Counting elliptic curves with a rational N-isogeny

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Junior Number Theory Days 2020

December 5, 2020

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- Such an isogeny is rational if $\operatorname{Ker} \phi$ is defined over $\mathbb{Q}$.

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More precise question

How many elliptic curves (up to $\mathbb{Q}$-isomorphism) of bounded naive height have a rational N-isogeny?

Some more notation

For two real valued functions $f(X)$ and $g(X)$, we say that $f(X) \asymp g(X)$ if there are two positive constants K_1 and K_2 such that

$$K_1 g(X) \leq f(X) \leq K_2 g(X).$$

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Example

If $N = 1$, we are counting integers in a box, and $\mathcal{N}(1, X) \asymp X^{5/6}$.

Main theorem

Theorem [BS, '20]

N	$h_N(X)$	N	$h_N(X)$
2	$X^{1/2}$	8	$X^{1/6}\log(X)$
3	$X^{1/2}$	9	$X^{1/6}\log(X)$
4	$X^{1/3}$	12	$X^{1/6}$
5	$X^{1/6}(\log(X))^2$	16	$X^{1/6}$
6	$X^{1/6}\log(X)$	18	$X^{1/6}$

Table 1: Values of $h_N(X)$, ordered by naive height

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- Therefore counting $\mathcal{N}(N, X) \leftrightarrow$ counting rational points on $\mathcal{X}_0(N)$.

Fun fact!

$\mathcal{X}_0(N)$ is not a scheme, but a stack.

Every point has the non-trivial automorphism $[-1]$. So, $\mathcal{X}_0(N)$ is actually a μ_2-gerbe.

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- Counting elliptic curves in quadratic twist families (generalizing work of Harron and Snowden),
- Counting points of bounded height on weighted projective stacks (using framework of Ellenberg, Satriano and Zureick-Brown).

Plan

The Harron and Snowden framework

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- Counting problem: count pairs $(A, B) \in \mathbb{Z}^2$ such that:
  - $\exists u, t \in \mathbb{Q}$ such that $A = u^4 f(t)$, $B = u^6 g(t)$,
  - $\max\{|A|^3, |B|^2\} < X$,
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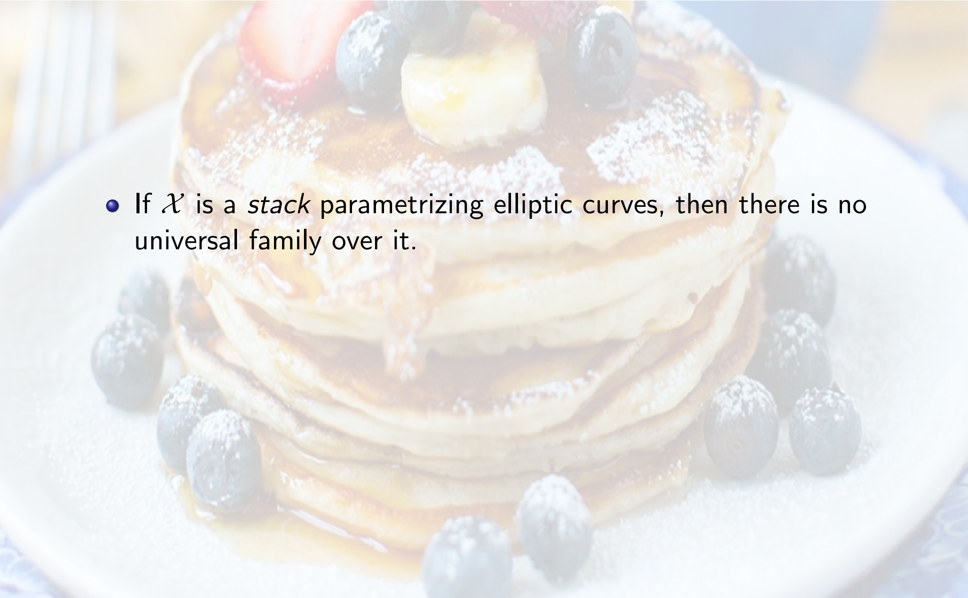

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The problem with stacks

- If $\mathcal{X}$ is a *stack* parametrizing elliptic curves, then there is no universal family over it.
- However, in our case, we can find a double cover that has one (at least generically).

Motivating example: $\mathcal{X}_0(3)$

The case for general N:

- For $N \in \{3, 4, 6, 8, 9, 12, 16, 18\}$, we consider the cover $\Phi_N : \mathcal{X}_1(N) \to \mathcal{X}_0(N)$, whose geometric fibers are isomorphic to $(\mathbb{Z}/N\mathbb{Z})^\times$.

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Proposition [BS, 2020]

Let $N \in \{3, 4, 6, 8, 9, 12, 16, 18\}$. Then for an appropriate choice of H in each case, $\mathcal{X}_{1/2}(N)$ is a stacky curve with at most one stacky point, whose coarse space is isomorphic to $\mathbb{P}^1$.

Modified counting problem

For $N \in \{3, 4, 6, 8, 9, 12, 16, 18\}$ we are able to find $f_N, g_N \in \mathbb{Q}[t]$ coprime, such that every elliptic curve giving a rational point on $\mathcal{X}_{1/2}(N)^{**}$ is isomorphic to one of the form:

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**for $N = 3$, we want an open substack of $\mathcal{X}_{1/2}(N)$.

