Anticyclotomic Euler systems for conjugate self-dual representations of $GL(2n)$

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Bloch–Kato conjecture

Let

- K be a number field
- $\rho\colon \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_d(\overline{\mathbb{Q}}_p)$ a geometric Galois representation (i.e. unramified almost everywhere and de Rham at all p-adic places)

$\Rightarrow$ L-function $L(\rho, s)$ and (conjectural) functional equation

$$\Lambda(\rho^*(1), -s) = \varepsilon(\rho, s)\Lambda(\rho, s)$$

Conjecture (Bloch–Kato)

$$\mathrm{ord}_{s=0} L(\rho^*(1), s) = \dim \mathrm{H}^1_f(K, \rho) - \dim \mathrm{H}^0(K, \rho)$$

Bloch–Kato conjecture

Suppose

- K is imaginary quadratic (or CM)
- $\rho^c \cong \rho^*(1)$ (i.e. ρ is polarised)
- $s = 0$ is critical and $\mathrm{H}^0(K, \rho) = 0$

Then zero is the centre of the functional equation and $\varepsilon := \varepsilon(\rho, 0) = \pm 1$.

Conjecture

1. *If $\varepsilon = +1$ and $L(\rho, 0) \neq 0$, then*

$$\mathrm{H}^1_f(K, \rho) = 0.$$

2. *If $\varepsilon = -1$ and $L'(\rho, 0) \neq 0$, then*

$$\dim \mathrm{H}^1_f(K, \rho) = 1.$$

Elliptic curves I

- p is an odd prime ≥ 5
- $E/\mathbb{Q}$ elliptic curve, good reduction at p, non-CM, square-free conductor
- $\rho\colon \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}(V_p(E))$ which satisfies $\rho^c \cong \rho^*(1) \cong \rho$

Assume $\varepsilon = +1$.

Theorem (Waldspurger, ...)

There exists a definite quaternion algebra D, cusp form $f : D_{\mathbb{A}}^{\times} \to \mathbb{C}$ and embedding $K \hookrightarrow D$ such that:

$$\left| \int_{[\mathbb{A}_K^{\times}]} f(x) dx \right|^2 \approx L(\rho, 0) = L(E/K, 1)$$

$\Rightarrow$ if $L(E/K, 1) \neq 0$ can construct ramified classes in $\mathrm{H}^1(K, \rho)$ which force $\mathrm{H}^1_f(K, \rho) = 0$

Elliptic curves II

Assume $\varepsilon = -1$. There exists

- Indefinite quaternion algebra D, with embedding $K \hookrightarrow D$
- morphism $Y(D) \to E$

$\Rightarrow$ Heegner points $P_m \in E(K[m])$ related under $\mathrm{tr}_{K[m]}^{K[\ell m]} \colon E(K[\ell m]) \to E(K[m])$

Theorem (Gross–Zagier, ...)

Let $P := \mathrm{tr}_{K}^{K[1]} P_1$. *Then*

$$\mathrm{height}(P) \approx L'(E/K, 1)$$

$\Rightarrow$ Euler system argument of Kolyvagin, ... implies

$$\mathrm{H}^1_f(K, \rho) = \mathbb{Q}_p \cdot \kappa(P)$$

when $L'(E/K, 1) \neq 0$

A possible generalisation

Idea: Replace

- The embedding $\mathrm{Res}_{K/\mathbb{Q}} \mathrm{GL}_1 \hookrightarrow \mathrm{GL}_2$ with the embedding

$$\mathrm{Res}_{K/\mathbb{Q}} \mathrm{GL}_n \hookrightarrow \mathrm{GL}_{2n}$$

- The elliptic curve with cuspidal automorphic representation Π of $\mathrm{GL}_{2n}(\mathbb{A}_K)$ satisfying $\Pi^c \cong \Pi^\vee \cong \Pi$.

Then set $\rho = \rho_\Pi(n)$, where ρ_Π is the Galois representation associated with Π.

