A Geometric Jacquet-Langlands Transfer for Automorphic Forms of Higher Weights

Jize Yu

Institute for Advanced Study

Plan of Presentation

- Main theorem 1:

  A geometric Jacquet-Langlands transfer

- The principal ingredient: main theorem 2

  The geometric Satake equivalence

- Discussion on the geometric Satake equivalence
- Sketch of the proof of main theorem 1

A Geometric Jacquet-Langlands Transfer

Set up

Let (G_1, X_1) and (G_2, X_2) be two Hodge type Shimura data:

- G_i: reductive group over $\mathbb{Q}$
- X_i: $G_i(\mathbb{R})$-conjugacy class of homomorphisms $\mu_i : \mathbb{S} := \mathrm{Res}_{\mathbb{R}}^{\mathbb{C}} \mathbb{G}_m \to G_{i\mathbb{R}}$.
- assumptions for Shimura data
- there exists an embedding $(G_i, X_i) \hookrightarrow (\mathrm{Sp}_{2n}, \mathcal{H}_n^{\pm})$, where

$$\mathcal{H}_n^{\pm} := \{A \in M_n(\mathbb{C}) | A^T = A, \mathrm{Im}(A) \neq 0\}.$$

- Let $\{\mu_i\}$ denote the conjugacy class of Hodge cocharacters determined by X_i.
- Let E_i denote the reflex field of (G_i, X_i).
- Let $K_i \subset G_i(\mathbb{A}_f)$ be a sufficiently small open compact subgroup.

Assumptions

- $\theta : G_{1,\mathbb{A}_f} \simeq G_{2,\mathbb{A}_f}$.
- there exists an inner twist $\Psi : G_1 \to G_2$.
- $K_{1,p} (\simeq K_{2,p})$ is *hyperspecial*.

- Let $\underline{G_i}$ denote the integral model of $G_{i,\mathbb{Q}_p}$ over $\mathbb{Z}_p$ determined by $K_{i,p}$. Then $\underline{G_1} \simeq \underline{G_2}$.
- We identify their Langlands dual groups and write it as $\hat{G}$.
- Choose an isomorphism $\iota : \mathbb{C} \simeq \bar{\mathbb{Q}}_p$.
- Write σ for the arithmetic Frobenius morphism of $\mathbb{F}_p$.
- Let $V_i := V_{\mu_i}$ be the irreducible representation of $\hat{G}/\mathbb{Q}_\ell$ of highest weight μ_i.

- Let $\nu \mid p$ be a place of E_1E_2 (determined by our choice of isomorphism ι).
- Let Sh_{μ_i} denote the mod p fiber of the canonical integral model for $\mathrm{Sh}_{K_i}(G_i, X_i)$, base changed to $\bar{k}_\nu$.
- Let $\mathcal{H}^p$ denote the prime-to-p Hecke algebra.

The Category $\mathrm{Coh}^{\hat{G}}(\hat{G}\sigma)$

- Let $\hat{G}\sigma$ be the coset in $\hat{G} \rtimes \langle\sigma\rangle$.
- Denote by $[\hat{G}\sigma/\hat{G}]$ the stack of unramified Langlands parameters over $\mathbb{Q}_\ell$.
- Let $\mathrm{Coh}^{\hat{G}}(\hat{G}\sigma)$ denote the abelian category of coherent sheaves on $[\hat{G}\sigma/\hat{G}]$.
- Let $\mathcal{J}$ denote the global section of the structure sheaf on $[\hat{G}\sigma/\hat{G}]$.

Main Theorem 1

Let $\mathcal{L}_i$ be an ℓ-adic the étale local system on Sh_{μ_i}. Under a mild technical assumption.

Theorem (Y.)

