# A Local Trace Formula for the Local Gan-Gross-Prasad Conjecture for Special Orthogonal Groups

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# Spherical harmonics

- $$\mathrm{SO}_2(\mathbb{R}) \hookrightarrow \mathrm{SO}_3(\mathbb{R}) \quad \text{compact.}$$
- $$m(\pi) = \dim \mathrm{Hom}_{\mathrm{SO}_2(\mathbb{R})}(\pi, \mathbb{C}), \quad \pi \in \mathrm{Irr}(\mathrm{SO}_3(\mathbb{R})).$$
- By Frobenius reciprocity,
$$m(\pi) = \dim \mathrm{Hom}_{\mathrm{SO}_3(\mathbb{R})}(\pi, \mathrm{Ind}_{\mathrm{SO}_2(\mathbb{R})}^{\mathrm{SO}_3(\mathbb{R})}(\mathbb{C})).$$
- $$\mathrm{Ind}_{\mathrm{SO}_2(\mathbb{R})}^{\mathrm{SO}_3(\mathbb{R})}(\mathbb{C}) \simeq L^2(S^2).$$
- $$L^2(S^2) \curvearrowleft \mathrm{SO}_3(\mathbb{R}) \quad \text{spectral decomposition.}$$

# Spherical harmonics

- By the theory of spherical harmonics,

$$L^2(S^2) \simeq \widehat{\bigoplus}_{l=0}^{\infty} H_l,$$

$$H_l = \text{spherical harmonics of deg. } l, \quad \dim = 2l+1,$$
$$H_l \text{ is an irr. rep. of } \mathrm{SO}_3(\mathbb{R}).$$

- $$\dim \mathrm{Hom}_{\mathrm{SO}_3(\mathbb{R})}(\pi, L^2(S^2)) = 1, \quad \text{for any } \pi \in \mathrm{Irr}(\mathrm{SO}_3(\mathbb{R})).$$

- $$m(\pi) = \int_{\mathrm{SO}_2(\mathbb{R})} \Theta_\pi(h)dh, \quad \text{by Schur's orthogonality.}$$

# Set up

- $F$ local field of char. zero.
- $W \hookrightarrow V$ quadratic spaces $/F$.
- $W^{\perp}$ split of odd dim.
- $N =$ unipotent radical of the parabolic subgroup of $\mathrm{SO}(V)$ stabilizing the full isotropic flag determined by $W^{\perp}$.
- $G = \mathrm{SO}(W) \times \mathrm{SO}(V)$.
- $H = \mathrm{SO}(W) \ltimes N \hookrightarrow G$, with $\Delta : \mathrm{SO}(W) \hookrightarrow G$.
- $\xi =$ a generic character of $N$ extending to $H$.
- $(G, H, \xi)$ is called a **Gan-Gross-Prasad** triple.

# Multiplicity one

- Set

$$m(\pi) = \dim \mathrm{Hom}_{H(F)}(\pi, \xi_F), \quad \pi \in \mathrm{Irr}(G(F))$$

**Theorem.**

$$m(\pi) \leq 1.$$

- For $F$ $p$-adic, proved by A. Aizenbud-D. Gourevitch-S. Rallis-G. Schiffmann for $r = 0$, and W. Gan-B.Gross-D.Prasad reducing the general case to $r = 0$.
- For $F$ Archimedean, proved by B. Sun-C. Zhu for $r = 0$, and D. Jiang-Sun-Zhu reducing the general case to $r = 0$.

# Local Gan-Gross-Prasad conjecture

- The local Gan-Gross-Prasad conjecture suggests that $m(\pi)$ has more stable behavior by considering the local **Vogan packet** attached to $(G, H, \xi)$.
- To introduce local Vogan packets, consider pure inner forms of $\mathrm{SO}(W)$, parametrized by $H^1(F, \mathrm{SO}(W)) \simeq H^1(F, H)$.
- For $\alpha \in H^1(F, H)$, there exists

$$(W_\alpha, V_\alpha = W_\alpha \oplus W^\perp)$$

dim $W_\alpha =$ dim $W$, $\mathrm{disc}\, W_\alpha = \mathrm{disc}\, W$,
with a GGP triple

$$(G_\alpha, H_\alpha, \xi_\alpha).$$

Moreover

$${}^L G_\alpha \simeq {}^L G.$$

# Local Gan-Gross-Prasad conjecture

**Conjecture**.(Gan-Gross-Prasad)
For any generic $L$-parameter $\varphi : \mathcal{W}_F \to {}^L G$ with $L$-packet $\Pi^G(\varphi)$,

$$\sum_{\alpha \in H^1(F,H)} \sum_{\pi \in \Pi^{G_\alpha}(\varphi)} m(\pi) = 1.$$