The case $\varepsilon = +1$

Assume $\varepsilon = +1$.

Conjecture

There exists a pair of unitary groups $U(n) \times U(n) \subset U(2n)$ such that Π descends to an automorphic representation π of $U(2n)$ and

$$\underset{\text{for some } \phi \in \pi}{\int_{[U(n)(\mathbb{A})\times U(n)(\mathbb{A})]} \phi(x)dx \neq 0} \quad \Leftrightarrow \quad L(\rho, 0) = L(\Pi, 1/2) \neq 0$$

Evidence:

- If $U(2n)$ is quasi-split then Pollack–Wan–Zydor prove the $\Rightarrow$ implication
- Also related to work of Wei Zhang, Spencer Leslie on comparison of RTFs

One expects to produce ramified classes in $\mathrm{H}^1(K, \rho)$ similar to the work of Liu–Tian–Xiao–Zhang–Zhu $\Rightarrow \mathrm{H}^1_f = 0$

The case $\varepsilon = -1$

Assume $\varepsilon = -1$. Consider the embedding

$$H = U(1, n-1) \times U(0, n) \hookrightarrow U(1, 2n-1) = G$$

Then under reasonable conditions:

- Π descends to an automorphic representation π of G
- ρ appears in the étale cohomology of the Shimura variety $Y(G)$

Theorem (G.–Shah)

For every m divisible by only primes that split in K, and every Galois stable lattice $T \subset \rho$, there exist classes $c_m \in \mathrm{H}^1(K[m], T)$ that are related under corestriction.

Forthcoming work of Jetchev–Nekovář–Skinner

$$\Rightarrow \quad \begin{array}{c} \text{if } c := \mathrm{cores}_K^{K[1]} c_1 \text{ is non-torsion (+ big image assumption)} \\ \text{then } \mathrm{H}^1_f(K, \rho) = \overline{\mathbb{Q}}_p \cdot c \end{array}$$

The construction – trivial coefficients

- $Y(G)$ = (pro-)Shimura variety associated with G, of dimension $2n-1$
- $\mathrm{H}^i_{\mathrm{et}}(Y(G)) := \mathrm{H}^i_{\mathrm{et}}(Y(G)_{\overline{K}}, \overline{\mathbb{Q}}_p(n))$ étale cohomology
- $\mathcal{Z} \in \mathrm{CH}^n(Y(G)_{\overline{K}})$ cohomologically trivial codimensional n cycle, obtained from (components of) Shimura varieties for H

Different perspective:

$$\pi_f = \left\{ \begin{array}{c} \text{Galois equivariant maps} \\ \mathrm{H}^{2n-1}_{\mathrm{et}}(Y(G)) \to \rho \end{array} \right\}$$

(assuming π_f appears with multiplicity one)

The construction – trivial coefficients

Definition

Define classes

$$c(\phi) := \phi\left(\mathrm{AJ}(\mathcal{Z})\right) \in \varinjlim_{\overline{K} \supset L \supset K} \mathrm{H}^1(L, \rho)$$

for $\phi \in \pi_f$.

- Then $c_m := c(\phi_m)$ for suitable choices $\phi_m \in \pi_f$.
- To prove the Euler system relations, we need to compare $c(\phi_{\ell m})$ and $c(\phi_m)$, where $\phi_{\ell m}$ and ϕ_m differ only at the prime ℓ.
- By working locally at ℓ, and passing to the χ-part, one has a linear map

  $$c(-)^{\chi} \colon \pi_\ell \to \mathbb{C}(\chi)$$

  which is $H(\mathbb{Q}_\ell)$-equivariant.

