$P-$*ADIC MEASURES FOR RECIPROCALS OF* $L-$*FUNCTIONS OF TOTALLY REAL FIELDS*

Razan Taha

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Goal

Construct a p-adic measure λ explicitly using the Fourier coefficients of Hilbert Eisenstein series such that

$$\int_{G(K(p^\infty)/K)} \chi(x)N(x)^{k-1}\, d\lambda \text{``}=\text{''} (1-\chi(\mathfrak{p})N(\mathfrak{p})^{k-1})^{-1}L(1-k,\chi)^{-1}$$

under good conditions

Outline

- Motivation
- Hilbert Eisenstein Series
- Fourier Coefficients
- Defining λ
- λ is a measure

Motivation

- new method to construct p-adic L-functions analogous to the Langlands-Shahidi method
- reciprocals of L-functions show up in the non-constant Fourier coefficients of Eisenstein series
- Gelbart-Miller-Panchishkin-Shahidi (2014): this is true for the case $K = \mathbb{Q}$ for regular primes
- Gelbart-Greenberg-Miller-Shahidi (2016): for irregular primes, the distribution is not a measure

Set-Up and Notation

- K totally real number field
- degree of the extension: $[K:\mathbb{Q}] = r$
  $K \hookrightarrow \mathbb{R}^r : \alpha \to (\sigma_i(\alpha))$ where σ_i runs through $Gal(K/\mathbb{Q})$
- $\mathcal{O}$ ring of integers of K
- $\mathfrak{d}$ different ideal
- N : norm map
- D discriminant of K, $N(\mathfrak{d}) = |D|$
- totally positive element $\alpha \gg 0 : \sigma_i(\alpha) > 0 \, \forall i$

Hilbert Modular Group

- Hilbert modular group $SL_2(\mathcal{O})$ acts discontinuously on $\mathfrak{h}^r$ by Mobius transformations componentwise:
  - $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathcal{O})$
  - $z = (z_1, \cdots, z_r) \in \mathfrak{h}^r$

$$A \cdot z = \left(\frac{a^{(1)} z_1 + b^{(1)}}{c^{(1)} z_1 + d^{(1)}}, \cdots, \frac{a^{(r)} z_r + b^{(r)}}{c^{(r)} z_r + d^{(r)}} \right).$$

- one - to - one correspondence

$$\{\text{cusps}\} \leftrightarrow \{\text{ideal classes}\}$$
$$(a : b) \leftrightarrow [\langle a, b \rangle]$$
$$\infty = (1 : 0) \leftrightarrow [\mathcal{O}]$$

Eisenstein Series

- congruence subgroup for integral ideal $\mathfrak{n}$
  $\Gamma_0(\mathfrak{n}) = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathcal{O}) \middle| \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \bmod \mathfrak{n} \right\}$
- Hecke character $\chi : (\mathcal{O}/\mathfrak{n})^* \to \mathbb{C}^*$
- Fix $\mathfrak{a} = \langle \alpha \rangle$
- at the cusp $\kappa = \infty$:

$$E_{k,\mathfrak{n}}(\chi, z, \infty) = \sum_{a_2 \in (\mathfrak{a}/\mathfrak{n}\mathfrak{a})^\times} \chi(\langle a_2 \rangle) \sum_{\substack{\{c,d\} \in \mathfrak{a} \times \mathfrak{a} \\ \gcd\left(\frac{c}{\mathfrak{a}}, \frac{d}{\mathfrak{a}}\right) = 1 \\ d \equiv a_2 \bmod \mathfrak{n}\mathfrak{a}}} \frac{N(\mathfrak{a})^k}{(\mathfrak{n}cz + d)^k}$$

(c, d) and (c', d') are identified together if there exists $\varepsilon \equiv 1 \bmod \mathfrak{n}$ such that $c = \varepsilon c'$ and $d = \varepsilon d'$.

