition that we gave contains the clumsy requirement that the representation is generated by its K_∞-finite vectors (a condition, though, that clearly does not depend on the choice of K_∞). This condition is equivalent to a *finite length* condition (for the K^S-invariants of the representation, where K^S is an open subgroup away from a finite number of places S), i.e., we do not allow for "automorphic forms" to be approximable by vectors belonging to an increasing sum of finite-length representations, without belonging to a finite-length subsum.

At this point, it is not clear from the definition that the subrepresentation of $C_{mg}([G])^\infty$ generated by an automorphic form has bounded growth. However, it is true, and follows from the existence of exponents [also behind Harish-Chandra's finiteness theorem, not included yet].

14.3. Modular and cusp forms (analytic theory)

Let $G = \mathrm{GL}_2(\mathbb{R})^0$, acting on the complex upper half plane $\mathcal{H}$ on the left by Möbius transformations. Let Γ be a discrete subgroup, so that $\Gamma\backslash\mathcal{H}$ has finite volume. We may assume $-I \in \Gamma$ and $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$. Let $k \in \mathbb{Z}_{>0}$ and let $\chi : \Gamma \to S^1$ be a character satisfying $\chi(-I) = (-1)^k$.

Definition 14.3.1. Define $\mathcal{M}_k(\Gamma, \chi)$ to be the set of all holomorphic functions f on $\mathcal{H}$ satisying the following two conditions: $f(\gamma z) = \chi(\gamma)(cz + d)^k f(z)$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$; and f is holomorphic at the cusps of Γ. Let $\mathcal{S}_k(\Gamma, \chi) := \{f \in \mathcal{M}_k(\Gamma, \chi) |\ f \text{ vanishes at the cusps}\}$. Elements of $\mathcal{M}_k(\Gamma, \chi)$ are called *modular forms*, while those of $\mathcal{S}_k(\Gamma, \chi)$ are called *cusp forms*.

Now, we present modular and cusp forms as sections of some line bundles, following Deligne [**Del73**]. Towards this purpose, we construct a line bundle on $\mathcal{H}$ by presenting it as a space equipped with a universal elliptic curve and pushing forward the sheaf of differentials. We will demonstrate an action of G on the sections of the sheaf, and then explain the relationship of modular forms and cusp forms to this line bundle.

Let $\mathrm{Isom}(\mathbb{R}^2, \mathbb{C})$ be the set of isomorphisms of $\mathbb{R}^2$ and $\mathbb{C}$, as $\mathbb{R}$-vector spaces, and $\mathrm{Hom}^-(\mathbb{R}^2, \mathbb{C})$ the subset of orientation-reversing ones.[1] The structure of a complex vector space on $\mathbb{C}$ endows it with a natural structure of a two-dimensional complex submanifold of $\mathbb{C}^2$, with a free action of $\mathbb{C}^\times$. Moreover, it has a left action of $\mathrm{GL}_2(\mathbb{R})^0$, induced from its right action on $\mathbb{R}^2$, whose elements we think of as row vectors: $g{\cdot}T(v) := T(vg)$. The quotient $\mathrm{Hom}^-(\mathbb{R}^2, \mathbb{C})/\mathbb{C}^\times$ is thus a one-dimensional complex manifold with a $\mathrm{GL}_2(\mathbb{R})^0$-action, which parametrizes orientation-reversing

[1] That's unfortunate, and contrary to the convention of Deligne, but reproduces the usual action on $\mathcal{H}$

complex structures on $\mathbb{R}^2$. We identify this quotient with the complex upper half plane $\mathcal{H}$, by sending the class of a homomorphism T to $\frac{T(1,0)}{T(0,1)} \in \mathcal{H}$, and then $\mathrm{GL}_2(\mathbb{R})^0$ acts on the left by Möbius transformations $\gamma \cdot z = \frac{az+b}{cz+d}$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

By construction, the space $\mathcal{H}$ comes equipped with a complex line bundle ω^{-1}, whose pullback to $\mathrm{Hom}^-(\mathbb{R}^2, \mathbb{C})$ is the structure sheaf $\mathcal{O}$ of holomorphic functions. Equivalently, sections of ω^{-1} are real analytic functions $\sigma : \mathcal{H} \to \mathbb{R}^2$, such that for one, equivalently any, complex analytic lift $\mathcal{H} \ni z \mapsto T_z \in \mathrm{Hom}^-(\mathbb{R}^2, \mathbb{C})$, the composition $z \mapsto T_z(\sigma(z))$ is holomorphic. (Note that such lifts T exist, for example by fixing that $T(0,1) = 1$, otherwise we would need to make the same statements locally.) The action of $\mathrm{GL}_2(\mathbb{R})^0$ induces a $\mathrm{GL}_2(\mathbb{R})^0$-equivariant structure on ω^{-1}; we define this as a *right* action on sections, by

$$\sigma|_1\gamma(z) = \sigma(\gamma z)^\gamma,$$

where the exponent denotes the right action of γ on $\mathbb{R}^2$ (and the number "1" stands for the first power of ω^{-1}).

