Calabi-Yau varieties and Shimura varieties

(based partly on joint work w/ Arnav Tripathy)

§1. Motivation + Statement of results

Defn. A smooth proj. variety $X/\mathbb{C}$ is Calabi-Yau (CY) if canonical bundle is trivial: $\underbrace{\Omega^d_X}_{\text{canonical bdle}} \cong \mathcal{O}_X$.

E.g. by adjunction:

Cubic hypersurface in $\mathbb{P}^2$ = Elliptic curve
quartic hypersurface in $\mathbb{P}^3$ = K3 surface
quintic hypersurface in $\mathbb{P}^4$ = "Quintic 3-fold", generic.
⋮

Hodge theory.

$$H^d(X,\mathbb{Q})\otimes\mathbb{C} \cong \underbrace{H^{d,0}}_{\substack{\cong H^0(\Omega^d_X)\\ \cong \mathbb{C}}} \oplus \underbrace{H^{d-1,1}}_{\substack{\cong \text{ tangent space}\\ \text{to } \mathrm{Def}(X)}} \oplus \cdots$$

Thm (Bogomolov-Tian-Todorov) Deformations of CY's always unobstructed. (of dim $h^{d-1,1}$.)

Recent (last few decades) interest comes from string theory ⇝ very rich algebraic geometry: $\underbrace{\text{curve counting, enumerative geometry}}_{\text{"A-side"}}$, $\underbrace{\text{derived categories}}_{\text{"B-side"}}$

$$\text{"A-side"} \xleftrightarrow[\text{"mirror symmetry"}]{} \text{"B-side"}$$

Arithmetic of CY's?

Possible Answer from String theory: Attractor mechanism

Defn (Ferrara-Kallosh, Brunner-Roggenkamp) X is an attractor variety if $\exists\, \gamma \in H^d(X, \mathbb{Q})$ s.t.

$$(\ast) \qquad \gamma \perp H^{d-1,1}.$$ [Attractor Condition]

Rmk. heuristically, for fixed γ,

$$\begin{aligned} \text{dim of this locus} &= \#\text{ moduli of } X - \#\text{ conditions imposed by } \ast \\ &= h^{d-1,1} - h^{d-1,1} \\ &= 0, \end{aligned}$$

