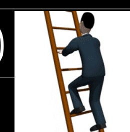
Control Theorems for Fine Selmer Groups 

(joint with Meng Fai Lim)

§ Introduction: We study arithmetic objects in infinite towers of number fields

objects: Class gps, Selmer groups, fine selmer gps

$$\mathbb{Q} = \mathbb{Q}_0 \subsetneq \mathbb{Q}_1 \subsetneq \dots \subsetneq \mathbb{Q}_n \subsetneq \dots \subsetneq \bigcup_n \mathbb{Q}_n = \mathbb{Q}_{cyc} \subsetneq \mathbb{Q}(\mu_{p^\infty})$$



$Gal(\mathbb{Q}_n/\mathbb{Q}) \simeq \mathbb{Z}/p^n$, $\mathbb{Q}_n \subsetneq \mathbb{Q}(\mu_{p^{n+1}})$

$\underset{\overset{\|}{\Gamma}}{Gal(\mathbb{Q}_{cyc}/\mathbb{Q})} = \varprojlim \mathbb{Z}/p^n \simeq \mathbb{Z}_p$

$F = nf$, you can always construct a cyclotomic $\mathbb{Z}_p$-extn

Iwasawa Algebra

$G =$ any pro-p p-adic analytic gp $(G = \Gamma \simeq \mathbb{Z}_p)$

$\Lambda(G) = \mathbb{Z}_p[[G]] = \varprojlim_{H \trianglelefteq_o G} \mathbb{Z}_p[G/H]$

$G = \Gamma$, $\mathbb{Z}_p[[\Gamma]] \xrightarrow{\sim} \mathbb{Z}_p[[T]]$ $(\gamma \mapsto 1+T)$

Structure Theorem: $M =$ fin gen $\Lambda(\Gamma)$-module

$$0 \to \text{fin} \to M \to \Lambda^r \oplus \bigoplus_{i=1}^{s} \Lambda/p^{m_i} \oplus \bigoplus_{j=1}^{t} \Lambda/f_j^{n_j} \to \text{fin} \to 0$$

$f_j =$ irreducible distinguished polynomials

$r = 0$; $\mu = \sum_i m_i$; $\lambda = \sum_j (\deg f_j) \cdot n_j$

Thm (Iwasawa) $p^{e_n} \| \underbrace{h(F_n)}_{\text{class number}}$, $e_n = \mu p^n + \lambda n + \nu$

$n \gg 0$, $\mu, \lambda \in \mathbb{Z}_{\geq 0}$, $\nu \in \mathbb{Z}$

$M = Gal(L_\infty/F_\infty)$, $L_\infty =$ max Ab unram p-extn$/F_\infty$

Conj (Iwasawa) $\mu_{cyc} = 0$ [known for $F/\mathbb{Q}$ = Abelian by Ferrero–Washington 1979]

Mazur (Selmer groups of Abelian Varieties: today we'll just talk about elliptic curves)

$\Gamma \simeq \mathbb{Z}_p$, $p \neq 2$

measures local global failure

$$0 \to \mathrm{Sel}(E[p^m]/F) \to H^1(G_S(F), E[p^m]) \to \bigoplus_{v \in S} H^1(F_v, E)[p^m]$$

$S = S_p \cup S_{bad} \cup S_\infty$

$G_S(F) = \mathrm{Gal}(F_S/F)$, F_S = max unram outside S extn of F

$$\mathrm{Sel}(E/F) = \mathrm{Sel}(E[p^\infty]/F) = \varinjlim_n \mathrm{Sel}(E[p^n]/F)$$

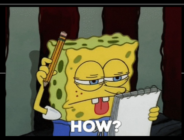
$\mathrm{Sel}(E/F_\infty) = \varinjlim_n \mathrm{Sel}(E/F_n)$; $\mathrm{Sel}^\vee(E/F_\infty)$ = $\Lambda(\Gamma)$-modules, fin gen

(Conj) p = good ord redn then $r = 0$; $\mu_{cyc} > 0$ possible

(Coates-Sujatha) 2005 $\quad 0 \to \underbrace{\mathrm{Sel}_0(E/F)}_{\text{fine Selmer gp}} \to \mathrm{Sel}(E/F) \to \bigoplus_{v \mid p} H^1(F_v, E[p^\infty])$

$$\mathrm{Sel}_0(E/F_\infty) = \varinjlim \mathrm{Sel}(E/F_n)$$

(Conjecture A) $\mathrm{Sel}_0^\vee(E/F_{cyc})$ is fin gen torsion $\Lambda(\Gamma)$-module and $\mu_{cyc} = 0$.



Philosophy: (Coates-Sujatha / Lim-Murty)

(*) Classical Iwasawa $\mu = 0$ conj is equivalent to Conj A for fine Selmer gps.

§ Control Theorems

$$s_{F_\infty/F'} : \mathrm{Sel}(E/F') \longrightarrow \mathrm{Sel}(E/F_\infty)^{\mathrm{Gal}(F_\infty/F')}$$

(Mazur) 1972 for $\mathbb{Z}_p$ - extensions

ker and coker of $s_{F_\infty/F'}$ is fin and bounded independent of n.

(Greenberg 2003) for a large class of p-adic Lie extns arising 'naturally' in number theory / arithmetic geometry

ker and cok are fin ; often possible to show that ker is bounded ;
but cok is not bounded.