There exists a map

$$\mathrm{Spc} : \mathrm{Hom}_{\mathrm{Coh}^{\hat{G}}(\hat{G}\sigma)}(\widetilde{V}_1, \widetilde{V}_2) \to \mathrm{Hom}_{\mathcal{H}^p \otimes \mathcal{J}}(\mathrm{H}^*_c(\mathrm{Sh}_{\mu_1}, \mathcal{L}_1), \mathrm{H}^*_c(\mathrm{Sh}_{\mu_2}, \mathcal{L}_2)),$$

which is compatible with compositions on the source and target. If $\mathrm{Sh}_{\mu_1} = \mathrm{Sh}_{\mu_2}$ *is a Shimura set, the action of* $\mathrm{End}_{\mathrm{Coh}^{\hat{G}}(\hat{G}\sigma)}(\widetilde{V}_\mu)$ *on*

$$\mathrm{H}^*_c(\mathrm{Sh}_\mu, \mathbb{Q}_\ell) \simeq C_c(G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K, \mathbb{Q}_\ell)$$

coincides with the usual Hecke algebra action under the Satake isomorphism.

- The theorem generalizes a previous construction of Xiao-Zhu
  - $\hat{G}_{\bar{\mathbb{Q}}_\ell}$.
  - $\mathcal{L}_i$ equals to the constant sheaf.
- This result is about $\mathbb{Q}_\ell$-sheaves, but the integral coefficient geometric Satake equivalence is indispensable.

The Geometric Satake Equivalence

Background: the Satake isomorphism

- Let F be a non-archimidean local field with $\mathcal{O}$ its ring of integers and $k = \mathbb{F}_q$ its residue field. In other words, F is a finite extension of $\mathbb{Q}_p$ or is isomorphic to $\mathbb{F}_q((\varpi))$.
- Let G be a split connected reductive group scheme over $\mathcal{O}$.
- We fix a choice of a Borel subgroup and a maximal torus $T \subset B \subset G$.
- Let $\mathbb{X}_\bullet := \mathrm{Hom}(\mathbb{G}_m, G)$, and $\mathbb{X}^\bullet := \mathrm{Hom}(G, \mathbb{G}_m)$ denote the group of cocharacters and characters of G.
- Let W denote the Weyl group of T, and $\mathbb{X}_\bullet^+ := \mathbb{X}_\bullet / W$ denote the semigroup of dominant cocharacters.

The Spherical Hecke Algebra

- We can endow a topology on $G(F)$ such that it is a locally compact topological group.
- Let $K := G(\mathcal{O})$, then K is a maximal compact subgroup of $G(F)$.
- Choose the unique Haar measure such that the volume of K equals to 1.

Definition
The *spherical Hecke algebra* of G is defined to be

$$\mathcal{H}_G := \mathcal{C}_c(K\backslash G/K, \mathbb{Z}).$$

The following results are well-known

- (*Gelfand's trick*$\Rightarrow$) the convolution of functions

$$(f * g)(x) := \int_G f(y)g(y^{-1}x)dy$$

  endows $\mathcal{H}_G$ with a commutative algebra structure.
- (*Cartan Decomposition theorem*$\Rightarrow$) the set of characteristic functions

$$\{\mathbf{1}_{KgK} \mid g \in \mathbb{X}_\bullet^+\}$$

  forms a set of $\mathbb{Z}$-basis of $\mathcal{H}_G$.

THE LANGLANDS DUAL GROUP

- Let E be an algebraically closed field.
- Associate the root datum $(\mathbb{X}^\bullet, \mathbb{X}_\bullet, \Delta, \Delta^\vee)$ to our choice of $T \subset B \subset G$.
- Let $(\mathbb{X}_\bullet, \mathbb{X}^\bullet, \Delta^\vee, \Delta)$ denote the dual root datum.

Definition

We define the Langlands dual group $\hat{G}/E$ of G to be the unique (up to isomorphism) reductive group over E determined by the dual root datum $(\mathbb{X}_\bullet, \mathbb{X}^\bullet, \Delta^\vee, \Delta)$.

Here is a list of examples to be kept in mind.