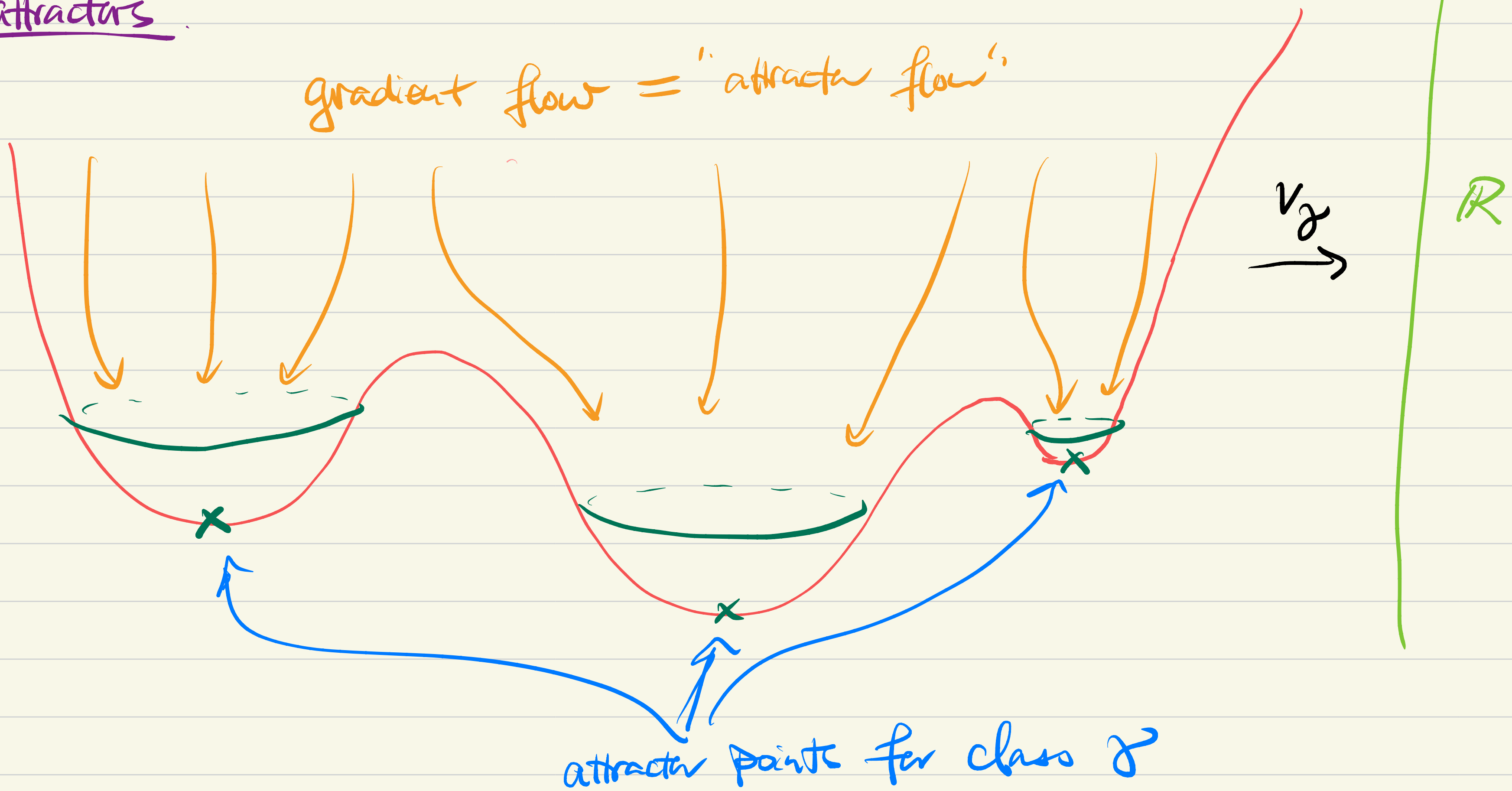
so expect points!

These are therefore sometimes called "attractor points", when we refer to the point corresp. to X in its moduli space.

Attractor Conjecture (Moore 98): if X is an attractor, then it is defined$/\overline{\mathbb{Q}}$. (has model $/ \overline{\mathbb{Q}}$).

Moore verified his conjecture in several cases: Abelian 3-folds, K3 × elliptic curve, their finite quotients, ...

Alternative Description of attractors



$$V_\gamma := \frac{\left|\int_\gamma \Omega\right|^2}{\int_X \Omega\wedge\bar{\Omega}}$$

for Ω an element of $H^0(\Omega_X)$

Note. Attractor points can equally well be defined for variations of Hodge structures (VHS):

$\gamma \perp H^{d-1,1}$ only a statement on Hodge structure.

Where do these come from?

Gross constructed "canonical" Calabi-Yau variations of Hodge structure (CYVHS) on many Shimura varieties. (including $O(2,n)$, ..., E_7)

Thm A (L) The attractor conjecture holds for Gross' CYVHS; more precisely, the attractor points are CM points on the Shimura variety.

Aside. variations of $\mathbb{Q}$-HS: over base M, consists of local system $\mathbb{H}$ of $\mathbb{Q}$-vector spaces + splitting

$$\mathbb{H} \otimes \mathbb{C} \simeq H^{d,0} \oplus H^{d-1,1} \oplus \cdots \qquad (+\text{"Griffiths transversality"})$$

For Shimura varieties attached to group G, $\mathbb{H}$ corresponds to a representation of G.

Example. $G = SL_2$, $H = \mathrm{Sym}^3\,\mathrm{Std} = \{\text{binary cubic forms } /\mathbb{Q}\}$

Thm A promises a map

$$\begin{tikzcd}
\{\gamma\} \arrow[rr] \arrow[d, equal] & & \{\text{CM points on } \mathcal{H}, \text{ upper half plane}\} \arrow[d, phantom, "\cup|"] \\
\{\text{binary cubic forms}\} \arrow[r, "\text{`` Eisenstein form''}"'] & \{\text{binary quadratic forms}\} \arrow[r] & \{\tau = \text{root of binary quadratic form}\}
\end{tikzcd}$$

For experts, these prehomogeneous vector spaces also appear in quaternionic modular forms; in particular, they index Fourier modes in theory of Fourier expansion in recent spectacular work of A. Pollack.