Eisenstein Series

- at a cusp $\kappa \neq \infty$: $A\kappa = \infty$
  - $A = \left(\begin{smallmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{smallmatrix}\right)$
  - $\det A = 1$
  - $\kappa = -\frac{\alpha_4}{\alpha_3}$, $\quad \mathfrak{b} = \langle \alpha_3, \alpha_4 \rangle$, $\quad \mathfrak{b}^{-1} = \langle \alpha_1, \alpha_2 \rangle$, $\quad \gcd(\mathfrak{n}, \mathfrak{b}) = 1$

$$E_{k,\mathfrak{n}}(\chi, z, \kappa) = (\alpha_3 A^{-1} z + \alpha_4)^k \sum_{a_2 \in (\mathfrak{a}/\mathfrak{n}\mathfrak{a})^\times} \chi(\langle a_2 \rangle) \sum_{\substack{\{c,d\} \in \mathfrak{a} \times \mathfrak{a} \\ \gcd\left(\frac{c}{\mathfrak{a}}, \frac{d}{\mathfrak{a}}\right) = 1 \\ d \equiv a_2 \bmod \mathfrak{n}\mathfrak{a}}} \frac{N(\mathfrak{a})^k}{(\mathfrak{n} c A^{-1} z + d)^k}$$

Fourier Series Expansion

$$G_k(z, a_1, a_2, \mathfrak{n}, \mathfrak{a}) = \sum_{\substack{\{c,d\}^+ \in \mathfrak{a}\times\mathfrak{a} \\ (c,d)\equiv(a_1,a_2) \bmod \mathfrak{n}\mathfrak{a}}} \frac{N(\mathfrak{a})^k}{(cz+d)^k}.$$

Fourier Expansion (Klingen, 1962)

$$\text{constant term} \;+\; A \sum_{\substack{c\equiv a_1^* \bmod \mathfrak{n}\mathfrak{a}\mathfrak{b} \\ \nu\equiv 0 \bmod \mathfrak{b}/\mathfrak{n}\mathfrak{a}\mathfrak{d} \\ c\nu\gg 0, \{c\}^+}} \operatorname{sgn} N(\nu) N(\nu)^{k-1} \exp\left(a_2^*\nu + c\nu z\right) \quad (1)$$

$$A = \frac{(-2\pi i)^{kr} N(\mathfrak{a})^{k-1} N(\mathfrak{b})}{(k-1)!^r N(\mathfrak{n}) |D|^{1/2}}$$

where $a_1^* = \alpha_4 a_1 - \alpha_1 a_2 \in \mathfrak{a}\mathfrak{b}$ and $a_2^* = \alpha_1 a_2 - \alpha_2 a_1 \in \mathfrak{a}\mathfrak{b}^{-1}$

Fourier Series Expansion

$$\sum_{\substack{\{c,d\}\in\mathfrak{a}\times\mathfrak{a}\\ \gcd\left(\frac{c}{\mathfrak{a}},\frac{d}{\mathfrak{a}}\right)=1\\ d\equiv a_2 \bmod \mathfrak{n}\mathfrak{a}}} \frac{N(\mathfrak{a})^k}{(cz+d)^k} = \sum_{a_1\in\mathfrak{a}/\mathfrak{a}\mathfrak{n}} \sum_{i=1}^{h^+(\mathfrak{n})} \sum_{\mathfrak{t}\in C_i} \frac{\mu(\mathfrak{t})}{N(\mathfrak{t})^k} G_k(z,\tau a_1,\tau a_2,\mathfrak{n},\mathfrak{a}\mathfrak{t}) \quad (2)$$

$\mu(\mathfrak{t})$: the Mobius function
$Cl(\mathfrak{n})^+ = \{\text{fractional ideals}\}/\{\langle\alpha\rangle : \alpha \gg 0, \alpha \equiv 1 \bmod \mathfrak{n}\}$
$C_1, \cdots, C_{h^+(\mathfrak{n})}$: the strict ideal classes of K modulo $\mathfrak{n}$
$\mathfrak{t}_1, \cdots, \mathfrak{t}_{h^+(\mathfrak{n})}$: $\mathfrak{t}_i \in C_i^{-1}$ fixed representatives
$\mathfrak{t} \in C_i$, $(\tau) = \mathfrak{t}\mathfrak{t}_i$ where $\tau \equiv 1 \bmod \mathfrak{n}$.
$h^+(\mathfrak{n})$: the strict ideal class number of K mod $\mathfrak{n}$.