Plan

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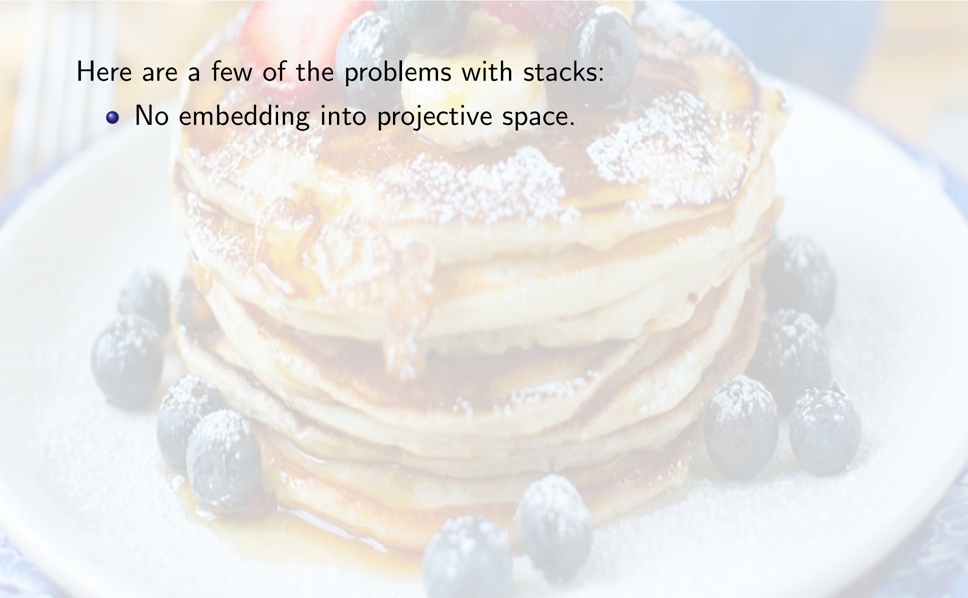
If $x \in X(\mathbb{Q})$ define the height of x as:

$$\mathrm{Ht}_{\mathcal{L}}(x) := \mathrm{Ht}(\phi_{\mathcal{L},n}(x))^{1/n}.$$

What to do about stacks

Here are a few of the problems with stacks:

- No embedding into projective space.



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Fixing these

In a forthcoming paper, Ellenberg, Satriano and Zureick-Brown suggest a definition of height that fixes all of these. We will denote their height as: $\mathrm{Ht}_{\mathcal{L},ESZB}(x)$.

Weighted projective stacks

Let $a_0, a_1 \dots a_k$ be positive integers. Consider the $\mathbb{G}_m$ action on $\mathbb{A}^{k+1}$ given by:

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Idea

We're going to map our stacks into weighted projective stacks.

ESZB Height on a nice enough stack

Proposition, [ESZB, '20]

Let $\mathcal{X}$ be a stack over $\operatorname{Spec}\mathbb{Z}$, let $\mathcal{L}$ be a line bundle on $\mathcal{X}$ such that $\mathcal{L}^{\otimes n}$ is generically globally generated by sections $s_0, s_1, s_2 \cdots s_k$. Let $x : \operatorname{Spec}\mathbb{Q} \to \mathcal{X}$ and for each i, let $x_i = x^*(s_i)$. Suppose you can scale $x_0, x_1, \ldots, x_k$ so that each $x_i \in \mathbb{Z}$ and for every prime p, there is some x_i such that $v_p(x_i) < n$. Then the height is given by:

$$\log \mathrm{Ht}_{\mathcal{L},ESZB}(x) = \frac{1}{n} \log \max\{|x_0|, |x_1|, \ldots |x_k|\} + O_{\mathcal{X}(\mathbb{Q})}(1).$$

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Corollary

Consider $(E, C) \in \mathcal{X}_0(N)(\mathbb{Q})$, then

$$\mathrm{Ht}_{naive}(E) = const \cdot \mathrm{Ht}_{\lambda, ESZB}(E)^{12}.$$

Rings of modular forms

Theorem, [HT '11]

Let $M(N)$ denote the ring of modular forms of $\mathcal{X}_0(N)$. The following are the generators and relations of $M(N)$:

N	Degrees of generators	Relations
2	$(2, 4)$	None
3	$(2, 4, 6)$	$b^2 - ac$
4	$(2, 2)$	None
5	$(2, 4, 4)$	$b^2 - c(a^2 + 4b - 8c)$
6	$(2, 2, 2)$	$b^2 - ac$
8	$(2, 2, 2)$	$b^2 - ac$
9	$(2, 2, 2)$	$b^2 - ac$

Table 2: Ring of modular forms of low level

The final problem

- Now we have reduced our counting integers in a box with certain relations between them, e.g. for $\mathcal{X}_0(3)$, we count triples (a, b, c) such that $|a| < X^{1/6}$, $|b| < X^{1/3}$ and $|c| < X^{1/2}$, $b^2 = ac$ and $\gcd\{a^6, b^3, c^2\}$ is 12th power free.

Thank you for listening!