Notice that, by construction we have an isomorphism of the associated real analytic vector bundle $\omega_{\mathbb{R}}^{-1}$ with the constant real analytic vector bundle with fiber $\mathbb{R}^2$, such that constant sections of $\mathbb{R}^2$ correspond to holomorphic sections. (Such a structure is called a *variation of complex structure* on $\mathbb{R}^2$ over $\mathcal{H}$.) This gives rise to a $\mathrm{GL}_2(\mathbb{R})^0$-equivariant surjection

$$\mathcal{O} \otimes \underline{\mathbb{C}^2} \to \omega^{-1} \tag{14.3.1.1}$$

of complex vector bundles on $\mathcal{H}$ (where $\underline{\mathbb{C}^2}$ denotes the constant sheaf).

Note that ω^{-1} can be trivialized as a complex vector bundle (after all, $\mathcal{H}$ is simply connected), *but not $GL_2(\mathbb{R})^0$-equivariantly so.* In fact:

Lemma 14.3.2. *The section $\mathcal{H} \ni z \mapsto T_z \in Hom^-(\mathbb{R}^2, \mathbb{C})$ determined by $T_z(0,1) = 1$ (hence $T_z(1,0) = z$) induces a trivialization of ω^{-1}, such that the (right) action of $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{R})^0$ on sections of ω^k is given by*

$$f|_k\gamma(z) = (cz+d)^{-k} f(\gamma z). \tag{14.3.2.1}$$

Moreover, there is an equivariant isomorphism of complex line bundles, $\omega^2 = \Omega^1$ (the bundle of holomorphic 1*-forms) on $\mathcal{H}$.*

Proof. For the first statement, if $\sigma = (\sigma_1, \sigma_2)$ is a section of ω^{-1}, and f is the corresponding section of the trivial line bundle induced by this trivialization, then f is given by $f(z) = T_z \circ \sigma(z) = z\sigma_1(z) + \sigma_2(z)$.

The action of an element γ as above is hence given by $f|_1\gamma(z) = T_z \circ (\sigma|_1\gamma)(z) = T_z(\sigma(\gamma z)^\gamma) = (az+b)\sigma_1(\gamma z) + (cz+d)\sigma_2(\gamma_z) = (cz+d)f(\gamma z)$, and the case of a general power of ω is immediate.

The sheaf Ω^1 of differential one-forms can be trivialized by the global section dz on $\mathcal{H}$, and then it is immediate to check that the right action of γ sends the form $f(z)dz$ to $f|_2\gamma(z)dz$, identifying the trivializations of Ω^1 and ω^2 equivariantly. $\square$

Now let Γ be as above, and assume additionally that Γ is torsion free. The latter assumption ensures that it acts properly discontinuously on $\mathcal{H}$ and $\Gamma\backslash\mathcal{H} =: Y_\Gamma$ has a unique complex manifold structure making $\mathcal{H} \to \Gamma\backslash\mathcal{H}$ a local analytic isomorphism.

If Y_Γ is not compact, we let X_Γ be the compactification that is obtained by adding cusps. [Cusps and their comparison to adelic reduction theory need to be added.]

The sheaves ω^{-1}, Ω^1 being equivariant, they extend to the quotient Y_Γ, and they admit a natural extension to X_Γ, to be denoted by the same symbols: the sheaf Ω^1 as the sheaf of one-forms; for the sheaf ω^{-1}, we may assume (by applying a Möbious transformation) that the cusp of interest is the one-point compactification of $\Gamma_\infty\backslash\mathcal{H}$ at $i\infty$, where Γ_∞ is a discrete subgroup of upper triangular unipotent matrices; then we declare the Γ_∞-invariant section $\sigma : \mathcal{H} \to \mathbb{R}^2$, $\sigma(z) = (0,1)$ of ω^{-1} to extend to a non-zero section at the cusp. The comparison of Lemma 14.3.2 extends to the cusps as follows:

Lemma 14.3.3. *In a neighborhood of a cusp ∞, we have*

$$\Omega = \omega^2(\infty). \tag{14.3.3.1}$$

Proof. Identifying a neighborhood of the cusp with a neighborhood of $i\infty$ in $\Gamma_\infty\backslash\mathcal{H}$, as above, and using the trivialization of Lemma 14.3.2, the non-zero section $\sigma(z) = (0,1)$ of ω^{-1} corresponds to the constant function $f_{-1}(z) = 1$, and therefore the function $f_2 = f_{-1}^{-2} = 1$ corresponds to a non-zero section of ω^2 in a neighborhood of the cusp. If $\begin{pmatrix} 1 & \alpha \\ & 1 \end{pmatrix}$ is a generator for Γ_∞, a holomorphic coordinate at the cusp is given by $q_\alpha = e^{\frac{2\pi i z}{\alpha}}$, so $dz = \frac{\alpha}{2\pi i}\frac{dq_\alpha}{q_\alpha}$, and we see that the corresponding differential form $f_2 dz = dz$ has a pole of order 1 at $q_\alpha = 0$. $\square$

Now we are ready to express modular forms and cusp forms in terms of these line bundles; the proposition below could also have been taken as the definition.