Non Constant Term

$$\frac{(-2\pi i)^{kr} N(\mathfrak{b})^k}{(k-1)!^r N(\mathfrak{n})^k |D|^{\frac{2k-1}{2}}} \sum_{i=1}^{h^+(\mathfrak{n})} \sum_{\mathfrak{t} \in C_i} \frac{\mu(\mathfrak{t})}{N(\mathfrak{t})^k}$$

$$\sum_{\substack{\nu \in \mathcal{O} \\ c\nu \gg 0, \{c\}^+}} \operatorname{sgn} N(\nu) N(\nu)^{k-1} \exp\left(\frac{\mathfrak{t}_i a_2' \nu \mathfrak{b}}{\mathfrak{n}\mathfrak{d}} + \frac{c\nu \mathfrak{b}^2 z}{\mathfrak{d}}\right)$$

$\frac{1}{L(s,\chi)} = \sum \frac{\mu(t)\chi(t)}{N(t)^s}$

Fourier Coefficients

The $\mathfrak{p}^m$th coefficient in the Fourier expansion is

$$\mathcal{C}_{\mathfrak{p}^m}(\varepsilon_{k,\mathfrak{p}^m,\mathfrak{b}_l}) = \frac{(-2\pi i)^{kr} N(\mathfrak{b}_l)^k}{(k-1)!^r |D|^{\frac{2k-1}{2}}} \sum_{u=0}^{m} N(\mathfrak{p})^{u(k-1)-mk} \sum_{i=1}^{h^+(\mathfrak{p}^m)} \sum_{\mathfrak{t} \in C_i} \frac{\mu(\mathfrak{t})}{N(\mathfrak{t})^k} \exp\left(\frac{\mathfrak{t}_i a_2' \mathfrak{p}^u \mathfrak{b}_l}{\mathfrak{p}^m \mathfrak{d}}\right)$$

p-adic Measures on Galois Groups

- $K(p^\infty)/K$: maximal abelian unramified-outside-p extension of K
- $\mathfrak{b}_1, \cdots, \mathfrak{b}_{h^+}$: the strict ideal classes of K
- Class Field Theory: $Gal\,(K(p^\infty)/K) = \sqcup_l \mathfrak{b}_l^{-1}(\mathcal{O}_\mathfrak{p}^*/U)$
  U : the closure in $\mathcal{O}_\mathfrak{p}^*$ of all totally positive units in $\mathcal{O}^*$
- for any continuous function ϕ on $Gal\,(K(p^\infty)/K)$, define

$$\int_{Gal(K(p^\infty)/K)} \phi\, d\mu = \sum_l \int_{\mathcal{O}_\mathfrak{p}} \phi(\mathfrak{b}_l^{-1}x) d\mu_l$$

Defining λ

Define a distribution on $\mathfrak{b}_l^{-1}(\mathcal{O}_\mathfrak{p}^*/U)$ by

$$\lambda_l(\mathfrak{a}+\mathfrak{p}^m\mathcal{O}_\mathfrak{p}) = \frac{1}{2^r N(\mathfrak{b}_l)^k}\mathcal{C}_{\mathfrak{p}^m}(\varepsilon_{k,\mathfrak{p}^m,\mathfrak{b}_l}) + \gamma(k)\lambda_{Haar}$$

where

$$\gamma(k) = \begin{cases} \frac{N(\mathfrak{p})^{2k-1}}{N(\mathfrak{p})^k-1}\frac{(1-N(\mathfrak{p})^{k-1})^{-1}}{\zeta(1-k)} & \text{for k even} \\ 0 & \text{otherwise} \end{cases}$$

$$\mathcal{C}_{\mathfrak{p}^m} = A^* N(\mathfrak{b}_l)^k \sum_{u=0}^{m} N(\mathfrak{p})^{u(k-1)-mk} \sum_{i=1}^{h^+(\mathfrak{p}^m)} \sum_{\mathfrak{t}\in C_i} \frac{\mu(\mathfrak{t})}{N(\mathfrak{t})^k} \exp\left(\frac{\mathfrak{t}_i a_2' \mathfrak{p}^u \mathfrak{b}_l}{\mathfrak{p}^m \mathfrak{d}}\right)$$

