# Bar-Cobar and Koszul duality for algebraic Operads

Naruki Masuda

JHU Category Seminar, December 6, 2018 & February 26, 2019

In this note, I will describe how to relax algebraic structures into homotopical ones, using homological algebra of operads. The main reference is [1].

## 1 Three definitions of operads

In this section $(\mathcal{M}, \otimes, I)$ is a cocomplete symmetric closed monoidal category. Examples include $(\mathbf{Set}, \times, *)$, $(\mathbf{Top}, \times, *)$, $(\mathbf{Mod}_{\mathbb{K}}, \otimes, \mathbb{K})$ for a commutative ring $\mathbb{K}$, $(\mathbf{grMod}_{\mathbb{K}}, \otimes, \mathbb{K})$, $(\mathbf{dgMod}_{\mathbb{K}}, \otimes, \mathbb{K})$, $(\mathbf{Top}_*, \wedge, S^0)$, $(\mathbf{Spec}, \wedge, \mathbb{S})$. The base ring $\mathbb{K}$ is usually fixed and omitted from the notation. See Remark 3.1.1 for more on these closed monoidal structures.

### 1.1 The first definition

Let $\Sigma$ be a category whose objects are finite sets $\{1, 2, \dots, n\}$ for each $n \geq 1$ and morphisms are bijections between them. Therefore the automorphism group $\mathrm{Aut}_\Sigma(n) = \Sigma_n$ is a symmetric group and $\Sigma$ is, as a groupoid, the direct sum of all of these. $\Sigma$ is sometimes called the *symmetric groupoid*.

**Definition 1.1.1.** A (reduced right) $\Sigma$-*module* [1] is a functor $M : \Sigma^{\mathrm{op}} \to \mathcal{M}$ i.e., a sequence of objects $M(n) \in \mathcal{M}$, $n \geq 1$, together with a right action of $\Sigma_n$ on each $M(n)$. For an element $\mu \in M(n)$, $n$ is called an *arity* of $\mu$. A morphism of $\Sigma$-module is a natural transformation. We denote the category of $\Sigma$-module in $\mathcal{M}$ by $\mathcal{M}^{\Sigma^{\mathrm{op}}}$.

Each term $M(n)$ of $\Sigma$-module should be considered as a space of $n$-ary operations, where the right $\Sigma_n$-action encodes permutation of inputs. Usually, objects in $\mathcal{M}$ are identified with $\Sigma$-modules concentrated in arity 1. The adjective "reduced" means that we do not consider 0-ary operations, i.e. constants such as units, and will be omitted in the rest of this note. We need to refine the theory to treat the bar-cobar duality for operads which governs algebraic structure with constants, see Theorem 6.3.4 for instance.

**Definition 1.1.2.** An *operad* in $\mathcal{M}$ is a triple $(\mathcal{P}, \gamma, \eta)$ such that

- $\mathcal{P} \in \mathcal{M}^{\Sigma^{\mathrm{op}}}$,
- $\gamma(n; k_1, \dots, k_n) : \mathcal{P}(n) \otimes_{\Sigma_n} (\mathcal{P}(k_1) \otimes \cdots \otimes \mathcal{P}(k_n)) \to \mathcal{P}(k_1 + \cdots k_n)$ for $n, k_1, \dots, k_n \geq 1$, called the *composition maps* which are $\Sigma_{k_1} \times \cdots \times \Sigma_{k_n} (\subset \Sigma_{k_1+\cdots+k_n})$-equivariant,
- $\eta : I \to \mathcal{P}(1)$ (or in concrete categories $\mathrm{id} \in \mathcal{P}(1)$), called the *identity operation.*
- $\gamma$ and $\eta$ are suitably associative and unital.

A morphism of operads is a morphism of $\Sigma$-modules which commutes with $\eta$ and $\gamma$. We denote the category of operads in $\mathcal{M}$ by $\mathbf{Op}(\mathcal{M})$.

This is a generalization of the notion of associative algebra in the following sense: when $\mathcal{P}(n) = 0$ (the initial object) for $n \neq 1$, we can identify $\mathcal{P}$ with an unital associative algebra (or monoid) $\mathcal{P}(1) \in \mathcal{M}$. In particular, the monoidal unit $I$ admits a canonical operad structure. The above definition has a slightly different formulation:

**Proposition 1.1.3.** *Let $(\mathcal{P}, \gamma, \eta)$ be an operad. It is characterized by the identity and the collection of* partial composition *maps $- \circ_i - : \mathcal{P}(m) \otimes \mathcal{P}(n) \to \mathcal{P}(m+n-1)$ for $m, n \geq 1, 1 \leq i \leq m$ given by $\gamma(m; 1, \dots, 1, n, 1, \dots, 1) \circ (\mathrm{id} \otimes \eta \otimes \cdots \otimes \eta \otimes \mathrm{id} \otimes \eta \otimes \cdots \otimes \eta)$ satisfying suitable unitality and associativity.*

[1] This terminology ($\mathbb{S}$-module in [1]) is popular in the algebraic context. In the topological context, it is often called symmetric sequence, since "$\mathbb{S}$-module" usually means the EKMM model of spectra.