Proposition 14.3.4. *Under the trivializations of Lemma 14.3.2, $\mathcal{M}_k(\Gamma,1)$ is the image of $H^0(X_\Gamma, \omega^{\otimes k}) \to H^0(Y_\Gamma, \omega^{\otimes k})$, and $\mathcal{S}_k(\Gamma,1)$ is the image of $H^0(X_\Gamma, \omega^{\otimes k}(-D)) \to H^0(Y_\Gamma, \omega^{\otimes k})$, where D denotes the divisor corresponding to the cusps of X_Γ.*

Proof. This is immediate from Lemma 14.3.2, except for the behavior at the cusps. We have seen that, by definition, the constant function corresponds under the above trivialization to a section of ω^k over $\Gamma_\infty\backslash\mathcal{H}$, hence modular forms of weight k extend to sections of ω^k at the cusps, and cusp forms extend to sections vanishing at the cusps. $\square$

Now let $\mathcal{F}_\mathbb{R}$ be the *dual* of the constant sheaf $\underline{\mathbb{R}^2}$ over $\mathcal{H}$. By using the standard symplectic form on $\mathbb{R}^2$, we can and will identify it with $\underline{\mathbb{R}^2} \otimes \det^{-1}$, equivariantly under the $\mathrm{GL}_2(\mathbb{R})^0$-action. Let $\mathcal{F} = \mathcal{F}_\mathbb{R} \otimes_\mathbb{R} \mathbb{C}$. The surjection (14.3.1.1) induces, dually, an injection $\omega \hookrightarrow \mathcal{O} \otimes \mathcal{F} \otimes \det^{-1}$ which, for every non-negative integer k, gives rise to an injection

$$\omega^k \hookrightarrow \mathcal{O} \otimes \mathrm{Sym}^k \mathcal{F} \otimes \det{}^{-k}. \tag{14.3.4.1}$$

In particular, the restriction to $\mathrm{SL}_2(\mathbb{R})$ is an equivariant injection to $\mathcal{O} \otimes \mathrm{Sym}^k \mathcal{F}$, and both sheaves descend to Y_Γ.

From these maps, (14.3.3.1), and de Rham cohomology, we obtain:

$$H^0(Y_\Gamma, \omega^{\otimes k}) \xrightarrow{\sim} H^0(Y_\Gamma, \omega^{\otimes k-2} \otimes \Omega^1_{Y_\Gamma}) \to H^1(Y_\Gamma, Sym^{k-2}\mathcal{F}) \tag{14.3.4.2}$$

Theorem 14.3.5 (Shimura isomorphism)**.** *The map (14.3.4.2) carries $\mathcal{S}_k(\Gamma,1)$* into *$H^1(Y_\Gamma, Sym^{k-2}\mathcal{F})$ and induces an isomorphism*

$$\mathcal{S}_k(\Gamma,1) \oplus \overline{\mathcal{S}_k(\Gamma,1)} \xrightarrow{\sim} \tilde{H}^1(Y_\Gamma, Sym^{k-2}\mathcal{F}), \tag{14.3.5.1}$$

where $\tilde{H}^\bullet$ denotes the image of $H_c^\bullet \to H^\bullet$ (cohomology with compact supports to cohomology without supports).

Proof. See [**Del73**] for references and further discussion. $\square$

Now we describe the above sheaves and constructions in terms of moduli of elliptic curves. The benefit of doing so is that it allows to endow the spaces Y_Γ, X_Γ, and the above sheaves, with algebro-geometric structure over the rational numbers or appropriate rings of integers.

Definition 14.3.6. An *elliptic curve* in the category of complex manifolds is a pair (E, e) consisting of a compact Riemann surface of genus one, and a point on it. More generally, an *elliptic curve over a complex manifold S* is a smooth (submersive) morphism of complex manifolds $E \to S$, whose fibers are elliptic curves, equipped with a section $e : S \to E$.

Equivalent definitions mention the abelian group structure on the elliptic curve, which arises by identifying it with its Jacobian, by sending a point x to the divisor $(x) - (e)$.

The space $S = \mathrm{Isom}(\mathbb{R}^2, \mathbb{C})$ comes equipped with an elliptic curve E_0, defined by the following short exact sequence of sheaves:

$$0 \to \underline{\mathbb{Z}^2} \to \omega^{-1} \to E_0 \to 0.$$

Moreover, this elliptic curve comes equipped with the following structure:

- an identification of its fundamental group with $\mathbb{Z}^2$; equivalently, an identification of the local system of homology groups $(R^1 f_* \underline{\mathbb{Z}})^\vee$ (where $R^1 f_*$ denotes the first derived functor of pushforward, i.e., fiberwise cohomology), or cohomology groups $R^1 f_* \underline{\mathbb{Z}}$, with the constant sheaf $\underline{\mathbb{Z}^2}$;
- an identification of the analytic sheaf $e^* \Omega^1_{E_0/S} = f_* \Omega^1_{E_0/S}$ with the sheaf ω.