Why is the control thm important / useful / interesting

* If you assume fin of Ш, CT predicts growth of Ш in towers !

Analogous question for fine Selmer groups

$$r_{\mathcal{L}/F'} : \mathrm{Sel}_0(E/F') \longrightarrow \mathrm{Sel}_0(E/\mathcal{L})^{\mathrm{Gal}(\mathcal{L}/F')}$$

$\mathbb{Z}_p^d$ $(d \geq 1)$

(Rubin) multi-$\mathbb{Z}_p$-extn where primes decompose finitely

$\mathcal{L}/F$ = p-adic Lie extn

fin ker & coker (not nec bdd)

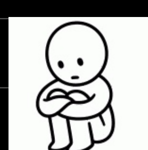
(Wuthrich 2004) all $\mathbb{Z}_p$-extns

ker & coker are fin & bounded

Our Results : Overview

* all p-adic extns, no hypothesis on the redn type of p, whenever we can show finiteness, we give precise growth estimates!
* multi $\mathbb{Z}_p$ extns
* Kummer extns
* trivializing extns.

$\Big\}$ say more

Unfortunately :(

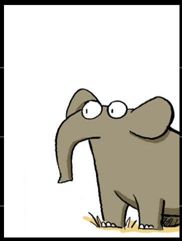
Can not say anything about growth of the fine analogue of Ш.

Hope: Control Thms can open a new line of investigation for studying growth questions.

Standard technique : $G_n = \mathrm{Gal}(F_\infty/F_n)$; $\# \mathrm{Gal}(F_n/F) = p^{dn}$

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Sel}_0(E/F_n) \arrow[r] \arrow[d, "r_n"'] & H^1(G_S(F_n), E[p^\infty]) \arrow[r] \arrow[d, "g_n"'] & \bigoplus_{v \in S} H^1(F_{n,v}, E[p^\infty]) \arrow[d, "\ell_n"'] \\
0 \arrow[r] & \mathrm{Sel}_0(E/F_\infty)^{G_n} \arrow[r] & H^1(G_S(F_\infty), E[p^\infty])^{G_n} \arrow[r] & \left(\bigoplus_{v \in S} H^1(F_{\infty,v}, E[p^\infty])\right)^{G_n}
\end{tikzcd}$$

* We have not been able to show that our growth estimates on ker & cok are sharp but they are "good enough"



Thm : F_∞/F = d-dim (uniform) p-adic Lie group

$$r_n : \mathrm{Sel}_0(E/F_n) \to \mathrm{Sel}_0(E/F_\infty)^{G_n}$$

has kernel and cokernel that are co-fin gen

$$\left.\begin{array}{l} \mathrm{corank}_{\mathbb{Z}_p}(\ker r_n) = O(1) \\ \mathrm{corank}_{\mathbb{Z}_p}(\mathrm{coker}\, r_n) = O(p^{(d-1)n}) \end{array}\right\}$$

Thm : (1) $\mathbb{Z}_p^d$ - extn : kernel and cokernel are fin
$d \geq 2$

$$\left.\begin{array}{l} \mathrm{ord}_p |\ker r_n| = O(n)^{*} \\ \mathrm{ord}_p |\mathrm{cok}\, r_n| = O(p^{(d-1)n}) \end{array}\right\}$$

If $F_\infty \supseteq F_{cyc}$ and Conj A holds for $\mathrm{Sel}_0(E/F_{cyc})$

$$\mathrm{ord}_p |\mathrm{Sel}_0(E/F_n)^\vee [p^\infty]| = O(p^{(d-1)n}) \leftarrow$$

* can do better when p = good redn :) 

(2) Trivializing extens, $F_\infty = F(E[p^\infty]) \supseteq F_{cyc}$

- E be an elliptic curve with CM

$\mathrm{Gal}(F(E[p^\infty])/F) \simeq \mathbb{Z}_p^2$ (upto fin base change)

$$\mathrm{ord}_p |\ker r_n| = O(n)^{*}$$
$$\mathrm{ord}_p |\mathrm{coker}\, r_n| = O(n)$$

If Conj A holds*, $\mathrm{ord}_p(\mathrm{Sel}_0^\vee(E/F_n)[p^\infty]) = O(np^n)$

- E be an elliptic curve w/o CM

$\mathrm{Gal}(F_\infty/F)$ is of dim 4 $\subseteq GL_2(\mathbb{Z}_p)$

$$\mathrm{ord}_p |\ker r_n| = O(n)^{*}$$
$$\mathrm{ord}_p |\mathrm{coker}\, r_n| = O(np^{2n})$$

If Conj A holds,* $\mathrm{ord}_p(\mathrm{Sel}_0^\vee(E/F_n)[p^\infty]) = O(np^{3n})$

(3) $\mathbb{Z}_p \rtimes \mathbb{Z}_p$ extns (Kummer extns) where we can get bounds as well ... we can chat about it separately if you're keen.

Rk. There are pseudo-nullity conjectures (Conj B) which suggest the fine Selmer group is "very small" in an infinite tower of $d \geq 2$. If you further assume those (some evidence), one can get better est.

* Special cases where you can get rid of this hypothesis :)

* no special cases where we can do away with this hypothesis :(