The Gross-Zagier-Zhang formula over function fields

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History: Gross-Zagier formula

- A: an elliptic curve over $\mathbb{Q}$: $A(\mathbb{Q})$: finitely generated abelian group.
- Birch and Swinnerton-Dyer conjecture: $\mathrm{rank}_{\mathbb{Z}} A(\mathbb{Q}) = \mathrm{ord}_{s=1} L(A, s)$.
- Question: If $L(A, s)$ is odd, how to find a non-torsion point?

Theorem (Gross and Zagier)

For some imaginary quadratic fields E such that $L(A_E, s)$ is odd, there is an explicit $P \in A(E)$ such that

$$\langle P, P \rangle_{\mathrm{NT}} = c \cdot L'(A_E, 1).$$

- Heegner point P: produced from $\phi(\mathcal{H} \bigcap E)$, where

$$\phi : \mathcal{H} = \{\tau \in \mathbb{C} : \mathrm{Im}\tau > 0\} \to X_0(N) \xrightarrow{\text{Wiles et al.}} A.$$

- Application: BSD in analytic rank 1 case (Kolyvagin's Euler system).

History: Analog over $F_q(T)$, q odd

- A: an elliptic curve over $F_q(T)$, split multiplicative reduction at $\infty = \left(\frac{1}{T}\right)$.

Theorem (Rück and Tipp)

For some separable quadratic extensions $E/F_q(T)$, nonsplit at ∞ (imaginary), such that $L(A_E, s)$ is odd, there is an explicit $P \in A(E)$ such that

$$\langle P, P \rangle_{\mathrm{NT}} = c \cdot L'(A_E, 1).$$

- Heegner point P: produced from $\phi(\Omega_\infty \bigcap E)$, where

$$\phi : \Omega_\infty = \text{Drinfeld upper half plane} \to M \xrightarrow{\text{Drinfeld}} A.$$

- M: Drinfeld modular curve.

Generalization

S.Zhang, Yuan-W.Zhang-S.Zhang:

- F : totally real field; E imaginary quadratic extension;
- σ: cuspidal automorphic representation of $\mathrm{GL}_{2,F}$, holomorphic of weight 2, such that $L(s, \sigma_E)$ is odd.

Q. 2019:

- F : arbitrary function field; E "arbitrary" quadratic extension;
- σ: arbitrary cuspidal automorphic representation of $\mathrm{GL}_{2,F}$ such that $L(s, \sigma_E)$ is odd.

Generalization

Q. 2019

- F : arbitrary function field; E arbitrary quadratic extension;
- σ: arbitrary cuspidal automorphic representation of $\mathrm{GL}_{2,F}$ such that $L(s, \sigma_E)$ is odd.
- Application: (full) BSD for abelian varieties (over F) of GL_2-type of "analytic rank 1" (Tate, Milne, Schneider, Kato and Trihan, Katz).

"odd" implies:

- there exists a place ∞ of F not split in E such that σ_∞ is a discrete series ($\mathrm{JL}(\sigma_\infty)$ exists).

Conditions on abelian varieties:

- not everywhere (potential) good reduction;
- trivial central character up to twist (though we do not have this condition on σ.

Gross-Zagier-Zhang formula over function fields

- $\mathbb{B}$: incoherent quaternion algebra over $\mathbb{A}_F$, i.e. division at odd number of places. $\mathbb{B}_\infty$ is division. Fix $\mathbb{A}_E \hookrightarrow \mathbb{B}$.
- M_U/F: Drinfeld modular curves, indexed by $U \subset \mathbb{B}^\times$. Hecke action: $\mathbb{B}^\times$ acts on $\varprojlim M_U$.
- A/F: abelian variety such that (up to twist by a character)

$$\pi_A := \varinjlim_U \mathrm{Hom}(J(M_U), A)_\mathbb{C} = \mathrm{JL}_{\mathbb{B}^\times}(\sigma).$$

- $t^\circ \in \left(\varprojlim_U J(M_U)\right)^{E^\times}$ where $t \in E^\times \backslash \mathbb{A}_E^\times$.
- For $\phi \in \pi = \pi_A$, let

$$P_\pi(\phi) := \int_{E^\times \backslash \mathbb{A}_E^\times / \mathbb{A}_F^\times} \phi(t^\circ) \in A(E^{\mathrm{ab}})_\mathbb{C}.$$