Proposition 14.3.7. (1) *The functor which associates to each complex manifold S the set of isomorphism classes of elliptic curves ($f : E \to S$, $e : S \to E$), equipped with isomorphisms $e^*\Omega^1_E \simeq \mathcal{O}$ and $R^1 f_* \underline{\mathbb{Z}} \simeq \underline{\mathbb{Z}^2}$ is representable by the complex manifold Isom$(\mathbb{R}^2, \mathbb{C})$, equipped with a universal elliptic curve E_0.*

(2) *The functor which associates to each analytic space S the set of isomorphism classes of elliptic curves over S, equipped with an isomorphism $R^1 f_* \underline{\mathbb{Z}} \simeq \underline{\mathbb{Z}^2}$ is represented by the complex manifold Isom$(\mathbb{R}^2, \mathbb{C})/\mathbb{C}^\times$.*

Proof. Omitted. $\square$

The exterior square $\wedge^2 R^1 f_* \underline{\mathbb{Z}}$ is canonically trivialized by the fundamental class corresponding to the complex orientation, and $\mathcal{H} = \mathrm{Hom}^-(\mathbb{R}^2, \mathbb{C})/\mathbb{C}^\times$ represents those isomorphisms for which the induced isomorphism $\wedge^2 R^1 f_* \underline{\mathbb{Z}} \xrightarrow{\sim} \wedge^2 \underline{\mathbb{Z}^2}$ sends this class to $e_2 \wedge e_1$, in the standard basis.

The map (14.3.1.1), now, and its dual (14.3.4.1) corresponds to the Hodge filtration

(14.3.7.1)

$$0 \to \omega = R^0 f_* \Omega^1_{E_0/S} \to \mathcal{O} \otimes R^1 f_* \underline{\mathbb{R}} \to \omega^{-1} = (R^0 f_* \Omega^1_{E_0/S})^\vee = R^1 f_* \mathcal{O}_{E_0} \to 0.$$

(Note that in this discussion we have been ignoring determinant factors when identifying the constant sheaf with its dual, since the factors above do not carry an action of $\mathrm{GL}_2(\mathbb{R})^0$, but only of its subgroup $\mathrm{SL}_2(\mathbb{Z})$.)

14.4. Maaß forms

Definition 14.4.1. For fixed k ("weight") $\in \mathbb{Z}$, define *Maaß differential operators* on $C^\infty(\mathcal{H})$ as follows:
$R_k := iy\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{k}{2}$
$L_k := -iy\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - \frac{k}{2}$
$\Delta_k := -y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) + iky\frac{\partial}{\partial x} = -R_{k-2}L_k + \frac{k}{2}(1-\frac{k}{2})$

Let $G = \mathrm{GL}_2(\mathbb{R})^0$ act on $C^\infty(\mathcal{H})$ by $(f|_k g)(z) = \left(\frac{c\overline{z}+d}{|cz+d|}\right)^k f(gz)$, $g = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in G$. Let Γ be a discontinuous subgroup of $\mathrm{SL}_2(\mathbb{R}) \subset G$ containing $-I$ with $\Gamma\backslash\mathcal{H}$ of finite volume, and let χ be a unitary character of Γ. We take $C^\infty(\Gamma\backslash\mathcal{H},\chi,k)$ to denote the set of functions, $f, \in C^\infty(\mathcal{H})$ satisfying $\chi(\gamma)f(z) = (f|_k\gamma)(z)$, for $\gamma\in\Gamma$. (Note that this forces $\chi(-I) = (-1)^k$). A short calculation gives us the following lemma:

Lemma 14.4.2. *R_k and L_k act as weight raising and lowering operators respectively, namely: $R_k : C^\infty(\Gamma\backslash\mathcal{H},\chi,k) \to C^\infty(\Gamma\backslash\mathcal{H},\chi,k+2)$; $L_k : C^\infty(\Gamma\backslash\mathcal{H},\chi,k) \to C^\infty(\Gamma\backslash\mathcal{H},\chi,k-2)$; $\Delta_k : C^\infty(\Gamma\backslash\mathcal{H},\chi,k) \to C^\infty(\Gamma\backslash\mathcal{H},\chi,k)$.*

Proof. Left to the reader. $\square$

Definition 14.4.3. A *Maaß form* of weight k for Γ is a smooth complex valued function f on $\mathcal{H}$ that satisfies:

(1) $f \in C^\infty(\Gamma\backslash\mathcal{H},\chi,k)$, for some χ;
(2) $\Delta_k f = \lambda f$, some $\lambda \in \mathbb{C}$;
(3) f has moderate growth at cusps of Γ.