Moreover, the non-vanishing of $m(\pi)$ is detected by representations of the component group $A_\varphi$ attached to $\varphi$, which is related to the sign of the relevant local symplectic root numbers.

- $$\varphi \text{ is } \begin{cases} \text{generic,} & L(s, \varphi, \mathrm{Ad}) \text{ is holomorphic at } s = 1 \\ \text{tempered,} & \mathrm{Im}(\varphi) \text{ is bounded} \end{cases}$$

# Local Gan-Gross-Prasad conjecture: $p$-adic

- J.-L. Waldspurger (tempered) and C. Moeglin-Waldspurger (generic) proved the conjecture completely when $F$ is $p$-adic (Assuming LLC for non quasi-split $\mathrm{SO}$ and quasi-split $\mathrm{SO}_{2n}$).
- The local GGP conjecture speculates parallel behaviors for unitary groups. R. Beuzart-Plessis (tempered) and Gan-A. Ichino (generic) proved the conjecture when $F$ is $p$-adic.
- There are parallel conjectures for skew-hermitian unitary groups and symplectic-metaplectic groups.
  Gan-Ichino proved the conjecture for skew-hermitian unitary groups, and H. Atobe for symplectic-metaplectic groups, via theta correspondence when $F$ is $p$-adic.

# Local Gan-Gross-Prasad conjecture: Archimedean

- For unitary groups, when $F = \mathbb{R}$,
  Beuzart-Plessis proved the multiplicity part of the conjecture for $\varphi$ tempered.
  H. He proved the conjecture for discrete series representations.
  H. Xue proved the conjecture for $\varphi$ tempered.
- For special orthogonal groups, when $F = \mathbb{C}$,
  J. Möllers proved the conjecture for $\mathrm{SO}(n) \times \mathrm{SO}(n+1)$.

# The theorem

In the special orthogonal groups setting, we prove the following theorem.

## Theorem (L.)

*For any tempered L-parameter* $\varphi : \mathcal{W}_F \to {}^L G$,

$$\sum_{\alpha \in H^1(F,H)} \sum_{\pi \in \Pi^{G_\alpha}(\varphi)} m(\pi) = 1.$$

- We follow the approach of Waldspurger and Beuzart-Plessis.

# Local trace formula

- For $\pi \in \mathrm{Temp}(G(F))$, by Frobenius reciprocity for unitary representations,

  $$\mathrm{Hom}_{H(F)}(\pi, \xi_F) \simeq \mathrm{Hom}_{G(F)}(\pi, \mathrm{Ind}_H^G \xi_F)$$

  where $\mathrm{Ind}_H^G \xi = L^2(H(F)\backslash G(F), \xi_F)$.
- $$L^2(H(F)\backslash G(F), \xi_F) \curvearrowleft G(F) \quad \text{spectral decomposition.}$$

# Local trace formula

- Following Arthur,

$$L^2(H(F)\backslash G(F), \xi_F) \curvearrowleft \mathcal{C}_c^\infty(G(F)) \quad \text{via convolution.}$$

- For $f \in \mathcal{C}_c^\infty(G(F))$, $x \in G(F)$, $\varphi \in L^2(H(F)\backslash G(F), \xi_F)$,

$$(R(f)\varphi)(x) = \int_{G(F)} f(g)\varphi(xg)dg = \int_{H(F)\backslash G(F)} K_f(x,y)\varphi(y)dy$$

where

$$K_f(x,y) = \int_{H(F)} f(x^{-1}hy)\xi_F(h)dh, \quad x, y \in G(F).$$

- $R(f)$ has an integral kernel $K_f(x,y)$.

