Analogues of Alladi's formula over global function fields

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joint work with Lian Duan and Shaoyun Yi

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Schedule

Alladi's formula and its generalizations

Alladi's formula

$\mu(n)$: Möbius function.
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Theorem (Alladi, 1977)

Let $k, \ell \geqslant 1$ be integers with $(\ell, k) = 1$. Then

$$-\sum_{\substack{n \geqslant 2 \\ p_{\min}(n) \equiv \ell (\mathrm{mod}\ k)}} \frac{\mu(n)}{n} = \frac{1}{\varphi(k)}.$$

Here $p_{\min}(n)$ is the smallest prime factor of n, and $\varphi(n)$ is Euler's totient function.

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PNT in arithmetic progressions: $\#\{p \leq x : p \equiv \ell (\mathrm{mod}\ k)\} \sim \frac{1}{\varphi(k)}\pi(x)$.

Two key ingredients

1. Alladi's duality idea on prime factors of integers
Let $p_{\max}(n)$ be the largest prime factor of n. If $f(1) = 0$, $p_{\min}(1) = p_{\max}(1) = 1$, then

$$\sum_{d|n} \mu(d) f(p_{\min}(d)) = -f(p_{\max}(n)),$$

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Theorem (Alladi, 1977)

For any bounded function f and constant δ, we have

$$\sum_{n \leqslant x} f(p_{\max}(n)) \sim \delta \cdot x \tag{1}$$

if and only if

$$-\sum_{n=2}^{\infty} \frac{\mu(n) f(p_{\min}(n))}{n} = \delta. \tag{2}$$

Two key ingredients

2. Distribution of largest prime factors

Theorem (Alladi, 1977)

If $(\ell, k) = 1$, *then*

$$\sum_{\substack{n \leqslant x \\ p_{\max}(n) \equiv \ell (\mathrm{mod}\ k)}} 1 = \frac{x}{\varphi(k)} + O\Big(x \exp(-c(\log x)^{\frac{1}{3}})\Big)$$

for some constant $c > 0$.

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Proof of Alladi's formula. It follows immediately by taking f to be the indicator function for primes in arithmetic progression $\ell (\operatorname{mod} k)$:

$$f(n) = \begin{cases} 1, & \text{if } n = p \equiv \ell (\operatorname{mod} k); \\ 0, & \text{otherwise.} \end{cases}$$

Dawsey's formula: Chebotarev density

$L/\mathbb{Q}$: finite Galois extension over $\mathbb{Q}$;
$G = \mathrm{Gal}(L/\mathbb{Q})$: Galois group;
$\left[\frac{L/\mathbb{Q}}{\mathfrak{p}}\right]$: Artin symbol;
$\left[\frac{L/\mathbb{Q}}{p}\right] := \left\{ \left[\frac{L/\mathbb{Q}}{\mathfrak{p}}\right] : \mathfrak{p} \subseteq \mathcal{O}_L \text{ and } \mathfrak{p} | p \right\}$ for p unramified.

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Chebotarev Density Theorem (CDT): for any conjugacy class $C \subset G$, we have

$$\#\left\{ p \leq x \text{ unramified} : \left[\frac{L/\mathbb{Q}}{p}\right] = C \right\} \sim \frac{|C|}{|G|}\pi(x).$$

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For any conjugacy class $C \subset G$, we have

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Alladi's formula follows by taking $L = \mathbb{Q}(\zeta_k)$ and $C =$ the conjugacy class of ℓ, where ζ_k is the k-th primitive unit root.

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Theorem (Sweeting-Woo, 2019)

For any conjugacy class $C \subset G$, we have

$$-\lim_{X\to\infty} \sum_{\substack{2\le|\mathfrak{a}|\le X \\ \mathfrak{a}\in\mathfrak{D}(L/K,C)}} \frac{\mu(\mathfrak{a})}{|\mathfrak{a}|} = \frac{|C|}{|G|}.$$

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Theorem (Kural-McDonald-Sah (KMS), 2019)

If $S \subseteq \mathcal{P}$ has natural density $\delta(S)$, then

$$-\lim_{X\to\infty} \sum_{\substack{2\leq|\mathfrak{a}|\leq X \\ \mathfrak{a}\in\mathfrak{D}(K,S)}} \frac{\mu(\mathfrak{a})}{|\mathfrak{a}|} = \delta(S).$$

Remarks:
1. For any conjugacy class $C \subset G$, take

$$S = \left\{ \mathfrak{p} \in \mathcal{P} : \mathfrak{p} \text{ unramified}, \left[\frac{L/K}{\mathfrak{p}}\right] = C \right\}.$$

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E.g. we can take $\mu * a$ to be the Liouville function or the Ramanujan sum.
Recently, Ono-Schneider-Wagner showed a partition-theoretic analogue to this kind of generalization.