Example

G	GL_n	SL_n	SO_{2n+1}	SO_{2n}	E_8
$\hat{G}$	GL_n	PGL_n	Sp_{2n}	SO_{2n}	E_8

The Classical Satake Isomorphism

- Let $\mathrm{Rep}_E(\hat{G})$ denote the category of finite-dimensional $\hat{G}$-representations over E.
- Let $R(\hat{G})$ denote the Grothendieck K-ring of $\mathrm{Rep}_E(\hat{G})$.

The Langlands' reinterpretation of the classical Satake isomorphism states the following.

Theorem

There exists an algebra isomorphism

$$\mathcal{H}_G \otimes E \simeq R(\hat{G}) \otimes E.$$

Slogan of geometric Satake

Motivation:

"Categorify" the classical Satake isomorphism.

- Categorifying $R(\hat{G})$ gives rise to $\mathrm{Rep}_E(\hat{G})$.
- Question: how to categorify $\mathcal{H}_G$?

The idea to answer the above question is

- Endow the quotient $G(F)/G(\mathcal{O})$ with an algebro-geometric structure.
- Consider the $G(\mathcal{O})$-equivariant perverse sheaves on $G(F)/G(\mathcal{O})$.

An Attempt

Let $G = GL_n$ and $F = \mathbb{Q}_p$.

- By a lattice in F^n, we mean
  - a finitely generated projective $\mathcal{O}$-module Λ, together with
  - an isomorphism $\Lambda \otimes_{\mathcal{O}} F \simeq F^n$.
- Write $\Lambda_0 := \mathcal{O}^n$ for the standard lattice.
- The quotient G/K maybe identified with the set of lattices via

$$gK \in G/K \longmapsto g\Lambda_0.$$

- Let $w_i := \text{diag}\{1^i 0^{n-i}\}$ be viewed as a dominant cocharater. The quotient $K\varpi^{w_i}K/K$ maybe identified with lattices

$$\{\Lambda \mid \varpi\mathcal{O}^n \subset \Lambda \subset \mathcal{O}^n, \text{ and length } (\mathcal{O}^n/\Lambda) = i\}.$$

There is a canonical bijection between $K\varpi^{w_i}K/K$ and the set of k-points of the Grassmannian variety $Gr(n-i,n)$ of $(n-i)$-planes in a fixed n-dimensional space.

WHY GEOMETRIC SATAKE

The geometric Satake equivalence has found profound applications in

- Langlands program: V. Lafforgue proved Langlands correspondence for reductive groups over function fields.
- number theory: Xiao-Zhu proved the "generic" case of the Tate conjecture on the mod p fibres of some Shimura varieties.

The Witt Vector Affine Grassmannian

We slightly change our setups.

- Let k be a fixed algebraically closed field of characteristic $p > 0$.
- Let $W(R)$ denote the ring of (p-typical) Witt vectors of any perfect k-algebra R.
- Let F be a totally ramified finite extension of $W(k)$, and $\mathcal{O}$ its ring of integers.
- Choose $\varpi \in \mathcal{O}$ to be a uniformizer.
- Let G be a split connected reductive group over $\mathcal{O}$.
- Let $\Lambda = \mathbb{Z}_\ell$ or $\mathbb{F}_\ell$ be the coefficient ring of sheaves considered later.

Let R be a perfect k-algebra, and H an affine group scheme of finite type defined over $\mathcal{O}$.

- $W_{\mathcal{O},n}(R) := W(R) \otimes_{W(k)} \mathcal{O}/\varpi^n$
- $W_{\mathcal{O}}(R) := \varprojlim_n W_{\mathcal{O},n}(R)$
- $D_R := \mathrm{Spec}(W_{\mathcal{O}}(R))$
- $D_R^\times := \mathrm{Spec}(W_{\mathcal{O}}(R)[1/\varpi])$

Definition

The the *(p-adic) jet group* (resp. *p-adic loop group*) of H is defined as the presheaf

$$L^+H(R) := H(W_{\mathcal{O}}(R))(\text{resp. } LH(R) := H(W_{\mathcal{O}}(R)[1/\varpi]))$$

over the opposite category of perfect k-algebras.