# Local trace formula

- Formally,
$$\mathrm{Tr}(R(f)) \sim \int_{H(F)\backslash G(F)} K(x,x)dx.$$
- In general, RHS is not absolutely convergent.
- Work with *strongly cuspidal* functions.
- $f \in \mathcal{C}_c^\infty(G(F))$ is called strongly cuspidal if
$$\int_{U(F)} f(mu)du = 0, \quad m \in M(F)$$
for any proper parabolic subgroup $P = MU$ of $G$.
- Similarly, define strongly cuspidal functions in the *Harish-Chandra Schwartz space* $\mathcal{C}(G(F))$ of $G(F)$, denoted as $\mathcal{C}_{\mathrm{scusp}}(G(F))$.

# Local trace formula

Theorem (L.)

*For* $f \in \mathcal{C}_{\mathrm{scusp}}(G(F))$,

$$J(f) = \int_{H(F)\backslash G(F)} K_f(x,x)dx$$

*is absolutely convergent.*

- Establish spectral and geometric expansions for $J(f)$ through comparing with Arthur's local trace formula.

# Spectral expansion

### Theorem (L.)

*For* $f \in \mathcal{C}_{\mathrm{scusp}}(G(F))$*, set*

$$J_{\mathrm{spec}}(f) = \int_{\mathcal{X}(G(F))} D(\pi)\theta_f(\pi)m(\pi)d\pi.$$

*Then* $J_{\mathrm{spec}}(f)$ *is absolutely convergent, and*

$$J(f) = J_{\mathrm{spec}}(f).$$

- $\mathcal{X}(G(F)) := \{(M,\sigma)| \quad \sigma \in T_{\mathrm{ell}}(M(F))\}/\mathrm{conj.}$, where $T_{\mathrm{ell}}(M(F)) =$ elliptic representations introduced by Arthur.
- For $\pi$ attached to $(M,\sigma)$, $\theta_f(\pi) = (-1)^{a_G - a_M} J_M^G(\sigma, f)$, where $J_M^G(\sigma, f)$ is the weighted character defined by Arthur.

# Geometric multiplicity formula

Theorem (L.)

*For* $\pi \in \mathrm{Temp}(G(F))$,

$$m(\pi) = m_{\mathrm{geom}}(\pi) = \int_{\Gamma(G,H)} c_\pi(x) D^G(x)^{1/2} \Delta(x)^{-1/2} dx.$$

- When $F$ is $p$-adic it was proved by Waldspurger.

# Geometric multiplicity formula: $\Gamma(G, H)$

- $$\Gamma(G,H) := \bigcup_{T\in\mathcal{T}} T_{\mathrm{reg}}(F).$$
  $\mathcal{T}$ is a set of subtori of $\mathrm{SO}(W)$.

- $T \in \mathcal{T}$ iff. $T$ max. ell. in $\mathrm{SO}(W'')$
  where $W'' \subset W$ non-degenerate and $\dim(W/W'')$ even.

# Geometric multiplicity formula: definition of $c_\pi$

**Theorem (Harish-Chandra for $p$-adic, Barbasch-Vogan for Archimedean)**

*For $x \in G_{\mathrm{ss}}$ and $X \in \omega \subset \mathfrak{g}_x$ a small neighborhood of 0, there exists constants $c_{\pi,\mathcal{O}}(x) \in \mathbb{C}$ such that*

$$\lim_{X \to 0} D^G(xe^X)^{1/2}\Theta_\pi(xe^X) = D^G(x)^{1/2} \sum_{\mathcal{O} \in \mathrm{Nil}_{\mathrm{reg}}(\mathfrak{g}_x)} c_{\pi,\mathcal{O}}(x)\widehat{j}(\mathcal{O}, X).$$

*Here $\widehat{j}(\mathcal{O}, X) = \mathcal{F}(J_{\mathcal{O}}(\cdot))$.*

- The definition of $c_\pi$, first appeared in the work of Waldspurger, is the main technical ingredient.
- $c_\pi$ is nonzero only when $G_x$ is quasi-split. When it is the case, $c_\pi = c_{\pi,\mathcal{O}}$ for a particular $\mathcal{O} \in \mathrm{Nil}_{\mathrm{reg}}(\mathfrak{g}_x)$.