Case above, binary cubic forms index Fourier mode for split G_2.

Rmk. in particular get parametrization of (certain) CM points on E_7- Shimura variety.

Rmk. recovers all previous known cases of attractor conjecture.

Thm B. (w/ A. Tripathy) Assuming a standard conjecture, for $d=1,3,5,9$ $\exists$ counterexamples to the Attractor Conjecture. For the examples we consider, in dim $d=1,3,5,9$ cases moduli spaces are Shimura varieties, attractor points are CM points.

Sketch of proof. We consider the Dolgachev Calabi-Yau's, which exist in all odd dimensions.

Important feature. For such a CY, $H^d(X,\mathbb{Q})$ comes from $H^1(C,\mathbb{Q})$ for curves C, and X and C determine each other.

Essentially, $H^d(X,\mathbb{Q}) \cong \bigwedge^d_{\mathcal{A}} H^1(C,\mathbb{Q})$ as $\mathbb{Q}$-HS, where $H^1(C,\mathbb{Q})$ has an $\mathcal{A}$-action.

[analogue: $H^g(\mathrm{Jac},\mathbb{Q}) \cong \bigwedge^g H^1(C,\mathbb{Q})$.]

Upshot. $$P: \underbrace{\mathcal{M}}_{\text{moduli of Dolgachev CY's}} \hookrightarrow \underbrace{Sh}_{\text{some unitary Shimura variety.}}$$

Need some transcendence result!

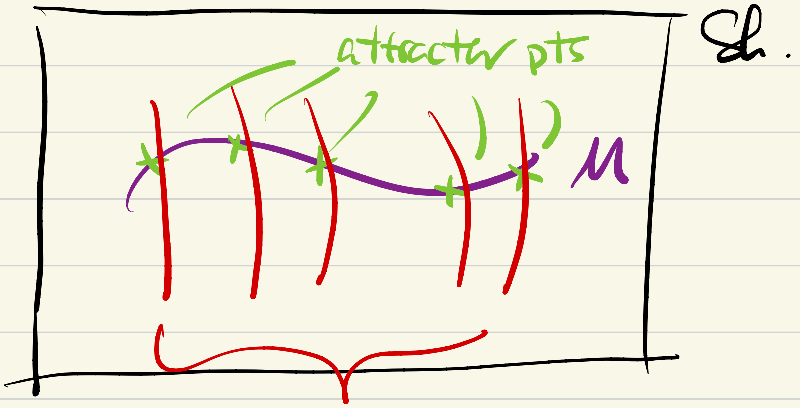
Lemma. Assuming the Attractor Conj, attractor points $x \in \mathcal{M}$ satisfy

$$P(x) \sim A_1 \times A_2$$

"isogenous to"

cm by $\mathbb{Q}(\zeta)$ for ζ a primitive n^{th} root of 1 $(d = 2n-3)$

Pictorially.



Sub-Shimura varieties, cut out by the splitting condition $P(x) \sim A_1 \times A_2$

Aside on Shimura varieties. Sh determined by group G.

Comes with a beautiful collection of subvarieties — Shimura subvarieties.

E.g. CM points, $\left\{\begin{array}{c}\text{Abelian surfaces w/}\\ \text{real multiplication}\end{array}\right\} \subset \left\{\begin{array}{c}\text{all Abelian}\\ \text{surfaces}\end{array}\right\} \cdots$

These may intersect where you may not expect them to, e.g.

Sh' has dense set of CM points, but naively would
$\cap|$
Sh. not expect discrete pts to intersect proper subvarieties.