Definition

The *Witt vector affine Grassmannian* of G over k is defined as the fpqc quotient

$$Gr_G := [LG/L^+G],$$

over the opposite the category of perfect k-algebras.

It also has the following moduli interpretation

$$Gr_G(R) := \left\{ (\mathcal{E}, \phi)/\cong \middle| \begin{matrix} \mathcal{E} \to D_R \text{ is a } G\text{-torsor, and} \\ \phi : \mathcal{E} \mid_{D_R^\times} \simeq \mathcal{E}^0 \mid_{D_R^\times} \end{matrix} \right\},$$

where $\mathcal{E}^0$ denotes the trivial G-torsor.

Question:

Is Gr_G *representable* by nice geometric objects?

Answer

YES!

Theorem (Bhatt-Scholze)

The Witt vector affine Grassmannian Gr_G is represented by an inductive limit of the perfection of projective varieties over $\mathbb{F}_p$.

The geometry of affine Grassmannian

- Let β be a modification of two G-torsors $\mathcal{E}_1$ and $\mathcal{E}_2$.
- Choose isomorphisms $\phi_i : \mathcal{E}_i \simeq \mathcal{E}^0$.

Considering the following diagram

$$\begin{array}{ccc} \mathcal{E}_1 & \overset{\beta}{\dashrightarrow} & \mathcal{E}_2 \\ \phi_1 \downarrow & & \downarrow \phi_2 \\ \mathcal{E}^0 & \overset{\phi_2\beta\phi_1^{-1}}{\dashrightarrow} & \mathcal{E}^0 \end{array}$$

The *Cartan decomposition* identifies $\phi_2\beta\phi_1^{-1}$ as an element in $\mathbb{X}_\bullet^+$.

Definition

We denote the dominant cocharacter $\phi_2\beta\phi_1^{-1}$ by $\mathrm{Inv}(\beta)$ and call it the *relative position*.

Definition

For each $\mu \in \mathbb{X}_\bullet^+$, we define the *Schubert cell* (resp. *Schubert variety*) Gr_μ (resp. $Gr_{\leq\mu}$) as

$$Gr_\mu := \{(\mathcal{E}, \phi) \mid \mathrm{Inv}(\phi) = \mu\} (\text{resp. } Gr_{\leq\mu} := \{(\mathcal{E}, \phi) \mid \mathrm{Inv}(\phi) \leq \mu\}).$$

Lemma

Let $\mu, \nu \in \mathbb{X}_\bullet^+$, then

- *the Schubert variety $Gr_{\leq\mu}$ is the perfection of a projective variety defined over k,*
- *L^+G acts on $Gr_{\leq\mu}$ through a finite type quotient,*
- *$Gr_\mu \subset Gr_\nu$ if and only if $\mu \leq \nu$,*
- *the Zariski closure of Gr_μ equals to $Gr_{\leq\mu}$, and $Gr_{\leq\mu} = \cup_{\mu' \leq \mu} Gr_{\mu'}$.*

Example (Minuscule case)

Let $G = GL_n$, and $w_i = \mathrm{diag}\{1^i 0^{n-i}\}$.
An element $(\mathcal{E}, \beta) \in Gr_{w_i}(R)$ maybe identified with a $W_{\mathcal{O}}(R)$-*lattice* i.e.

- a finitely generated projective $W_{\mathcal{O}}(R)$-module Λ, with
- an isomorphism $\beta : \Lambda \otimes_{W_{\mathcal{O}}(R)} W_{\mathcal{O}}(R)[1/\varpi] \simeq W_{\mathcal{O}}(R)[1/\varpi]^n$.