Here, moderate growth at ∞ means that $f(x+iy)$ is bounded by a polynomial in y as $y\to\infty$. For a general cusp of Γ, $a \in \mathbb{R}\cup\infty$, let $\xi \in \mathrm{SL}_2(\mathbb{R})$ be such that $\xi(\infty) = a$. Then f is said to be of moderate growth at a if $f|_k\xi \in C^\infty(\xi^{-1}\Gamma\xi\backslash\mathcal{H},\xi\chi\xi^{-1},k)$ is of moderate growth at ∞.

Remark 14.4.4. Observe that the vanishing of a Maaß form f under the operator L_k is equivalent to $y^{-k/2}f$ satisfying the Cauchy–Riemann equations. This defines an embedding

$$\mathcal{M}_k(\Gamma,\chi) \hookrightarrow C^\infty(\Gamma\backslash\mathcal{H},\chi,k). \tag{14.4.4.1}$$

As $y^{k/2}f' \in \ker(L_k)$, it is Δ_k-eigenfunction with eigenvalue $\frac{k}{2}(1-\frac{k}{2})$. Moderate growth is automatic, hence the embedding above identifies holomorphic modular forms with a subspace of the Maaß forms, determined by the vanishing of the weight-lowering operators.

When we pass to $G = \mathrm{GL}_2(\mathbb{R})^0$-representations in the next section, it will turn out that the vanishing of the weight-lowering operator places the image of holomorphic modular forms in the *discrete series* representation with a certain eigenvalue for the Casimir operator corresponding to k.

14.5. Classical automorphic forms

With notation as in the previous section, namely $G = \mathrm{GL}_2(\mathbb{R})^0$ and χ a character of the lattice $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$, let $C^\infty(\Gamma\backslash G, \chi)$ denote the space of complex-valued smooth functions F on G satisfying $F(\gamma g) = \chi(\gamma)F(g)$ for $\gamma \in \Gamma$, $g \in G$. G acts on this space by right translation. Let $C^\infty(\Gamma\backslash G, \chi, k) \subset C^\infty(\Gamma\backslash G, \chi)$ be the set of functions, F, additionally satisfying $F(gk_\theta) = e^{ik\theta}F(g)$, for $\kappa_\theta \in SO_2(\mathbb{R})$ that gives clockwise rotation by θ.

Proposition 14.5.1. *There exists an inclusion* $\sigma_k : C^\infty(\Gamma\backslash \mathcal{H}, \chi, k) \to C^\infty(\Gamma\backslash G, \chi, k)$ *given by* $f \mapsto (F : g \mapsto (f|_k g)(i))$. *Furthermore, there exist elements* R, L *and* Δ *of* $U(\mathfrak{g}_\mathbb{C})$ *acting on* $C^\infty(\Gamma\backslash G, \chi, k)$, *that commute with the action of* R_k, L_k *and* Δ_k *respectively.* Δ *is (up to a scalar) the Casimir element of* $U(\mathfrak{g})$.

Proof. That the image F of f belongs to $C^\infty(\Gamma\backslash G, \chi, k)$ is quickly checked. For the rest of the statements, notice first that every element $g \in G$ can be uniquely written as $\begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix}\begin{pmatrix} y^{1/2} & xy^{-1/2} \\ 0 & y^{-1/2} \end{pmatrix}\kappa_\theta$. Here, x, y and u are uniquely determined, while θ is uniquely determined mod 2π. Define the following elements of $U(\mathfrak{g}_\mathbb{C})$: $R := \frac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}$ and $L := \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}$.

Computing the action of R, L and the Casimir element Δ in terms of x, y, u and θ, we get that $dR = e^{2i\theta}(iy\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{1}{2i}\frac{\partial}{\partial \theta})$, $dL = e^{2i\theta}(-iy\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - \frac{1}{2i}\frac{\partial}{\partial \theta})$ and $d\Delta = -y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) + y\frac{\partial^2}{\partial x \partial \theta}$. One checks that these operators commute with the action of R_k, L_k and Δ_k in the desired fashion.

For details of the calculation, refer to [**Bum97**, Theorem 2.2.5]. $\square$

Let ω be a (unitary) character of the center $Z(G)$ of G, agreeing with χ on $-I$. Consider $C^\infty(\Gamma\backslash G, \chi, \omega) \subset C^\infty(\Gamma\backslash G, \chi)$ denoting functions, F, that additionally satisfy $F(zg) = \omega(z)F(g)$.

Definition 14.5.2. $\mathcal{A}(\Gamma\backslash G, \chi, \omega)$ is the subspace of those elements of $C^\infty(\Gamma\backslash G, \chi, \omega)$ that are $Z(U(\mathfrak{g}_\mathbb{C}))$-finite, K-finite, and satisfy the condition of moderate growth below. Elements of $\mathcal{A}(\Gamma\backslash G, \chi, \omega)$ are called *automorphic forms*.