Zilber-Pink Conj is the converse:

If $V \subseteq \mathrm{Sh}$ proper, and

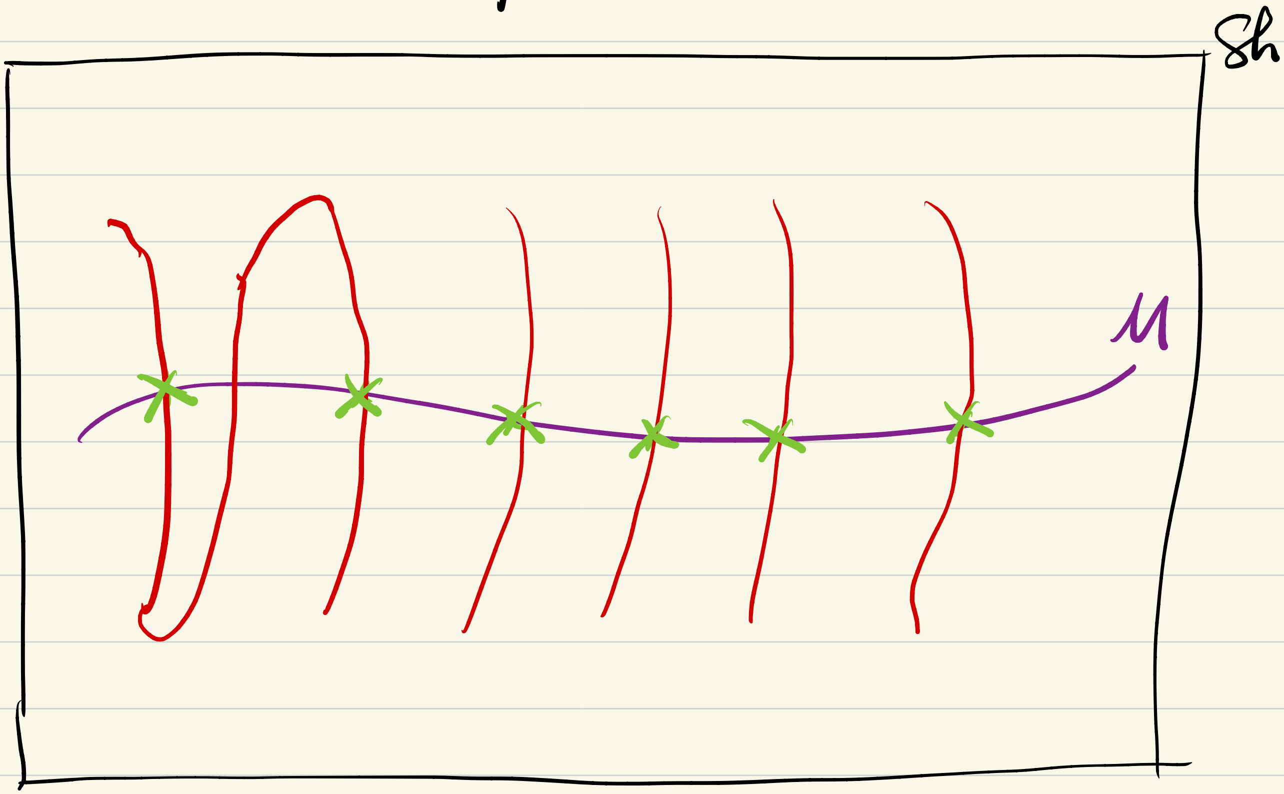
$$\bigcup_{\mathrm{Sh}' \in \Sigma} V \cap \mathrm{Sh}' \text{ dense in } V,$$

then V contained in a proper Shimura subvariety of Sh.

Σ: a set of Shimura subvarieties with codim $>$ dim V.

unlikely intersection condition.

Back to proof. We already have $\mathcal{M}$ intersects proper Sh' at attractor pts.