# Geometric multiplicity formula: definition of $c_\pi$

- For unitary groups, $\mathrm{Nil}_{\mathrm{reg}}(\mathfrak{g}_x)$ can be permuted by scaling. The geometric multiplicity is independent of the orbit chosen. Therefore set

$$c_\pi(x) := \frac{\sum_{\mathcal{O}\in\mathrm{Nil}_{\mathrm{reg}}(\mathfrak{g}_x)} c_{\pi,\mathcal{O}}}{|\mathrm{Nil}_{\mathrm{reg}}(\mathfrak{g}_x)|}.$$

- Benefit:

$$D^G(x)^{1/2}c_\pi(x) = \lim_{x'\in T_{\mathrm{qd},x}(F)\to x} \frac{D^G(x')\Theta_\pi(x')}{|W(G_x, T_{\mathrm{qd},x})|}$$

  where $T_{\mathrm{qd},x} \subset B_x \subset G_x$.
- It is NOT the case for special orthogonal groups, really need to pick up a particular regular nilpotent orbit.

# Geometric multiplicity formula: definition of $c_\pi$

- $\mathrm{Nil}_{\mathrm{reg}}(\mathfrak{so}(V)) \neq \emptyset$ iff. $(V, q)$ is quasi-split.
  For $\dim V$ is odd or $\leq 2$, $|\mathrm{Nil}_{\mathrm{reg}}(\mathfrak{so}(V))| = 1$.
- For $\dim V = 2m$ is even and $\geq 4$, set

$$\mathcal{N}^V = \begin{cases} F^\times / F^{\times 2}, & \text{split} \\ \mathrm{Im}(q_{\mathrm{an}})/F^{\times 2}, & \text{non-split.} \end{cases}$$

  Then $\mathcal{N}^V \leftrightarrow \mathrm{Nil}_{\mathrm{reg}}(\mathfrak{so}(V))$.
- Therefore

$$\mathrm{Nil}_{\mathrm{reg}}(\mathfrak{g}) \leftrightarrow \begin{cases} \mathcal{N}^V, & \dim V \text{ is even } \geq 4, \\ \mathcal{N}^W, & \dim W \text{ is even } \geq 4. \end{cases}$$

# Geometric multiplicity formula: definition of $c_\pi$

- Recall $V = W \oplus \langle v_0 \rangle \oplus Z$.
- Set $\nu_0 = q(v_0)$. When $\dim V$ is even $\geq 4$, $\nu_0 \in \mathcal{N}^V$; When $\dim W$ is even $\geq 4$, $-\nu_0 \in \mathcal{N}^W$.
- For $x \in T_{\mathrm{reg}} \in \mathcal{T}$, set $V'_x$ (resp. $W'_x$) $= \ker(1-x)$ in $V$ (resp. $W$).
- Then
$$G_x = G'_x \times G''_x$$
with $G'_x = \mathrm{SO}(V'_x) \times \mathrm{SO}(W'_x)$, $G''_x = T \times T$.
- When $G'_x$ is quasi-split, set

$$c_\pi(x) = \begin{cases} c_{\pi,\mathcal{O}_{\nu_0}}, & \dim V'_x \geq 4 \text{ even} \\ c_{\pi,\mathcal{O}_{-\nu_0}}, & \dim W'_x \geq 4 \text{ even} \\ c_{\pi,\mathcal{O}_{\mathrm{reg}}}, & \text{otherwise.} \end{cases}$$

# Geometric multiplicity formula: definition of $c_\pi$

## Lemma (L.)

*For any* $\mathcal{O} \in \mathrm{Nil}_{\mathrm{reg}}(\mathfrak{g}_x)$, *define*

$$c_{\varphi,\mathcal{O}}(x) := \sum_{\pi \in \Pi^G(\varphi)} c_{\pi,\mathcal{O}}(x).$$

*Then*

$$c_{\varphi,\mathcal{O}}(x) = c_{\varphi,\mathcal{O}'}(x)$$

*for any* $\mathcal{O}, \mathcal{O}' \in \mathrm{Nil}_{\mathrm{reg}}(\mathfrak{g}_x)$.
*In particular,*

$$D^G(x)^{1/2} c_{\varphi,\mathcal{O}}(x) = |W(G_x, T_{\mathrm{qd},x})|^{-1} \lim_{x' \in T_{\mathrm{qd},x}(F) \to x} D^G(x') \sum_{\pi \in \Pi^G(\varphi)} \Theta_\pi(x').$$