Analogues for function fields

Divisors of function fields

Goal:

Find the analogue of KMS' formula over function fields:

$$-\lim_{X\to\infty} \sum_{\substack{2\le|\mathfrak{a}|\le X\\ \mathfrak{a}\in\mathfrak{D}(K,S)}} \frac{\mu(\mathfrak{a})}{|\mathfrak{a}|} = \delta(S).$$

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$\mathbb{F}_q$: a finite field of q elements. In $\mathbb{F}_q[x]$, we have

$$F = \prod_{i=1}^{r} P_i^{\alpha_i} \longleftrightarrow \sum_{i=1}^{r} \alpha_i \cdot P_i$$

Multiplicative semigroup $\longleftrightarrow$ Abelian semigroup.

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E.g.,

$$F=(1+x)^{2}(3+x)^{1}(1+x-x^2)^{5}$$

$$\longleftrightarrow F=2\cdot(1+x)+1\cdot(3+x)+5\cdot(1+x-x^2)$$

Notation

$K/\mathbb{F}_q(x)$: a global function field (or simply a function field).
$\mathcal{P}$: all prime divisors of K. The letter P always denotes a prime divisor.
$\mathcal{D}$: the free Abelian semigroup generated by $\mathcal{P}$ (i.e., the set of effective divisors).
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$|D| = \prod_P |P|^{a_P}$, where $|P| = q^{\deg P}$.
$\mu(D)$: the Möbius function for divisors.

$$\mu(D) = \begin{cases} 1 & \text{if } D = 0, \\ (-1)^k & \text{if } D = P_1 + \cdots + P_k, P_i' s \text{ are distinct}, \\ 0 & \text{otherwise.} \end{cases}$$

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D is *distinguishable*: if there is a unique prime factor, say $P_{\min}(D)$, of D attaining the minimal degree $d_-(D)$.

e.g. $(1+x)(2+x)(1+x+x^5)$ is NOT distinguishable;
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$\delta(S) = \lim_{n\to\infty} \frac{\#\{P\in S:\deg P=n\}}{\#\{P\in\mathcal{P}:\deg P=n\}}$: natural density of S.

Main Theorem

Theorem (Duan-W.-Yi (DWY), 2020)

Given a global function field K, if $S \subset \mathcal{P}$ has a natural density $\delta(S)$, then

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Remark. Duan later found that the main theorem also holds for varieties.

Application 1: Chebotarev Density Theorem for function fields

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By Chebotarev Density Theorem for function fields, we have

Corollary 1 (DWY, 2020)

For any conjugacy class $C \subseteq G$, we have

$$-\lim_{n\to\infty} \sum_{\substack{D\in\mathfrak{D}(K,\mathcal{P}),1\leq \deg D\leq n \\ (P_{\min}(D),L/K)=C}} \frac{\mu(D)}{|D|} = \frac{\#C}{\#G}.$$

Application 2: Prime Polynomial Theorem for arithmetic progressions

S: a subset of monic irreducible polynomials of $\mathbb{F}_q[x]$.
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Let $q \geq 2$ be fixed. If $S \subset \mathcal{P}$ has a natural density $\delta(S)$, then we have that

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By Prime Polynomial Theorem for arithmetic progressions, we get that

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For relatively prime $f, g \in \mathbb{F}_q[x]$, we have

$$-\lim_{n\to\infty} \sum_{\substack{F\in\mathfrak{D}(q,\mathcal{P}),1\leq \deg F\leq n \\ p_{\min}(F)\equiv f (\mathrm{mod}\ g)}} \frac{\mu(F)}{|F|} = \frac{1}{\varphi(g)}.$$

Here $\varphi(g)$ is the function field Euler totient function.

Key ingredients in the proof of Main Theorem

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1. Duality between prime factors of divisors

Recall that S is a set of prime divisors of K.

For a divisor $A \in \mathcal{D}$, we let

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$$Q_S(A) := \# \left\{P \in S : \deg P = d^+(A), P|A\right\}$$

Key ingredients in the proof of Main Theorem

Strategy: follow Alladi's work and KMS' work.