**Definition 1.1.4.** In algebraic contexts such as $\mathcal{M} = \mathbf{Mod}, \mathbf{grMod}, \mathbf{dgMod}$, an operad $\mathcal{P}$ is *augmented* when it is equipped with a map of operads $\varepsilon : \mathcal{P} \to I$ called an augmentation. In this case, $\mathcal{P} = \bar{\mathcal{P}} \oplus I$ as $\Sigma$-modules for $\bar{\mathcal{P}} := \ker \varepsilon$. Morphisms of augmented operads are morphisms of operads which respects the augmentations. We denote the category of augmented operads by $\mathbf{aug}\text{-}\mathbf{Op}(\mathcal{M})$.

## 1.2 First examples and algebras over operads

Here I give some of the examples. The prototypical example is the endomorphism operad:

**Example.** For any object $X \in \mathcal{M}$, we define the *endomorphism operad* $\mathrm{End}_X$ by

$$\mathrm{End}_X(n) := \mathcal{M}(X^{\otimes n}, X).$$

The composition maps and identity are given by the actual composition and the identity. Dually the *co-endomorphism operad* $\mathrm{coEnd}_X$ is given by

$$\mathrm{coEnd}_X(n) := \mathcal{M}(X, X^{\otimes n}).$$

Using this (co-)endomorphism operad, we can define the notion of (co)algebras over an operad:

**Definition 1.2.1.** $\mathcal{P}$-algebra (resp. coalgebra) is an object $X \in \mathcal{M}$ together with a map of operads $\mathcal{P} \to \mathrm{End}_X$ (resp. $\mathcal{P} \to \mathrm{coEnd}_X$). Morphisms are those in $\mathcal{M}$ which commute with structure morphisms. We denote the category of $\mathcal{P}$-algebra by $\mathcal{P}\text{-}\mathbf{Alg}$.

A morphism of operads $f : \mathcal{P} \to \mathcal{Q}$ induces a functor $f^* : \mathcal{Q}\text{-}\mathbf{Alg} \to \mathcal{P}\text{-}\mathbf{Alg}$. The structure morphism $f : \mathcal{P} \to \mathrm{End}_X$ can be considered as a "multilinear representation" and allows $\mu \in \mathcal{P}(n)$ to act on $X$ as an actual $n$-ary operation. We denote $f(\mu)(x_1, \ldots, x_n)$ simply by $\mu(x_1, \ldots, x_n)$. When $\mathcal{P}$ is identified with an associative algebra, the notion of $\mathcal{P}$-algebra reduces to the usual representation.

The following Ass and Com are also fundamental:

**Example.** The *associative operad* Ass is given by $\mathrm{Ass}(n) := I \cdot \Sigma_n$. Here the dot means the copower over $\mathbf{Set}$. The category Ass-$\mathbf{Alg}$ is the category of associative algebras without units.

**Example.** The *commutative operad* Com is given by $\mathrm{Com}(n) := I$. The category Com-$\mathbf{Alg}$ is the category of commutative algebra without units.

## 1.3 The second definition: operad as a monoid

"Operad" is a portmanteau word made by combining "operation" and "monad." The following definitions give rise to the monadic viewpoint:

**Definition 1.3.1.** The *Schur functor* associated to a $\Sigma$-module $M = \{M(n)\}$ in $\mathcal{M}$ is the endofunctor $\tilde{M}$ of $\mathcal{M}$ given by

$$\tilde{M}(X) := \int^{n \in \Sigma} M(n) \otimes X^{\otimes n} := \coprod_{n \geq 0} (M(n) \otimes X^{\otimes n})_{\Sigma_n}$$

for each $X \in \mathcal{M}$. This gives a functor $\tilde{(-)} : \mathcal{M}^{\Sigma^{\mathrm{op}}} \to \mathrm{End}(\mathcal{M})$, which is faithful in $\mathbf{Set}$ and algebraic examples.

$\tilde{M}(X)$ should be interpreted as a space of tuples $(f; x_1, \ldots, x_n)$, where $f$ is an $n$-ary operation and $x_1, \ldots, x_n$ are inputs, identified up to equivariance. The $\Sigma$-module $I$ is defined by $I(1) = I$ and $I(n) = 0$ otherwise. The associated Schur functor $\tilde{I}$ is the identity functor.