# Geometric expansion

## Theorem (L.)

*For* $f \in \mathcal{C}_{\mathrm{scusp}}(G(F))$, *set*

$$J_{\mathrm{geom}}(f) = \int_{\Gamma(G,H)} c_f(x) D^G(x)^{1/2} \Delta(x)^{-1/2} dx.$$

*Then* $J_{\mathrm{geom}}(f)$ *is absolutely convergent, and*

$$J(f) = J_{\mathrm{geom}}(f).$$

# Geometric expansion: definitions

- Set
$$\theta_f(x) = (-1)^{a_G - a_{M(x)}} D^G(x)^{-1/2} J^G_{M(x)}(x, f).$$
Then $\theta_f(x)$ is conjugation invariant.
- It is a **quasi-character**, i.e.
$$\lim_{X \to 0} D^G(xe^X)^{1/2} \theta_f(xe^X) = D^G(x)^{1/2} \sum_{\mathcal{O} \in \mathrm{Nil}_{\mathrm{reg}}(\mathfrak{g}_x)} c_{\theta_f, \mathcal{O}}(x) \widehat{j}(\mathcal{O}, X).$$
- Define
$$c_f(x) = \begin{cases} c_{\theta_f, \mathcal{O}_{\nu_0}}, & \dim V'_x \geq 4 \text{ even} \\ c_{\theta_f, \mathcal{O}_{-\nu_0}}, & \dim W'_x \geq 4 \text{ even} \\ c_{\theta_f, \mathcal{O}_{\mathrm{reg}}}, & \text{otherwise.} \end{cases}$$

# Geometric expansion: localization

- By partition of unity,

$$\text{supp } \theta_f \subset \begin{cases} \text{neighborhood of } x \neq 1 \\ \text{neighborhood of } x = 1 \end{cases}$$

- For $x \in \mathrm{SO}(W)_{\mathrm{ss}}$, when $x \neq 1$,

$$(G_x, H_x, \xi_x) = (G'_x, H'_x, \xi'_x) \times (G''_x, H''_x, 1).$$

  $(G'_x, H'_x, \xi'_x)$ is a GGP triple of smaller dimension, and $(G''_x, H''_x, 1)$ is $\Delta : H''_x \hookrightarrow H''_x \times H''_x = G''_x$.
- Induction on dim $G$ and Arthur's local trace formula.

# Geometric expansion: Lie algebra variant

- For supp $\theta_f \subset$ neighborhood of $x = 1$, via exponential, descent to Lie algebra variants $J_{\mathrm{geom}}^{\mathrm{Lie}}(f)$ and $J^{\mathrm{Lie}}(f)$.
- $J_{\mathrm{geom}}(f)$ contains asymptotic of weighted orbital integrals near singular locus, but Arthur's local trace formula only has regular semi-simple locus. Cannot compare directly.

# Geometric expansion: Lie algebra variant

- Perform a Fourier transform on $\mathfrak{h} = \mathrm{Lie} H$ to resolve the possible singularities,

$$K^{\mathrm{Lie}}(f, x) = \int_{\mathfrak{h}} f(gXg^{-1})\xi_F(X)dX = \int_{\Xi+\mathfrak{h}^{\perp}} \widehat{f}(g^{-1}Xg)dX.$$

- $$J^{\mathrm{Lie}}(f) = \int_{H(F)\backslash G(F)} dg \int_{\Xi+\mathfrak{h}^{\perp}} \widehat{f}(g^{-1}Xg)dX.$$

# Geometric expansion: Lie algebra variant

- After truncation and changing integration order, compare with Arthur's weighted orbital integrals.
- For $f \in \mathcal{S}_{\mathrm{scusp}}(\mathfrak{g}(F))$,

$$J^{\mathrm{Lie}}(f) = \int_{\Gamma(\Xi+\mathfrak{h}^{\perp})} D^{G}(X)^{1/2}\theta_{\widehat{f}}(X)dX.$$

- $\Gamma(\Xi + \mathfrak{h}^{\perp}) = G(F)$-conjugacy classes of regular semi-simple elements in $\Xi + \mathfrak{h}^{\perp}$.