Main Theorem

Let $p \in \mathbb{Q}$ be an odd prime number. Let $\mathfrak{p}$ be a prime ideal of $\mathcal{O}$ lying above p and h^+ the strict class number of K. Under suitable conditions,

1. There exists a p-adic measure λ on $G\left(K(p^\infty)/K\right)$ whose Mellin transform is the reciprocal of the p-adic L-function of the totally real field K

$$\int_{G(K(p^\infty)/K)} N(x)^{k-1}\, d\lambda = h^+(1 - N(\mathfrak{p})^{k-1})^{-1}\zeta(1-k)^{-1}$$

   for all even positive integers k.

2. For any non-trivial Hecke character of finite type $\chi \mod \mathfrak{p}^m$ with $m > 0$, we have

$$\int_{G(K(p^\infty)/K)} \chi(x)N(x)^{k-1}\, d\lambda = h^+L(1-k,\chi)^{-1}$$

   for any positive integer k satisfying the parity condition $\chi(-1) = (-1)^k$.

Regularity Conditions

- $p \nmid [K : \mathbb{Q}]$
- p does not divide the class number of $K(e^{2\pi i/p})$
- no prime $\wp$ of the field $K(e^{2\pi i/p} + e^{-2\pi i/p})$ lying above p splits in $K(e^{2\pi i/p})$

Proof: Computations

- Gauss sum

$$\tau(\chi) = \sum_{\mathfrak{x} \in \mathcal{O}/\mathfrak{p}^m} \chi(\mathfrak{x}) e^{2\pi i Tr(\mathfrak{x}/\mathfrak{p}^m \mathfrak{d})}$$

Property: If χ is primitive and $x \in \mathcal{O}$, then

$$\sum_{\mathfrak{x} \in \mathcal{O}/\mathfrak{p}^m} \chi(\mathfrak{x}) e^{2\pi i Tr(\mathfrak{x}x/\mathfrak{d})} = \begin{cases} \overline{\chi}(x)\tau(\chi) & \text{if } \gcd(x, \mathfrak{p}^m) = 1 \\ 0 & \text{otherwise} \end{cases}.$$

- Functional Equation of Hecke L-function

p-adic Measures of Totally Real Fields

Let $\mathfrak{c}$ be an ideal of K which is relatively prime to p. Then there exists a $\mathbb{Z}_p$-valued measure $\mu_{1,\mathfrak{c}}$ on $G\left(K(p^\infty)/K\right)$ such that for every integer $k \geq 0$ and every locally constant character χ, we have

$$\int_{G(K(p^\infty)/K)} \chi(x)N(x)^{k-1}\, d\mu_{1,\mathfrak{c}} = (1-\chi(\mathfrak{c})N(\mathfrak{c})^k))L(1-k,\chi).$$

(Cassou-Nogues, 1979, Deligne-Ribet, 1980)

Invertibility of $d\mu_{1,\mathfrak{c}}$

- $d\mu_{1,\mathfrak{c}} \in \Lambda(G)$ (the Iwasawa algebra)
- $\omega(x) = \lim_{n\to\infty} x^{p^n}$ Teichmuller character
- $d\mu_{1,\mathfrak{c}}$ is invertible whenever $\int_{G(K(p^\infty)/K)} \omega(x)^{-i}\, d\mu_{1,\mathfrak{c}}$ is invertible
- Integral$= \left(1 - \omega^{-i}(\mathfrak{c})N(\mathfrak{c})\right) L(0, \omega^{-i})$
- Kummer's Criterion (Greenberg, 1973): $L(0, \omega^{-i})$ is invertible whenever p is regular

Proof: λ is a measure

- Regularization: $\mu^*(U) = \mu_{1,\mathfrak{c}}^{-1}(U) - N(\mathfrak{c})\mu_{1,\mathfrak{c}}^{-1}(\mathfrak{c}U)$
- $\int_{G(K(p^\infty)/K)} \chi(x)N(x)^{k-1}\, d\mu^* = L(1-k,\chi)^{-1}$
- $\lambda(U^{-1}) = h^+ N(x)^{k-1}\mu^*(U)$ as distributions, hence they are both measures

Thank you!