Then $\mathrm{Inv}(\beta) = w_i$ if and only if

- β extends to a genuine map $\Lambda \subset \Lambda_0 := W_{\mathcal{O}}(R)^n$
- $\varpi\Lambda_0 \subset \Lambda$
- length $(\Lambda_0/\Lambda) = i$.

Then $Gr_{w_i} \cong Gr(n-i, n)$ which agrees with our previous discussion.

Example (Non-minuscule case)

Let $G = GL_2$, and $\mu = (2, 0)$.

- $Gr_{\leq\mu}$ has the following moduli interpretation

$$Gr_{\leq\mu}(R) = \{\Lambda_2 \subset \Lambda_0 \mid \text{length } (\Lambda_0/\Lambda_2) = 2\}.$$

- $Gr_{\leq\mu}$ admits a resolution $\widetilde{Gr_{\leq\mu}} \to Gr_{\leq\mu}$, where

$$\widetilde{Gr_{\leq\mu}}(R) := \left\{ \Lambda_2 \subset \Lambda_1 \subset \Lambda_0 \middle| \begin{matrix} \text{length } (\Lambda_0/\Lambda_1) = 1 \\ \text{length } (\Lambda_1/\Lambda_2) = 1 \end{matrix} \right\},$$

In fact, the resolution $\widetilde{Gr_{\leq\mu}}$ is isomorphic to the Hirzebruch surface $\Sigma_2 := \mathbb{P}(\mathcal{O}_{\mathbb{P}_1} \bigoplus \mathcal{O}_{\mathbb{P}_1}(-2))$.

THE SATAKE CATEGORY

It makes sense to consider the category $P_{L^+G}(Gr_{\leq\mu}, \Lambda)$ of Λ-coefficient L^+G equivariant perverse sheaves on $Gr_{\leq\mu}$. Finally, we define our principal object of study.

Definition

We define the category of Λ-coefficient L^+G-equivariant perverse sheaves on Gr_G to be

$$P_{L^+G}(Gr_G, \Lambda) := \varinjlim_{\mu} P_{L^+G}(Gr_{\leq\mu}, \Lambda).$$

We call this category the *Satake category* and will denote it by $\mathrm{Sat}_{G,\Lambda}$ for simplicity. In fact, $\mathrm{Sat}_{G,\Lambda}$ can be endowed with a monoidal structure "$\star$".

The Geometric Satake Equivalence

Now we have two monoidal categories

- The Satake category $\mathrm{Sat}_{G,\Lambda}$
- The category of finitely generated $\hat{G}$-modules over Λ $\mathrm{Rep}_{\Lambda}(\hat{G})$.

The expected relation between these two categories is established by the following theorem.

Theorem (Y.)

There is an equivalence of monoidal categories

$$\mathrm{Sat}_{G,\Lambda} \simeq \mathrm{Rep}_{\Lambda}(\hat{G}).$$

Historical Remark

The geometric Satake equivalence in different settings are proved by

- Lusztig-Ginzburg-Beilinson-Drinfeld-Mirković-Vilonen
  - G is a connected reductive group over $\mathbb{C}$.
  - Λ is a Noetherian local domain of finite global dimension.
- Zhu
  - G is a connected split reductive group scheme over the ring of integers of a mixed characteristic local field.
  - $\Lambda = \bar{\mathbb{Q}}_\ell$.
- in current setting, Scholze announced the geometric Satake equivalence for integral coefficients using the theory of diamonds.

Difficulties

The following new difficulties arise in our setting

- Lack of mixed characteristic analogue of *Beilinson-Drinfeld Grassmannians* (without appealing to Scholze's theory of diamonds).
- The integral coefficient Satake category is **NOT** semisimple.
- For a general object $\mathcal{F} \in \mathrm{Sat}_{G,\Lambda}$, $\mathrm{H}^*(Gr_G, \mathcal{F})$ may have **torsions**.

Sketch of Proof of Main Theorem 1

We explain some of the main ingredients of the proof

- *Local Hecke stacks*
- *Moduli of local Shtukas*
- *Cohomological correspondences*

From now on R will denote a perfect k-algebra.