Moderate growth here means that $\exists k > 0$ such that $\forall D \in U(\mathfrak{g}_\mathbb{C})$, $|Df(g)|$ has order of growth less than $\|g\|^k$ where $\|g\|$ can be defined to be a height function obtained by pulling back the maximum function along the embedding $G = \mathrm{GL}_2(\mathbb{R})^0 \hookrightarrow \mathbb{R}^5$. This is the embedding which sends a matrix to its 4 coordinates and the determinant. Observe that since ω is a unitary character, $|Df(g)|$ is infact a well defined function on $\Gamma\backslash\mathrm{SL}_2(\mathbb{R})$.

Below, we explain the relationship between Maaß forms and Classical automorphic forms.

Remark 14.5.3. Notice that this will also cover the relationship between modular forms and classical automorphic forms since modular forms give rise to certain Maaß forms (kernels of L_k operators) (Remark 14.4.4).

Let f be a Maaß form of weight k, for character χ. Let $F(g) := (f|_k g)(i) = \left(\frac{-ci+d}{|ci+d|}\right)^k f(gi)$, $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$.

Proposition 14.5.4. *The above map $f \mapsto F$ identifies Maaß forms of weight k for character χ with $\mathcal{A}(\Gamma\backslash G, \chi, \omega) \cap C^\infty(\Gamma\backslash G, \chi, k)$, where ω is the character which is trivial on the identity component of $\mathbb{R}^\times$ and equal to $(-1)^k$ on -1.*

Proof. Let $z = \begin{pmatrix} r & \\ & r \end{pmatrix} \in Z(G)$.

$F(zg) = ((f|_k z)|_k g)(i) = (\frac{-ci+d}{|ci+d|})^k (f|_k z)(gi) = (\frac{r}{|r|})^k (\frac{-ci+d}{|ci+d|})^k f(gi) = \omega(r) F(g)$

Now, let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. $F(\gamma g) = ((f|_k \gamma)|_k g)(i) = \chi(\gamma)(f|_k g)(i) = \chi(\gamma) F(g)$.

Therefore, $F \in \mathcal{A}(\Gamma\backslash G, \chi, \omega)$.

By Proposition 14.5.1, as f is an eigenvector of Δ_k, so is $F = \sigma_k(f)$ of Δ. Therefore, F is $Z(U(\mathfrak{g}_{\mathbb{C}}))$-finite, as the latter is generated by Δ (by Theorem 6.5.7, Theorem 6.4.2 and Definition 6.4.4). Additionally, being an element of $C^\infty(\Gamma\backslash G, \chi, k)$, F is also $K = SO_2(\mathbb{R})$-finite. Finally, the moderate growth conditions for f and F turn out to be equivalent, as in Remark 14.2.2. $\square$

14.6. Classical and adelic automorphic forms

Let $K_0(N) = \prod_p \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_p) | c \equiv 0 \bmod N \}$, and $K_1(N) = \prod_p \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_p) | c \equiv 0, d \equiv 1 \bmod N \}$. $\mathbb{A}$ refers to the ring of adeles of $\mathbb{Q}$.

Our objective now is to present a classical automorphic form (which is, in particular, a function on $\Gamma_0(N)\backslash \mathrm{GL}_2(\mathbb{R})^0$) as an automorphic form on $\mathrm{GL}_2(\mathbb{A})$. We will first construct an isomorphism between $\Gamma_0(N)\backslash \mathrm{GL}_2(\mathbb{R})^0$ and a quotient of $\mathrm{GL}_2(\mathbb{A})$, and then demonstrate a way to pull back elements of $\mathcal{A}(\Gamma\backslash G, \chi, \omega)$ to automorphic forms on $\mathrm{GL}_2(\mathbb{A})$ with suitably defined characters of $K_0(N)$ and $Z(\mathrm{GL}_2(\mathbb{A}))$, where Z denotes the center of GL_2.

Lemma 14.6.1.
(1) *For $i = 0, 1$, $\Gamma_i(N)\backslash GL_2(\mathbb{R})^0$ and $GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A})/K_0(N)$ are isomorphic as $GL_2(\mathbb{R})^0$-spaces, under the map induced by $GL_2(\mathbb{R})^0 \hookrightarrow GL_2(\mathbb{A})$.*
(2) *The above induces an isomorphism of $SL_2(\mathbb{R})$-spaces $\Gamma_0(N)\backslash SL_2(\mathbb{R})$ and $Z(\mathbb{A})GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A})/K_0(N)$*

Proof. Consider the determinant map $\mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A}) \to \mathbb{Q}^\times \backslash \mathbb{A}^\times$. Every fiber is represented by an element of $\mathrm{GL}_2(\mathbb{R})$, and by the strong approximation Theorem 13.6.5, applied to the group SL_2, each fiber is acted upon transitively by $\mathrm{SL}_2(\mathbb{R})S$, for any open subgroup S of $\mathrm{SL}_2(\mathbb{A}_f)$. Therefore, the map of double coset spaces

$$\mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A})/K_i(N)\mathrm{GL}_2(\mathbb{R})^0 \to \mathbb{Q}^\times \backslash \mathbb{A}^\times / \det(K_i(N))\mathbb{R}_+^\times$$

is a bijection. The right hand side represents the narrow class group of $\mathbb{Q}$, hence has only one element. Therefore, $\mathrm{GL}_2(\mathbb{A}) = \mathrm{GL}_2(\mathbb{Q})\mathrm{GL}_2(\mathbb{R})^0 K_i(N)$.