# Geometric expansion: Lie algebra variant

- Take Fourier inversion back for $J^{\mathrm{Lie}}(f)$.
- For any $\mathcal{O} \in \mathrm{Nil}_{\mathrm{reg}}(\mathfrak{g})$,

$$c_{\theta_f,\mathcal{O}}(0) = \int_{\Gamma(\mathfrak{g})} D^G(X)^{1/2}\theta_{\widehat{f}}(X)\Gamma_{\mathcal{O}}(X)dX.$$

- $\widehat{j}(X,\cdot) = \mathcal{F}(J(X,\cdot))$ and

$$\lim_{t\in F^{\times 2}, t\to 0} D^G(X,tY)\widehat{j}(X,Y) = D^G(Y)^{1/2} \sum_{\mathcal{O}\in\mathrm{Nil}_{\mathrm{reg}}(\mathfrak{g})} \Gamma_{\mathcal{O}}(X)\widehat{j}(\mathcal{O},Y).$$

(Shalika when $F$ is $p$-adic, Beuzart-Plessis when $F$ Archimedean)

# Regular germ formula

## Theorem (L.)

*For $G$ a quasi-split reductive algebraic group, $X \in \mathfrak{g}^{\mathrm{rss}}(F)$ and $\mathcal{O} \in \mathrm{Nil}_{\mathrm{reg}}(\mathfrak{g})$, set $T_G = G_X$. Then*

$$\Gamma_{\mathcal{O}}(X) = \begin{cases} 1, & \mathrm{inv}(X)\mathrm{inv}(T_G) = \mathrm{inv}_{T_G}(\mathcal{O}), \\ 0, & \textit{otherwise.} \end{cases}$$

*When $F$ is p-adic the result was already proved by D. Shelstad.*

- We also compute the invariants $\frac{\mathrm{inv}(T_G)\mathrm{inv}(X)}{\mathrm{inv}_{T_G}(\mathcal{O})}$ explicitly for any $X \in \mathfrak{g}^{\mathrm{rss}}$ without eigenvalue 0, following the work of Waldspurger.

# Regular germ formula

- Fix an $F$-splitting for $G$.
- The invariants $\mathrm{inv}(T_G), \mathrm{inv}(X)$ and $\mathrm{inv}_{T_G}(\mathcal{O})$ all lie in $H^1(F, T_G)$.
- $\mathrm{inv}_{T_G}(\mathcal{O})$ measures the difference between $\mathcal{O}$ and the regular nilpotent determined by the fixed $F$-splitting.
- $\mathrm{inv}(T_G)$ is connected with the Langlands-Shelstad transfer factor $\Delta_{\mathrm{I}}$.
- $\mathrm{inv}(X)$ is connected with the Langlands-Shelstad transfer factor $\Delta_{\mathrm{II}}$.

# Relation with the Kostant's sections

Based on a result of Kottwitz, we also prove the following theorem.

**Theorem (L.)**

$\Gamma_{\mathcal{O}}(X) = 1$ *if and only if the* $G(F)$*-orbit of* $X$ *and* $\mathcal{O}$ *lie in the* $G(F)$*-orbit of a common Kostant's section.*

- Kostant constructed a section for $\mathfrak{g} \to \mathfrak{g} /\!/ G \simeq \mathfrak{t}/W$, whose image in $\mathfrak{g}$ contains only regular elements, and meets every regular stable $\mathrm{Ad}(G)$-orbit exactly once.
- $\mathfrak{g}^{\mathrm{reg}} := \{X \in \mathfrak{g} | \quad \dim \mathrm{Cent}_{\mathfrak{g}}(X) = \dim \mathfrak{t}\}$. Regular elements are not necessarily semi-simple, e.g. regular nilpotent elements.
- The restriction of $\mathfrak{g} \to \mathfrak{t}/W$ to an $\mathrm{Ad}(G)$-orbit of a Kostant's section is a smooth submersion. The measures on the fibers are given by the relevant orbital integrals.

*Thank you!*