Relation to spherical varieties

Locally at ℓ:

- $G(\mathbb{Q}_\ell) = \mathrm{GL}_{2n}(\mathbb{Q}_\ell)$
- $H(\mathbb{Q}_\ell) = \mathrm{GL}_n(\mathbb{Q}_\ell) \times \mathrm{GL}_n(\mathbb{Q}_\ell)$

$\Rightarrow$ spherical pair of reductive groups

Proposition

1. $\mathrm{Hom}_H(\pi_\ell, \mathbb{C}(\chi))$ *is at most one-dimensional*
2. *If non-zero, then* π_ℓ *admits a Shalika model and we have an explicit basis*
$$Z_\chi \in \mathrm{Hom}_H(\pi_\ell, \mathbb{C}(\chi))$$
obtained from a zeta integral.

$\Rightarrow$ Can prove Euler system relations explicitly using this zeta integral and a combination of U_ℓ Hecke operators.

Some remarks

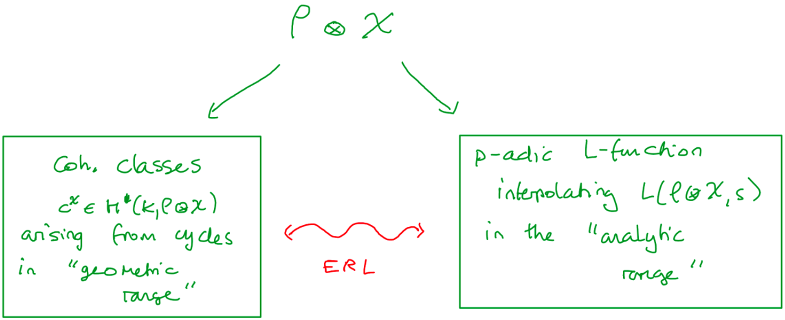
1. We actually use unitary similitude groups
2. $\mathscr{Z}$ is only cohomologically trivial after passing to ordinary parts for a Siegel U_p-operator
3. We have Euler system relations in the "p-direction", i.e. classes $c_{p^r} \in \mathrm{H}^1(K[p^r], T)$ for $r \geq 1$ which are compatible under corestriction.

   The relations in this case are related to the work of Loeffler on norm relations in Iwasawa theory via spherical varieties

Future work

Goal: Relate the non-vanishing of $c \in \mathrm{H}^1_f(K, \rho)$ to the L-function/p-adic L-function for ρ

Consequence: Obtain cases of the Bloch–Kato conjecture for ρ



Anticyclotomic characters

Definition

Let Σ^{ac} denote the set of all *anticyclotomic* characters, i.e. all Hecke characters $\chi\colon \mathbb{A}_K^\times \to \mathbb{C}^\times$ with infinity type $(j, -j)$.

Recall:

- Π is a regular algebraic cuspidal automorphic representation of $\mathrm{GL}_{2n}(\mathbb{A}_K)$ satisfying $\Pi^c \cong \Pi^\vee \cong \Pi$
- Π corresponds to an algebraic representation $V \boxtimes V^*$ of $\mathrm{GL}_{2n} \times \mathrm{GL}_{2n}$, where highest weight of V is

$$\mu = (a_1, \dots, a_n, -a_n, \dots, -a_1)$$

Idea: Study the behaviour of $L(\rho \otimes \chi, s)$ and V in different subsets of Σ^{ac}

Geometric region

Definition

$$\Sigma^{\mathrm{geom}} = \{\chi \in \Sigma^{\mathrm{ac}} : |j| \leq a_n\}$$

For $\chi \in \Sigma^{\mathrm{geom}}$, the sign of $L(\rho \otimes \chi, s)$ is -1 and there exist Euler system classes

$$c^{\chi} \in \mathrm{H}^1_f(K, \rho \otimes \chi)$$

e.g. $c^{\mathrm{triv}} = c$ as before

- Follows from the fact one has a $\mathrm{GL}_n \times \mathrm{GL}_n$-equivariant embedding

$$\det{}^j \boxtimes \det{}^{-j} \hookrightarrow V$$

- Construct classes by passing to the map on local systems

Geometric region: expectation

Expectation: One can p-adically interpolate $\{c^\chi : \chi \in \Sigma^{\mathrm{geom}}\}$ to obtain classes

$$c^\chi \in \mathrm{H}^1(K, \rho \otimes \chi) \qquad \chi \in \Sigma^{\mathrm{ac}}$$