Local Hecke stacks

Let $\mu_\bullet = (\mu_1, \mu_2, \cdots, \mu_n) \in (\mathbb{X}_\bullet^+)^n, \nu_\bullet = (\nu_1, \cdots, \nu_m) \in (\mathbb{X}_\bullet^+)^m$.

Definition

- The *local Hecke stack* $\mathrm{Hk}^{\mathrm{loc}}_{\mu_\bullet}(R)$ classifies the chain of modifications

$$\mathcal{E}_n \dashrightarrow \mathcal{E}_{n-1} \dashrightarrow \cdots \dashrightarrow \mathcal{E}_0$$

of G-torsors on D_R with relative positions $\leq \mu_n, \cdots, \leq \mu_1$.

Definition

The stack $\mathrm{Hk}^{0,\mathrm{loc}}_{\mu_\bullet|\nu_\bullet}(R)$ classifies the rectangles of modifications

$$\begin{array}{ccccccc} \mathcal{E}_n & \dashrightarrow & \mathcal{E}_{n-1} & \dashrightarrow & \cdots & \dashrightarrow & \mathcal{E}_0 \\ \big\downarrow{=} & & & & & & \big\downarrow{=} \\ \mathcal{E}'_m & \dashrightarrow & \mathcal{E}'_{m-1} & \dashrightarrow & \cdots & \dashrightarrow & \mathcal{E}'_0 \end{array}$$

of G-torsors on D_R with relative positions $\leq \mu_n, \cdots, \leq \mu_1$ and $\leq \nu_m, \cdots, \leq \nu_1$.

We need the finite type version of the local Hecke stacks

- $\mathrm{Hk}_{\mu_\bullet}^{\mathrm{loc}(s)}$.
- $\mathrm{Hk}_{\mu_\bullet|\nu_\bullet}^{0,\mathrm{loc}(s)}$.

to apply the ℓ-adic formalism.

Moduli of Local Shtukas

Let $\mu_\bullet = (\mu_1, \mu_2, \cdots, \mu_n) \in (\mathbb{X}_\bullet^+)^n, \nu_\bullet = (\nu_1, \cdots, \nu_m) \in (\mathbb{X}_\bullet^+)^m$.

Definition

The *moduli of local Shtukas* $\mathrm{Sht}^{\mathrm{loc}}_{\mu_\bullet}(R)$ classifies sequences of modifications

$$\mathcal{E}_n \dashrightarrow \mathcal{E}_{n-1} \dashrightarrow \cdots \dashrightarrow \mathcal{E}_0 \xrightarrow{\simeq} {}^\sigma\mathcal{E}_n$$

of G-torsors over D_R with relative positions $\leq \mu_n, \cdots, \leq \mu_1$.

There is a natural morphism

$$\psi : \mathrm{Sht}^{\mathrm{loc}}_{\mu_\bullet} \longrightarrow \mathrm{Hk}^{\mathrm{loc}}_{\mu_\bullet}.$$

Similar to the local Hecke stacks, we define

- $\mathrm{Sht}^{\mathrm{loc}}_{\mu_\bullet|\nu_\bullet}$ and its substack $\mathrm{Sht}^{\lambda,\mathrm{loc}}_{\mu_\bullet|\nu_\bullet}$.
- $\mathrm{Sht}^{\mathrm{loc}(m,n)}_{\mu_\bullet}$ and $\mathrm{Sht}^{\lambda,\mathrm{loc}(m,n)}_{\mu_\bullet|\nu_\bullet}$.

