The Tate conjecture for a concrete family of elliptic surfaces

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(Joint work with Lian Duan)

Outline

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Slogan

Tate classes are algebraic!

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- The Hodge conj. + The Mumford-Tate conj.$\Rightarrow$ The Tate conj.

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Definition (Geometric)

A Galois representation is geometric if it satisfies (a) and (b).

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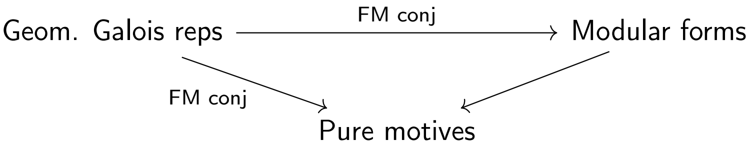

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Let $r : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ be a geometric Galois representation. Suppose that $\ell > 7$, and, furthermore, that

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Main player

$$\mathcal{S} := \mathcal{S}_a$$

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- $\dim H^2(\mathcal{S}) = 46, \dim H^{2,0}(\mathcal{S}) = \dim H^{0,2}(\mathcal{S}) = 3$, and Picard number $\rho(\mathcal{S}_{\overline{\mathbb{Q}}}) = \mathrm{rank}(\mathrm{NS}^1(\mathcal{S}_{\overline{\mathbb{Q}}})) = 37$, 38, 39, or 40 (depending on a).

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Conjecture (van Geemen and Top, 1995)

For each a,

$$V_\ell(-1) \cong \delta \operatorname{Sym}^2 T_\ell E$$

for some elliptic curve E and some quadratic character δ.

Geemen and Top's conjecture

Several properties of V_ℓ

- $V_\ell \cong \overline{V}_\ell$
- $h^{1,-1}(V_\ell) = h^{0,0}(V_\ell) = h^{-1,1}(V_\ell) = 1$
- $V_\ell \cong V_\ell^*$, i.e., V_ℓ self-dual

Conjecture (van Geemen and Top, 1995)

For each a,

$$V_\ell(-1) \cong \delta \, \mathrm{Sym}^2 \, T_\ell E$$

for some elliptic curve E and some quadratic character δ.

Selfdual and non-selfdual 3-dimensional Galois representations, 1995

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Theorem (Duan and W., 2019)

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Theorem (Duan and W., 2019)

If $a \equiv 2, 3 \mod 5$ and none of $2(1+a)$ or $2(1-a)$ is a square in $\mathbb{Q}$, then, for a density one subset of primes ℓ, the Tate conjecture for $\mathcal{S}$ is true, i.e.,

$$\mathrm{NS}^1(\mathcal{S}) \otimes \mathbb{Q}_\ell \xrightarrow{\cong} H^2_{et}(\mathcal{S}_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell(1))^{G_{\mathbb{Q}}}.$$

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Goal

To prove $\rho := V_\ell^{\mathrm{ss}}$ is absolutely irreducible.

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Lemma

If $\operatorname{tr} \rho(g^2) \neq 3$ *or* $\operatorname{tr} \rho(g) \neq \pm 1$, *then* $\det r(g) = 1$.

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Proposition

If $\left(\frac{2(1+a)}{p}\right) = \left(\frac{2(1-a)}{p}\right) = -1$, *then* $\operatorname{tr}(\rho(\mathrm{Frob}_p^2)) = -1 \mod 8.$

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- van Geemen and Top's conjecture? Potential automorphy.

Thank you for your listening!