We obtain a surjection $f : \mathrm{GL}_2(\mathbb{R})^0 \twoheadrightarrow \mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A})/K_i(N)$. Suppose $f(g) = f(g')$. Then, $g = \gamma g' k$ for some $\gamma \in \mathrm{GL}_2(\mathbb{Q})$, $k \in K_i(N)$. Writing $\gamma = \gamma_f \gamma_\infty$ (where γ_f corresponds to the part in finite adeles and γ_∞ is the part corresponding to archimedean places), $g = \gamma_f k \gamma_\infty g' = \gamma_\infty g'$. Therefore, $\gamma_f = k^{-1} \in K_i(N)$ and $\gamma_\infty = g' g^{-1}$ has positive determinant. So, $\gamma = \gamma_f \gamma_\infty \in \mathrm{GL}_2(\mathbb{Q}) \cap \mathrm{GL}_2(\mathbb{R})^0 K_i(N) = \Gamma_i(N) \Rightarrow g \in \Gamma_i(N) g'$. We get the first isomorphism in the statement of the proposition.

Taking quotients by $Z(\mathbb{R}^+)$ on both sides, and using the fact that $Z(\mathbb{A}) = Z(\mathbb{R}^+)Z(\mathbb{Q})(Z(\mathbb{A}) \cap K_0(N))$ (again by the triviality of the narrow class group), we get the second statement of the proposition. $\square$

Next, we explain how to produce from χ and ω characters λ and $\tilde{\omega}$ of $K_0(N)$ and $Z(\mathrm{GL}_2(\mathbb{A}))$ respectively. For this, we use again the fact that

$$\mathbb{A}^\times/\mathbb{Q}^\times \cong \mathbb{R}_+^\times \prod_{p<\infty} \mathbb{Z}_p^\times,$$

as in the proof of Lemma 14.6.1.

To construct a character $\lambda : K_0(N) \to \mathbb{C}^\times$, we proceed as follows. Projection to the places dividing N and the chinese remainder theorem gives us $\mathbb{A}^\times/\mathbb{Q}^\times \cong \mathbb{R}_+^\times \prod_{p<\infty} \mathbb{Z}_p^\times \twoheadrightarrow \prod_{p|N} \mathbb{Z}_p^\times \twoheadrightarrow (\mathbb{Z}/N\mathbb{Z})^\times$. Composing this projection with χ, we get a character $\tilde{\chi} : \mathbb{A}^\times/\mathbb{Q}^\times \to \mathbb{C}^\times$. Let $\rho : K_0(N) \to \mathbb{A}^\times/\mathbb{Q}^\times$ be the map given by sending $(\begin{pmatrix} a_p & b_p \\ c_p & d_p \end{pmatrix})_p$ to $(d_p)_p$. Define the character λ to be $\tilde{\chi}^{-1} \circ \rho$. Observe that if l is a prime not dividing N, $l = (l)_{p\le\infty} \equiv (1, (a_p)_p) \in \mathbb{R}_+^\times \prod \mathbb{Z}_p^\times$, where for $p \neq l$, $a_p = l^{-1}$ and $a_l = 1$. Therefore, l projects to $\overline{l}^{-1} \in (\mathbb{Z}/N\mathbb{Z})^\times$. By multiplicativity, any d coprime to N projects to $\overline{d}^{-1} \in (\mathbb{Z}/N\mathbb{Z})^\times$. Composing this map with χ, we observe that $\tilde{\chi}(d) = \chi^{-1}(\overline{d})$. If $\gamma = (\begin{pmatrix} a & b \\ c & d \end{pmatrix})_{p<\infty} \in K_0(N)$ such that $a, b, c, d \in \mathbb{Z}$, then the above discussion shows that $\lambda(\gamma^{-1}) = \tilde{\chi}^{-1}(\rho(\gamma^{-1})) = \chi(\overline{d})$.

The central character $\tilde{\omega}$ is given as follows: $Z(\mathrm{GL}_2(\mathbb{A})) \cong \mathbb{A}^\times \twoheadrightarrow \mathbb{A}^\times/\mathbb{Q}^\times \cong \mathbb{R}_+^\times \prod_{p<\infty} \mathbb{Z}_p^\times \xrightarrow{\mu} \mathbb{C}^\times$. Here μ is the map sending $(a_p)_{p\le\infty} \mapsto \omega(a_\infty)\tilde{\chi}^{-1}((a_p))$.

