The Gross-Zagier-Zhang formula over function fields

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History: Gross-Zagier formula

- A: an elliptic curve over \mathbb{Q} : $A(\mathbb{Q})$: finitely generated abelian group.
- Birch and Swinnerton-Dyer conjecture: $\operatorname{rank}_{\mathbb{Z}}A(\mathbb{Q}) = \operatorname{ord}_{s=1}L(A, s)$.
- Question: If L(A, s) is odd, how to find a non-torsion point?

Theorem (Gross and Zagier)

For some imaginary quadratic fields E such that $L(A_E, s)$ is odd, there is an explicit $P \in A(E)$ such that

$$\langle P, P \rangle_{\mathrm{NT}} = c \cdot L'(A_E, 1).$$

• Heegner point *P*: produced from $\phi(\mathcal{H} \cap E)$, where

$$\phi: \mathcal{H} = \{\tau \in \mathbb{C} : \mathrm{Im}\tau > 0\} \to X_0(N) \xrightarrow{\text{Wiles et al.}} A$$

• Application: BSD in analytic rank 1 case (Kolyvagin's Euler system).

History: Analog over $F_q(T)$, q odd

• A: an elliptic curve over $F_q(T)$, split multiplicative reduction at $\infty = (\frac{1}{T})$.

Theorem (Rück and Tipp)

For some separable quadratic extensions $E/F_q(T)$, nonsplit at ∞ (imaginary), such that $L(A_E, s)$ is odd, there is an explicit $P \in A(E)$ such that

$$\langle P, P \rangle_{\mathrm{NT}} = c \cdot L'(A_E, 1).$$

• Heegner point *P*: produced from $\phi(\Omega_{\infty} \bigcap E)$, where

$$\phi: \Omega_{\infty} = \mathsf{Drinfeld} \text{ upper half plane} \to M \xrightarrow{\mathsf{Drinfeld}} A.$$

• M: Drinfeld modular curve.

- S.Zhang, Yuan-W.Zhang-S.Zhang:
 - F : totally real field; E imaginary quadratic extension;
 - σ : cuspidal automorphic representation of $\operatorname{GL}_{2,F}$, holomorphic of weight 2, such that $L(s, \sigma_E)$ is odd.
- Q. 2019:
 - F : arbitrary function field; E "arbitrary" quadratic extension;
 - σ : arbitrary cuspidal automorphic representation of $GL_{2,F}$ such that $L(s, \sigma_E)$ is odd.

Q. 2019

- F : arbitrary function field; E arbitrary quadratic extension;
- σ : arbitrary cuspidal automorphic representation of $\operatorname{GL}_{2,F}$ such that $L(s, \sigma_E)$ is odd.
- Application: (full) BSD for abelian varieties (over F) of GL_2 -type of "analytic rank 1" (Tate, Milne, Schneider, Kato and Trihan, Katz).

"odd" implies:

there exists a place ∞ of F not split in E such that σ_∞ is a discrete series (JL(σ_∞) exists).

Conditions on abelian varieties:

- not everywhere (potential) good reduction;
- trivial central character up to twist (though we do not have this condition on σ .

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- B: incoherent quaternion algebra over A_F, i.e. division at odd number of places. B_∞ is division. Fix A_E → B.
- *M_U*/*F*: Drinfeld modular curves, indexed by *U* ⊂ B[×]. Hecke action: B[×] acts on lim *M_U*.
- A/F: abelian variety such that (up to twist by a character)

$$\pi_{\mathcal{A}} := \varinjlim_{U} \operatorname{Hom}(J(M_{U}), \mathcal{A})_{\mathbb{C}} = \operatorname{JL}_{\mathbb{B}^{\times}}(\sigma).$$

•
$$t^{\circ} \in \left(\varprojlim_{U} J(M_{U}) \right)^{E^{\times}}$$
 where $t \in E^{\times} \setminus \mathbb{A}_{E}^{\times}$.
• For $\phi \in \pi = \pi_{A}$, let

$$P_{\pi}(\phi) := \int_{E^{ imes} \setminus \mathbb{A}_{E}^{ imes} / \mathbb{A}_{F}^{ imes}} \phi(t^{\circ}) \in A(E^{\mathrm{ab}})_{\mathbb{C}}.$$

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•
$$\mathcal{P}_{\pi} := \left(\phi \otimes \tilde{\phi} \mapsto \langle P_{\pi}(\phi), P_{\pi}(\tilde{\phi}) \rangle_{\mathrm{NT}} \right) \in \mathrm{Hom}_{\mathbb{A}_{E}^{\times} \times \mathbb{A}_{E}^{\times}}(\pi \otimes \tilde{\pi}, \mathbb{C}).$$

- Tunnell-Saito: dim $\operatorname{Hom}_{\mathcal{E}_{\nu}^{\times}}(\pi_{\nu},\mathbb{C})=1$ by choosing \mathbb{B} .
- $\alpha_{\pi_{\nu}} \in \operatorname{Hom}_{E_{\nu}^{\times} \times E_{\nu}^{\times}}(\pi_{\nu} \otimes \tilde{\pi}_{\nu}, \mathbb{C})$: an explicit generator.

Theorem (Q.)

There is an explicit constant $c \neq 0$ such that

$$\mathcal{P}_{\pi} = \boldsymbol{c} \cdot L'(1/2, \pi_E) \cdot \prod_{\mathbf{v}} \alpha_{\pi_{\mathbf{v}}}.$$

• Note that $\pi_E \cong \sigma_E$ up to a twist. The general form of this theorem applies to $L'(1/2, \pi_E \otimes \Omega)$ where Ω is a Hecke character of E^{\times} .