1. Duality between prime factors of divisors

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Lemma (Duality Lemma, DWY, 2020)

Suppose $f : \mathbb{N} \to \mathbb{C}$, $f(0) = 0$. Then the following identity holds

$$\sum_{A \geq B} \mu(B) 1_{\mathfrak{D}(K,S)}(B) f(d_-(B)) = -Q_S(A) f(d^+(A)).$$

Here $1_{\mathfrak{D}(K,S)}$ is the indicator function on $\mathfrak{D}(K,S)$.

Theorem (DWY, 2020)

For any bounded arithmetic function f with $f(0) = 0$, we have that

$$\sum_{\deg A = n} Q_S(A) f(d^+(A)) \sim \delta(S) c_K q^n \tag{3}$$

if and only if

$$-\lim_{n\to\infty} \sum_{\substack{1 \le \deg A \le n \\ A \in \mathfrak{D}(K,S)}} \frac{\mu(A) f(d_-(A))}{|A|} = \delta(S), \tag{4}$$

where $c_K > 0$ is a constant.

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2. Asymptotic estimate for $Q_S(A)$

Theorem (DWY, 2020)

For any fixed subset $S \subseteq \mathcal{P}$ with natural density $\delta(S)$, we have

$$\sum_{\deg A = n} Q_S(A) = \delta(S) c_K q^n + o(q^n). \tag{5}$$

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Main tools: PNT for function fields, smooth divisors (analogous to smooth numbers), zero-free region of zeta function, elementary sieve.

Conjecture: Ultimate Alladi's formula

Elements of Alladi's formula

$(G, |\cdot|)$: a normed free Abelian semigroup generated by prime elements $\mathcal{P}$.
Norm $|\cdot|$: either archimedean or nonarchimedean.
$S \subseteq \mathcal{P}$: a subset of $\mathcal{P}$ with natural density $\delta(S)$.
$\mu(g)$: the Möbius function.
$\mathfrak{D}(G, S)$: the set of distinguishable elements whose $p_{\min}(g) \in S$.
(One may add more natural conditions on G for the need of the proof.)

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The Riemann zeta function on G:

$$\zeta_G(s) := \sum_{g \in G} \frac{1}{|g|^s} = \prod_{p \in \mathcal{P}} \left(1 - \frac{1}{|p|^s}\right)^{-1}.$$

Conjecture

Suppose

1. Numerator function $b(g)$: the Dirichlet series $\sum_{g\in G} b(g)|g|^{-s}$ is "near" to $1/\zeta_G^r(s)$ for some positive real number r (or even complex). The power r is called the order of $\sum_{g\in G} b(g)|g|^{-s}$.
2. Denominator function $h(g)$: it is multiplicative and "near" to $|g|$, say $\varphi(g), \sigma(g)$. The case $h(g) = |g|$ is essential.

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Motivation/Observation.

1. Analytic number theory aspect: when $\sum_{g\in G} b(g)|g|^{-s}$ is "near" to $1/\zeta_G^r(s)$, it is "near" to the Euler product $\prod_{p\in\mathcal{P}}\left(1-\frac{r}{|p|^s}\right)$, the probability of each prime p of the same weight r is equal in the Euler product.
2. Combinatorics aspect: Alladi's duality idea on prime factors holds for G using the principle of inclusion-exclusion .

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Key variables. The norm $|\cdot|$ and the function $b(g)$.

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3. Duan-W.-Yi, 2020: $G = \mathcal{D}$, the effective divisors of a global function field, $b(g) = \mu(g)$. (It may be generalized to the general free Abelian semigroup with nonarchimedean norm, at least for the norm $|g| = q^{\deg g}$ ($q > 1$ is real).)

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Unknown cases.

1. The case $r = 1$ for general $(G, |\cdot|)$ and $b(g)$. This would unify all of currently known results.
2. Until now, there is no any case known for $r \neq 1$. E.g., prove that

$$-\sum_{\substack{n\geqslant 2\\ p_{\min}(n)\equiv \ell (\operatorname{mod} k)}} \frac{\lambda(n)2^{\omega(n)}}{n} = \frac{1}{\varphi(k)}.$$

Note that

$$\sum_{n=1}^{\infty} \frac{\lambda(n)2^{\omega(n)}}{n^s} = \frac{\zeta(2s)}{\zeta(s)^2}.$$

Thank you!