Proposition 14.6.2. *Let $\tilde{\omega}$ be as defined above. The isomorphisms of Lemma 14.6.1 give rise to an inclusion of $\mathcal{A}(\Gamma\backslash G, \chi, \omega)$ into the set of automorphic forms on $GL_2(\mathbb{A})$ with central quasi-character $\tilde{\omega}$.*

Proof. Let $\lambda : K_0(N) \to \mathbb{C}^\times$ be as defined above. Let $F \in \mathcal{A}(\Gamma\backslash G, \chi, \omega)$. Let $g \in \mathrm{GL}_2(\mathbb{A})$. We have then that $g = \gamma g_\infty k$ for some $\gamma \in \mathrm{GL}_2(\mathbb{Q})$, and $k \in K_0(N)$. Consider the function $\phi : g \mapsto F(g_\infty)\lambda(k)$. To show this is well-defined, we need to show that if $g'_\infty = \gamma g_\infty k$ then $F(g'_\infty) = F(g_\infty)\lambda(k)$.

Notice that $g'_\infty = \gamma g_\infty k \Rightarrow g'_\infty = \gamma_f \gamma_\infty g_\infty k$ (writing γ as a product of the finite part and the archimedean part). Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Therefore, $g'_\infty = \begin{pmatrix} a & b \\ c & d \end{pmatrix}_\infty g_\infty$ and $k = \gamma_f^{-1} = (\begin{pmatrix} a & b \\ c & d \end{pmatrix})^{-1}_{p<\infty}$. This implies that $F(g'_\infty) = F(g_\infty)\chi(d)$, and now all we have to show is that $\chi(d) = \lambda((\begin{pmatrix} a & b \\ c & d \end{pmatrix})_{p<\infty})^{-1}$. But this follows from the construction λ.

$Z(\mathrm{GL}_2(\mathbb{A})) \simeq \mathbb{A}^\times = \mathbb{Q}^\times \mathbb{R}_+^\times \prod_p \mathbb{Z}_p^\times$ can be verified to be acting via the quasicharacter $\tilde{\omega}$ by computing separately the action of $\mathbb{Q}^\times$, $\mathbb{R}_+^\times$ and $\prod_p \mathbb{Z}_p^\times$. Let g again be equal to $g = \gamma g_\infty k$ for some $\gamma \in \mathrm{GL}_2(\mathbb{Q})$, and $k \in K_0(N)$. For $z = (z)_{p\le\infty} \in \mathbb{Q}^\times$, $\phi(zg) = F(g_\infty)\lambda(k) = \tilde{\omega}(z)\phi(g)$. For $z_\infty \in \mathbb{R}_+$, $\phi(z_\infty g) = F(z_\infty g_\infty)\lambda(k) = \omega(z_\infty)F(g_\infty)\lambda(k) = \tilde{\omega}(z)\phi(g)$. For $z = (z_p)_p \in \prod_p \mathbb{Z}_p^\times$, $\phi(zg) = F(g_\infty)\lambda(k)\lambda((\begin{pmatrix} z_p & \\ & z_p \end{pmatrix})_{p<\infty}) = \phi(g)\tilde{\chi}^{-1}((z_p)) = \tilde{\omega}(z)\phi(g)$. $\square$

Proposition 14.6.3. *Suppose $p \nmid N$. If $F \in \mathcal{A}(\Gamma\backslash G, \chi, \omega)$ is an eigenfunction of the classical Hecke operator T_p, then it is an eigenfunction of the measure given by the charasteristic function of $GL_2(\mathbb{Z}_p)\begin{pmatrix} p & \\ & 1 \end{pmatrix}GL_2(\mathbb{Z}_p)$, with the same eigenvalue. Furthermore, F is an eigenfunction of (the characteristic function of) $GL_2(\mathbb{Z}_p)\begin{pmatrix} p & \\ & p \end{pmatrix}GL_2(\mathbb{Z}_p)$ with eigenvalue $\chi(p)$.*

Proof. [Discussion of classical Hecke operators to be added.] □

14.7. Other chapters

(1) Introduction
(2) Basic Representation Theory
(3) Representations of compact groups
(4) Lie groups and Lie algebras: general properties
(5) Structure of finite-dimensional Lie algebras
(6) Verma modules
(7) Linear algebraic groups
(8) Forms and covers of reductive groups, and the L-group
(9) Galois cohomology of linear algebraic groups
(10) Representations of reductive groups over local fields
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CHAPTER 15

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Preamble

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CHAPTER 16

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In 5.3.18: *dual root system*
In 5.4.1: *based root system*, *simple roots*, *basis*
In 5.4.3: *Borel subalgebra*, *parabolic subalgebra*
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16.3. Other chapters

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