Arithmetic variants of Jacquet's relative trace formulas

• Consider all such π 's together, get a distribution H on $C_c^{\infty}(\mathbb{B}^{\times})$:

$$H: C^{\infty}_{c}(\mathbb{B}^{\times}) \xrightarrow{\text{Hecke action}} \bigoplus_{\pi} \pi \otimes \tilde{\pi} \xrightarrow{\bigoplus_{\pi} \mathcal{P}_{\pi}} \mathbb{C}.$$

• Z(f): Hecke correspondence. Then

$$H(f) = \int_{E^{\times} \setminus \mathbb{A}_{E}^{\times} / \mathbb{A}_{F}^{\times}} \int_{E^{\times} \setminus \mathbb{A}_{E}^{\times}}^{*} \langle Z(f)_{*} t_{1}^{\circ}, t_{2}^{\circ} \rangle_{\mathrm{NT}} dt_{2} dt_{1},$$

- Find another group with subgroup actions, which
 - give the same quotient space $E^{\times} \setminus B^{\times} / E^{\times}$. Here B is an F-quaternion algebra.
 - Encode the problem we are looking at.

Arithmetic variants of Jacquet's relative trace formulas

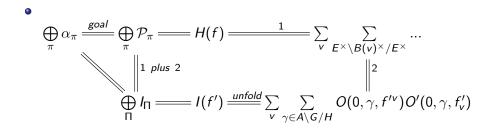
•
$$G = GL_{2,E}$$
, A the diagonal torus, $H = GU \begin{pmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{pmatrix}$. Then

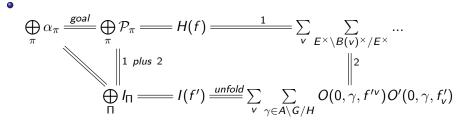
$$E^{\times} \setminus B^{\times} / E^{\times}$$
 " = " $A \setminus G / H$.

For f' ∈ C[∞]_c(G(A_E)), a kernel function K_{f'} on (G(E)\G(A_E))² represents the Hecke action. Define another distribution

$$I(f'):=rac{d}{ds}|_{s=0}\int_{[A]}\int_{[H]}K_{f'}(a,h)|a|^sdadh$$

• The integral over [A] encodes L-function; The integral over [H] tests base change.





• Intersection theory on integral models \mathcal{M}_U of modular curves:

$$H(f) = \sum_{v} i(f)_{v} + j(f)_{v},$$

where $i(f)_v$ is the horizontal intersection and $j(f)_v$ the rest.

1: Local computation of $i(f)_{\infty}$ (and $j(f)_{\infty}$)

- $\hat{\Sigma}_n$: Drinfeld's *n*-th formal covering of the upper half plane over F_{∞} .
- let $B = B(\infty)$ (the modification of ${\mathbb B}$ at ∞ , then

$$\hat{\mathcal{M}}_U = B^{\times} \setminus \left(\hat{\Sigma}_n \times \mathbb{B}^{\infty, \times} / U^{\infty} \right),$$

• Decompose $i(f)_{\infty}$ into a sum along $E^{\times} \setminus B^{\times} / E^{\times}$. For $f = f^{\infty} f_{\infty}$, $\delta \in B^{\times}$, the summand is $O(\delta, f^{\infty})$ times

$$i(\delta, f_{\infty}) := \int_{E_{\infty}^{\times}/F_{\infty}^{\times}} \int_{E_{\infty}^{\times}} \left(\int_{\mathbb{B}_{\infty}^{\times}} f_{\infty}(g) m_{\infty}(t_1^{-1} \delta t_2, g^{-1}) dg \right) dt_2 dt_1,$$

Define

$$m_{\infty} \in \mathcal{C}^{\infty}(B_{\infty}^{ imes} imes \mathbb{B}_{\infty}^{ imes} - \{(1,1)\})$$

as follows: $B_{\infty}^{\times} \times \mathbb{B}_{\infty}^{\times}$ acts on $\hat{\Sigma}_n$, and $m_{\infty}(g_1, g_2)$ is the intersection of $(g_1, g_2)z$ and z. Here $z \in \hat{\Sigma}_n$ is a CM point

2: Orbital comparison

- $E_{\infty}^{\times} \setminus B_{\infty}^{\times} / E_{\infty}^{\times}$ " = " $A_{\infty} \setminus G_{\infty} / H_{\infty}$, $\delta \leftrightarrow \gamma$ (regular orbits).
- Compare $O(\delta, f^{\infty})i(\delta, f_{\infty})$ with $O(0, \gamma, f'^{\infty})O'(0, \gamma, f'_{\infty})$.

• Also need
$$O(\delta, f_{\infty}) = O(0, \gamma, f'_{\infty})!$$

Proposition

Let $f_{\infty} = 1_{U_{\infty}}$. Then there exists f'_{∞} such that for all $\delta \leftrightarrow \gamma$ (regular orbits), we have

$$O(\delta, f_{\infty}) = O(0, \gamma, f'_{\infty}),$$

and

$$i(\delta, f_\infty) = O(0, \gamma, f'^\infty) + ext{ an orbital integral on } B_\infty^ imes$$

Proposition

Let $f_{\infty} = 1_{U_{\infty}}$. Then $j(\delta, f_{\infty})$ equals an orbital integral on B_{∞}^{\times} .

- Why the RTF method? If we follow Yuan-Zhang-Zhang and use theta lifting, we get the same results only in odd characertistics.
- The same method should work over number fields, and reprove Yuan-Zhang-Zhang's result.
- Potential application to higher weights representations of $\operatorname{GL}_2(\mathbb{Q})$ (with more ramifications).
- Potential application to higher derivatives of L-functions of GL_2 (not just PGL_2) over function fields (with more ramifications).

The End Thank you