Idea: p-adically interpolate the branching law $\det^j \boxtimes \det^{-j} \hookrightarrow V$

- Follows a similar strategy to Jetchev–Loeffler–Zerbes on Heegner points in Coleman families
- Also similar to work of Bertolini–Seveso–Venerucci on Reciprocity Laws for Balanced Diagonal Classes

Analytic region

Definition

$$\Sigma^{\mathrm{an}} = \{\chi \in \Sigma^{\mathrm{ac}} : a_n + 1 \leq |j| \leq \max(a_n + 1, a_{n-1})\}$$

For $\chi \in \Sigma^{\mathrm{an}}$ the sign of $L(\rho \otimes \chi, s)$ is $+1$ and we expect

$$\exists \phi \in \pi \text{ s.t. } \int_{[H]} \phi(x)\chi(x)dx \neq 0 \qquad \Leftrightarrow \qquad L(\rho \otimes \chi, 0) \neq 0$$

Analytic region: expectation

Expectation: There exists a p-adic L-function $\mathscr{L} \colon \Sigma^{\mathrm{ac}} \to \mathbb{C}_p$ such that

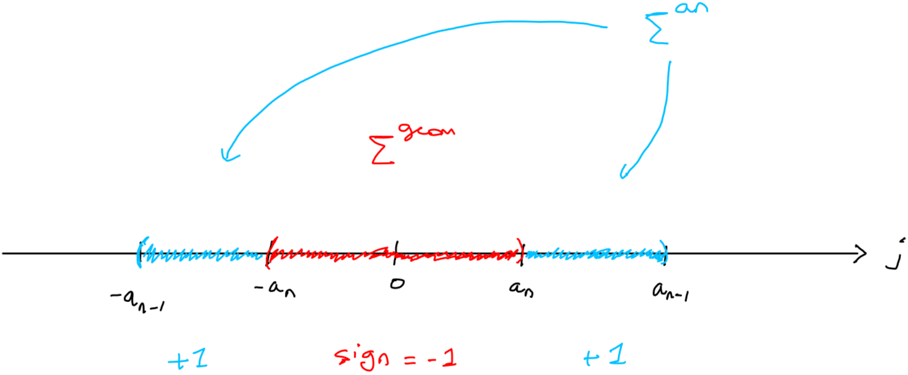
$$\mathscr{L}(\chi) \approx \int_{[H]} \phi(x)\chi(x)dx \qquad\qquad \chi \in \Sigma^{\mathrm{an}}$$

for suitable $\phi \in \pi$

To construct this:

1. Express automorphic period as a pairing in coherent cohomology (Su, 2019). The pairing will be in degrees $n-1, n$
2. p-adically interpolate this pairing using Higher Hida/Coleman theory of Boxer–Pilloni

The two regions



Explicit reciprocity law

Following the strategy of Loeffler–Zerbes (On the Bloch–Kato conjecture for GSp(4)) we expect

Expectation: For $\chi \in \Sigma^{\mathrm{ac}}$ one has

$$\mathscr{L}(\chi) \approx \langle \log_{\mathrm{BK}}(c^{\chi}), \eta_{\phi} \rangle \qquad \text{some } \eta_{\phi} \in \mathrm{D}_{\mathrm{dR}}(\rho)$$

In particular

$$\mathscr{L}(1) \neq 0 \quad \Rightarrow \quad c \neq 0 \quad \Rightarrow \quad \mathrm{H}^1_f(K, \rho) = \overline{\mathbb{Q}}_p \cdot c$$

Remark

Also expect applications to the anticyclotomic main conjecture for ρ.