Gross-Zagier-Zhang formula over function fields

- $\mathcal{P}_\pi := \left(\phi \otimes \tilde{\phi} \mapsto \langle P_\pi(\phi), P_\pi(\tilde{\phi})\rangle_{\mathrm{NT}}\right) \in \mathrm{Hom}_{\mathbb{A}_E^\times \times \mathbb{A}_E^\times}(\pi \otimes \tilde{\pi}, \mathbb{C})$.
- Tunnell-Saito: $\dim \mathrm{Hom}_{E_v^\times}(\pi_v, \mathbb{C}) = 1$ by choosing $\mathbb{B}$.
- $\alpha_{\pi_v} \in \mathrm{Hom}_{E_v^\times \times E_v^\times}(\pi_v \otimes \tilde{\pi}_v, \mathbb{C})$: an explicit generator.

Theorem (Q.)

There is an explicit constant $c \neq 0$ such that

$$\mathcal{P}_\pi = c \cdot L'(1/2, \pi_E) \cdot \prod_v \alpha_{\pi_v}.$$

- Note that $\pi_E \cong \sigma_E$ up to a twist. The general form of this theorem applies to $L'(1/2, \pi_E \otimes \Omega)$ where Ω is a Hecke character of $E^\times$.

Arithmetic variants of Jacquet's relative trace formulas

- Consider all such π's together, get a distribution H on $C_c^\infty(\mathbb{B}^\times)$:

$$H : C_c^\infty(\mathbb{B}^\times) \xrightarrow{\text{Hecke action}} \bigoplus_\pi \pi \otimes \tilde{\pi} \xrightarrow{\oplus_\pi \mathcal{P}_\pi} \mathbb{C}.$$

- $Z(f)$: Hecke correspondence. Then

$$H(f) = \int_{E^\times \backslash \mathbb{A}_E^\times / \mathbb{A}_F^\times} \int^*_{E^\times \backslash \mathbb{A}_E^\times} \langle Z(f)_* t_1^\circ, t_2^\circ \rangle_{\mathrm{NT}} dt_2 dt_1,$$

- Find another group with subgroup actions, which
  - give the same quotient space $E^\times \backslash B^\times / E^\times$. Here B is an F-quaternion algebra.
  - Encode the problem we are looking at.

Arithmetic variants of Jacquet's relative trace formulas

- $G = \mathrm{GL}_{2,E}$, A the diagonal torus, $H = \mathrm{GU}\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right)$. Then

$$E^\times \backslash B^\times / E^\times \text{``}=\text{''} A \backslash G / H.$$

- For $f' \in C_c^\infty(G(\mathbb{A}_E))$, a kernel function $K_{f'}$ on $(G(E)\backslash G(\mathbb{A}_E))^2$ represents the Hecke action. Define another distribution

$$I(f') := \frac{d}{ds}|_{s=0} \int_{[A]} \int_{[H]} K_{f'}(a,h)|a|^s \, da dh$$

- The integral over $[A]$ encodes L-function; The integral over $[H]$ tests base change.

Strategy

- $$\begin{array}{ccccccc}
\bigoplus_{\pi} \alpha_{\pi} & \overset{\textit{goal}}{=\!=\!=} & \bigoplus_{\pi} \mathcal{P}_{\pi} & = & H(f) & \overset{1}{=\!=\!=\!=\!=} & \sum_{v} \sum_{E^{\times} \backslash B(v)^{\times} / E^{\times}} \cdots \\
 & \diagdown\!\!\diagdown & \Big\| \; 1 \;\; \textit{plus} \;\; 2 & & & & \Big\| \; 2 \\
 & & \bigoplus_{\Pi} I_{\Pi} & = & I(f') & \overset{\textit{unfold}}{=\!=\!=} & \sum_{v} \sum_{\gamma \in A \backslash G / H} O(0, \gamma, f'^{v}) O'(0, \gamma, f'_{v})
\end{array}$$

Strategy

- $$\begin{array}{ccccccc}
\bigoplus_\pi \alpha_\pi & \overset{goal}{=\!=\!=} & \bigoplus_\pi \mathcal{P}_\pi & =\!=\!= & H(f) & \overset{1}{=\!=\!=\!=\!=\!=} & \sum_v \sum_{E^\times \backslash B(v)^\times / E^\times} \cdots \\
 & \diagdown\!\!\diagdown & \Big\| \; 1 \;\; plus \;\; 2 & & & & \Big\| \; 2 \\
 & & \bigoplus_\Pi I_\Pi & =\!=\!= & I(f') & \overset{unfold}{=\!=\!=} & \sum_v \sum_{\gamma \in A \backslash G / H} O(0, \gamma, f'^v) O'(0, \gamma, f'_v)
\end{array}$$
- Intersection theory on integral models $\mathcal{M}_U$ of modular curves:

$$H(f) = \sum_v i(f)_v + j(f)_v,$$

where $i(f)_v$ is the horizontal intersection and $j(f)_v$ the rest.