Cohomological Correspondences

- Let k be a perfect filed of characteristic $p > 0$
- All stacks will be algebraic stacks which are perfectly of finite presentation in the opposite category of perfect k-algebras

Definition

Let X_1, X_2 be two stacks and $\mathcal{F}_i \in D(X_i, \Lambda)$. A *cohomological correspondence* $(C, u) : (X_1, \mathcal{F}_1) \to (X_2, \mathcal{F}_2)$ is a stack $C \xrightarrow{c_1 \times c_2} X_1 \times X_2$, and $u : c_1^* \mathcal{F}_1 \to c_2^! \mathcal{F}_2$. We define the space of cohomological correspondences by $\mathrm{Corr}_C((X_1, \mathcal{F}_1), (X_2, \mathcal{F}_2))$, and

$$\mathrm{Corr}_C((X_1, \mathcal{F}_1), (X_2, \mathcal{F}_2)) \cong \mathrm{Hom}_{D(C,\Lambda)}(c_1^* \mathcal{F}_1, c_2^! \mathcal{F}_2).$$

Key Theorem

Let $V \in \mathrm{Rep}_\Lambda(\hat{G}_\Lambda)$, we produce a perverse sheaf $S(\widetilde{V})$ on $\mathrm{Sht}^{\mathrm{loc}}$ as follows

- *integral geometric Satake equivalence* $\Rightarrow S(V) \in \mathrm{Sat}_{G,\Lambda}$.
- *descent* $\Rightarrow S(V) \in \mathrm{P}(\mathrm{Hk}^{\mathrm{loc}})$.
- pullback along $\psi \Rightarrow$

$$\Psi : \mathrm{P}(\mathrm{Hk}^{\mathrm{loc}}) \longrightarrow \mathrm{P}(\mathrm{Sht}^{\mathrm{loc}})$$

$S(\widetilde{V}) := \Psi(S(V))$.

We write the support of $S(\widetilde{V})$ as $\mathrm{Sht}_V^{\mathrm{loc}(m,n)}$ for some proper choice of integers (m,n). We similarly define $\mathrm{Sht}_{V|W}^{\mathrm{loc}(m',n')}$.

Theorem (Y.)

For any projective Λ-modules $V_1, V_2 \in \mathrm{Rep}_\Lambda(\hat{G}_\Lambda)$, choose appropriate integers (m_1, n_1, m_2, n_2) and a dominant cocharacter λ. Consider the Hecke correspondence

$$\mathrm{Sht}_{V_1}^{\mathrm{loc}(m_1,n_1)} \xleftarrow{h^{\leftarrow}_{V_1}} \mathrm{Sht}_{V_1|V_2}^{\lambda,\mathrm{loc}(m_1,n_1)} \xrightarrow{h^{\rightarrow}_{V_2}} \mathrm{Sht}_{V_2}^{\mathrm{loc}(m_2,n_2)}$$

Then there exists the following map

$$\mathrm{Hom}_{\mathrm{Coh}^{\hat{G}}(\hat{G}\sigma)}(\widetilde{V}_1, \widetilde{V}_2) \xrightarrow{\mathcal{S}_{V_1,V_2}} \mathrm{Corr}_{\mathrm{Sht}^{\mathrm{loc}}_{V_1|V_2}}(S(\widetilde{V}_1), S(\widetilde{V}_2)), \qquad (1)$$

which is in independent of auxiliary choices.

We make the following remarks

- for $\Lambda = \bar{\mathbb{Q}}_\ell$, the operators are constructed by Xiao-Zhu.
- the map $\mathcal{S}_{V,W}$ may be regarded as a local analogue of V. Lafforgue's S-operators.

Proof of the main theorem 1

Recall our main theorem 1.

Theorem (Y.)