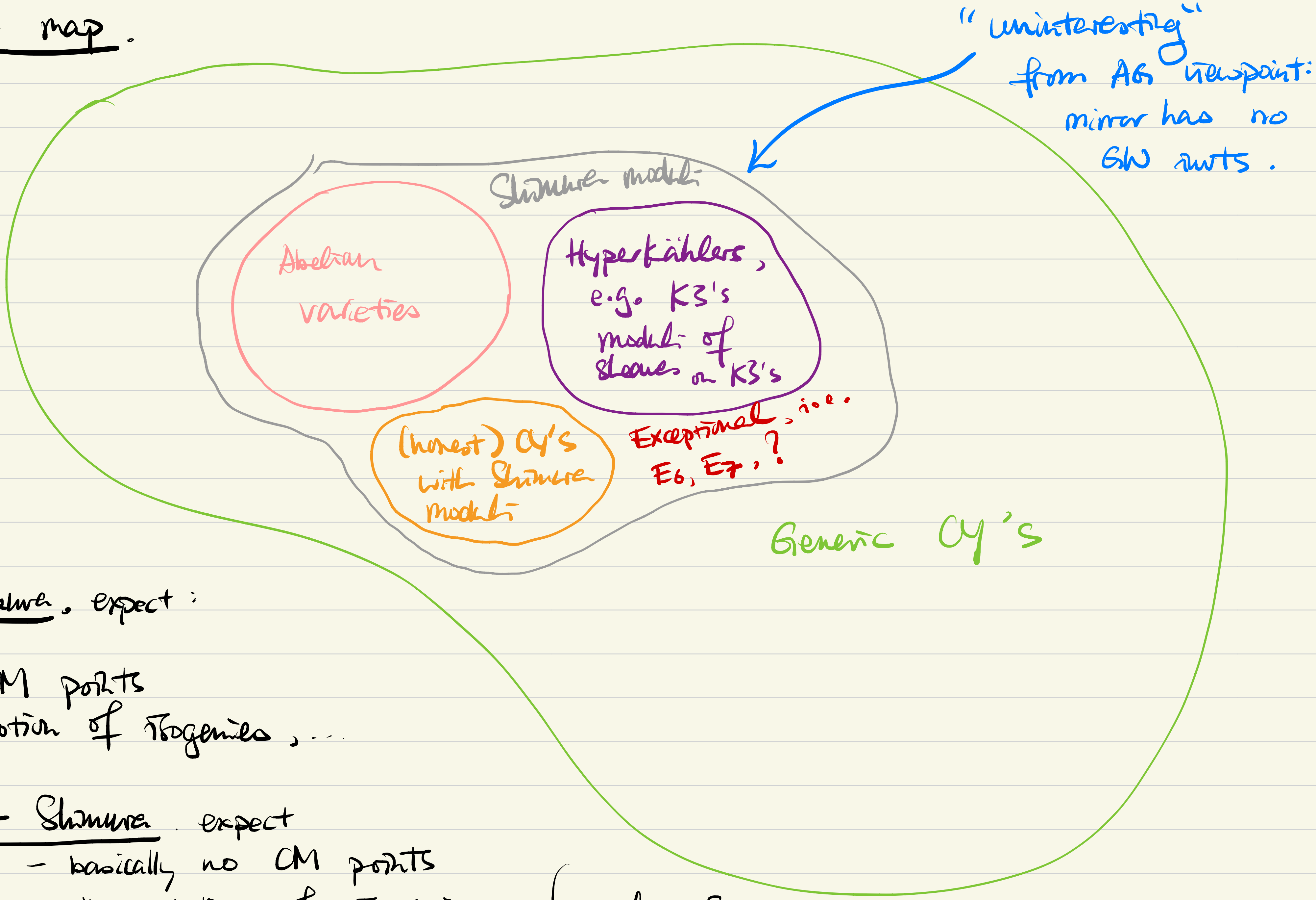
The following finishes the proof

Prop. (i) Attractor points are dense in $\mathcal{M}$;

(ii) the intersections between $\mathcal{M}$ and Sh' are unlikely;

(iii) $\mathcal{M}$ not contained in proper Shimura subvariety
"Hodge generic".

A map.



Shimura. expect:

- CM points
- notion of isogenies, ...

Not Shimura. expect

- basically no CM points
- no notion of isogenies

(analogues of Coleman conj.)

Thank you for your attention, and
to Yiannis and Dave for organizing
this event!