1: Local computation of $i(f)_\infty$ (and $j(f)_\infty$)

- $\hat{\Sigma}_n$: Drinfeld's n-th formal covering of the upper half plane over F_∞.
- let $B = B(\infty)$ (the modification of $\mathbb{B}$ at ∞, then

$$\hat{\mathcal{M}}_U = B^\times \backslash \left(\hat{\Sigma}_n \times \mathbb{B}^{\infty,\times} / U^\infty \right),$$

- Decompose $i(f)_\infty$ into a sum along $E^\times \backslash B^\times / E^\times$. For $f = f^\infty f_\infty$, $\delta \in B^\times$, the summand is $O(\delta, f^\infty)$ times

$$i(\delta, f_\infty) := \int_{E_\infty^\times / F_\infty^\times} \int_{E_\infty^\times} \left(\int_{\mathbb{B}_\infty^\times} f_\infty(g) m_\infty(t_1^{-1}\delta t_2, g^{-1}) dg \right) dt_2 dt_1,$$

- Define

$$m_\infty \in C^\infty(B_\infty^\times \times \mathbb{B}_\infty^\times - \{(1,1)\})$$

as follows: $B_\infty^\times \times \mathbb{B}_\infty^\times$ acts on $\hat{\Sigma}_n$, and $m_\infty(g_1, g_2)$ is the intersection of $(g_1, g_2)z$ and z. Here $z \in \hat{\Sigma}_n$ is a CM point

2: Orbital comparison

- $$\begin{array}{ccccccc}
\bigoplus_{\pi} \alpha_{\pi} & \overset{goal}{=\!=\!=} & \bigoplus_{\pi} \mathcal{P}_{\pi} & =\!=\!= & H(f) & \overset{1}{=\!=\!=\!=\!=\!=} & \sum_{v} \sum_{E^{\times}\backslash B(v)^{\times}/E^{\times}} \cdots \\
 & \diagdown\!\!\diagdown & \Big\| \; 1 \;\; plus \;\; 2 & & & & \Big\| \; 2 \\
 & & \bigoplus_{\Pi} I_{\Pi} & =\!=\!= & I(f') & \overset{unfold}{=\!=\!=} & \sum_{v} \sum_{\gamma \in A\backslash G/H} O(0,\gamma,f'^{v}) O'(0,\gamma,f'_{v})
\end{array}$$
- $E_{\infty}^{\times}\backslash B_{\infty}^{\times}/E_{\infty}^{\times}$ "$=$" $A_{\infty}\backslash G_{\infty}/H_{\infty}$, $\delta \leftrightarrow \gamma$ (regular orbits).
- Compare $O(\delta, f^{\infty}) i(\delta, f_{\infty})$ with $O(0, \gamma, f'^{\infty}) O'(0, \gamma, f'_{\infty})$.
- Also need $O(\delta, f_{\infty}) = O(0, \gamma, f'_{\infty})$!

2: Orbital comparison

Proposition

Let $f_\infty = 1_{U_\infty}$*. Then there exists* f'_∞ *such that for all* $\delta \leftrightarrow \gamma$ *(regular orbits), we have*

$$O(\delta, f_\infty) = O(0, \gamma, f'_\infty),$$

and

$$i(\delta, f_\infty) = O(0, \gamma, f'^{\infty}) + \text{ an orbital integral on } B_\infty^\times.$$

Proposition

Let $f_\infty = 1_{U_\infty}$*. Then* $j(\delta, f_\infty)$ *equals an orbital integral on* $B_\infty^\times$*.*

Remarks

- Why the RTF method? If we follow Yuan-Zhang-Zhang and use theta lifting, we get the same results only in odd characertistics.
- The same method should work over number fields, and reprove Yuan-Zhang-Zhang's result.
- Potential application to higher weights representations of $\mathrm{GL}_2(\mathbb{Q})$ (with more ramifications).
- Potential application to higher derivatives of L-functions of GL_2 (not just PGL_2) over function fields (with more ramifications).

The End
Thank you