Under a mild technical assumption, there exists a map

$$\mathrm{Spc}:\mathrm{Hom}_{\mathrm{Coh}^{\hat{G}}(\hat{G}\sigma)}(\widetilde{V}_1,\widetilde{V}_2)\to\mathrm{Hom}_{\mathcal{H}^p\otimes\mathcal{J}}(\mathrm{H}_c^*(\mathrm{Sh}_{\mu_1},\mathcal{L}_1),\mathrm{H}_c^*(\mathrm{Sh}_{\mu_2},\mathcal{L}_2)),$$

which is compatible with compositions on the source and target. If $\mathrm{Sh}_{\mu_1}=\mathrm{Sh}_{\mu_2}$ *is a Shimura set, the action of* $\mathrm{End}_{\mathrm{Coh}^{\hat{G}}(\hat{G}\sigma)}(\widetilde{V}_\mu)$ *on*

$$\mathrm{H}_c^*(\mathrm{Sh}_\mu,\mathbb{Q}_\ell)\simeq C_c(G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K,\mathbb{Q}_\ell)$$

coincides with the usual Hecke algebra action under the Satake isomorphism.

Sketch of Proof

- Choose a lattice $\Lambda_i \in \mathrm{Rep}_{\mathbb{Z}_\ell}(\hat{G}_{\mathbb{Z}_\ell})$ of V_i.
- $\mathrm{Hom}_{\mathrm{Coh}^{\hat{G}}(\hat{G}\sigma)}(\widetilde{V}_1, \widetilde{V}_2) \cong \mathrm{Hom}_{\mathrm{Coh}^{\hat{G}}(\hat{G}\sigma)}(\widetilde{\Lambda}_1, \widetilde{\Lambda}_2) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$
- Key Theorem $\Rightarrow$

$$\mathrm{Hom}_{\mathrm{Coh}^{\hat{G}}(\hat{G}\sigma)}(\widetilde{V}_1, \widetilde{V}_2) \to \mathrm{Corr}_{\mathrm{Sht}^{\mathrm{loc}}}(S(\widetilde{\Lambda}_1), S(\widetilde{\Lambda}_2)) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$$

- *"the mild technical assumption"* $\Rightarrow$

$$\mathrm{Corr}_{\mathrm{Sht}^{\mathrm{loc}}}(S(\widetilde{\Lambda}_1), S(\widetilde{\Lambda}_2)) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$$
$$\xrightarrow{\text{pullback}} \mathrm{Corr}_{\mathrm{Sh}_{\mu_1|\mu_2}}((\mathrm{Sh}_{\mu_1}, \mathbb{Z}_\ell\langle d_1\rangle), (\mathrm{Sh}_{\mu_2}, \mathbb{Z}_\ell\langle d_2\rangle)) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$$

- *"construct on each level"* $\rightsquigarrow$

$$\mathrm{Corr}_{\mathrm{Sh}_{\mu_1|\mu_2}}((\mathrm{Sh}_{\mu_1}, \mathbb{Z}_\ell\langle d_1\rangle), (\mathrm{Sh}_{\mu_2}, \mathbb{Z}_\ell\langle d_2\rangle)) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$$
$$\to \mathrm{Corr}_{\mathrm{Sh}_{\mu_1|\mu_2}}((\mathrm{Sh}_{\mu_1}, \mathcal{L}_{W,\mathbb{Q}_\ell}\langle d_1\rangle), (\mathrm{Sh}_{\mu_2}, \mathcal{L}_{W,\mathbb{Q}_\ell}\langle d_2\rangle))$$

- *pushforward the cohomological correspondence*

$$\mathrm{Corr}_{\mathrm{Sh}_{\mu_1|\mu_2}}((\mathrm{Sh}_{\mu_1}, \mathcal{L}_{W,\mathbb{Q}_\ell}\langle d_1\rangle), (\mathrm{Sh}_{\mu_2}, \mathcal{L}_{W,\mathbb{Q}_\ell}\langle d_2\rangle))$$
$$\to \mathrm{Hom}_{\mathcal{H}^p \otimes \mathcal{J}}(\mathrm{H}^*_c(\mathrm{Sh}_{\mu_1}, \mathcal{L}_{W,\mathbb{Q}_\ell}\langle d_1\rangle), \mathrm{H}^*_c(\mathrm{Sh}_{\mu_2}, \mathcal{L}_{W,\mathbb{Q}_\ell}\langle d_2\rangle))$$

Thank You!