**Definition 1.3.2.** For two $\Sigma$-modules $M, N$ we define

1. the *sum* $(M \amalg N)(n) := M(n) \amalg N(n)$,
2. the *tensor product* $(M \otimes N)(n) := \int^{i,j \in \Sigma} \Sigma(i+j, n) \otimes M(i) \otimes N(j) = \coprod_{i+j=n} (M(i) \otimes N(j)) \otimes_{\Sigma_i \times \Sigma_j} \Sigma_n$, aka the Day convolution,
3. the *composition product* $M \circ N := \int^{k \in \Sigma} M(k) \otimes N^{\otimes k} = \coprod_{k \geq 0} (M(k) \otimes N^{\otimes k})_{\Sigma_k}$. This can also be written as

$$\begin{aligned} M \circ N(n) &:= \int^{k \in \Sigma} \int^{i_1, \ldots, i_k \in \Sigma} \Sigma(i_1 + \cdots + i_k, n) \otimes M(k) \otimes N(i_1) \otimes \cdots \otimes N(i_k) \\ &\cong \coprod_{k} \coprod_{\substack{i_1 + \cdots + i_k \\ = n}} M(k) \otimes_{\Sigma_k} (N(i_1) \otimes \cdots \otimes N(i_k)) \otimes_{\Sigma_{i_1} \times \cdots \times \Sigma_{i_k}} \Sigma_n. \end{aligned}$$

These terminologies are justified by the following proposition.

**Proposition 1.3.3.** *The Schur functor of the above constructions gives the counterparts in the category of endofunctors, i.e.*

1. $\widetilde{M \amalg N} \cong \tilde{M}(\text{-}) \amalg \tilde{N}(\text{-})$,
2. $\widetilde{M \otimes N} \cong \tilde{M}(\text{-}) \otimes \tilde{N}(\text{-})$,
3. $\widetilde{M \circ N} \cong \tilde{M} \circ \tilde{N}$,

*where* $\amalg$ *and* $\otimes$ *on the right-hand side are taken objectwise. It follows that* $(\mathcal{M}^{\Sigma^{\mathrm{op}}}, \circ, I)$ *forms a monoidal category.*

*Proof.* This is a standard coend calculus. ☐

**Proposition 1.3.4.** *Operad structures on a* $\Sigma$*-module* $\mathcal{P}$ *correspond to monoid structures on* $\mathcal{P}$ *in* $(\mathcal{M}^{\Sigma^{\mathrm{op}}}, \circ, I)$ *(i.e. the monad structure on* $\tilde{\mathcal{P}}$*).*

*Proof.* $(\mu; \nu_1, \dots, \nu_k) \in (\mathcal{P}(k) \otimes \mathcal{P}^{\otimes k})_{\Sigma_k}$ can be considered as a "composable tuple of operations." Therefore a morphism $\mathcal{P} \circ \mathcal{P} \to \mathcal{P}$ of $\Sigma$-modules encodes appropriately equivariant composition maps. The unit map $I \to \mathcal{P}$ corresponds to the identity operation. ☐

**Proposition 1.3.5.** *Let* $\mathcal{P}$ *be an operad.* $\mathcal{P}$*-algebra (resp.* $\mathcal{P}$*-coalgebra) structures are equivalent to the algebra (resp. coalgebra) structures over the monad* $\tilde{\mathcal{P}}$*.*

*Proof.* Under the adjunction $\mathcal{M}(\tilde{\mathcal{P}}(X), X) \cong \prod_{n \geq 1} \mathcal{M}_{\Sigma_n}(\mathcal{P}(n), \mathcal{M}(X^{\otimes n}, X))$, two structures determines each other. ☐

From now on we will omit the notation for the Schur functor.

## 1.4 The third definition: operad as an algebra over the tree monad

We introduce "dendroidal" approach and give the third definition of operads[2]. In this note, we assume that all the trees are rooted, have no external vertices, and therefore every vertex has at least one input. For any nonempty finite set $X$, Let $\mathrm{T}(X)$ be the set of trees with a bijection between its leaves and $X$. We also define $\tilde{\mathrm{T}}(X)$ by the set of nontrivial (i.e. have at least one vertex) trees in $\mathrm{T}(X)$. The set of vertices of $t \in \mathrm{T}(X)$ is denoted by $\mathrm{vert}(t)$. For a vertex $v \in \mathrm{vert}(t)$, let us denote the set of incoming edges (from leaves to the root) by $\mathrm{in}(v)$. The number $|\mathrm{in}(v)|$ of incoming edges is called the arity of the vertex $v$.

**Definition 1.4.1.**

1. For $M \in \mathcal{M}^{\Sigma^{\mathrm{op}}}$ and $t \in \mathrm{T}(n)$, we define the *treewise tensor product* by $M(t) = \bigotimes_{v \in \mathrm{vert}(t)} M(|\mathrm{in}(v)|)$.
2. The *tree functor* $\mathbb{T} : \mathcal{M}^{\Sigma^{\mathrm{op}}} \to \mathcal{M}^{\Sigma^{\mathrm{op}}}$ (resp. the *reduced tree functor* $\tilde{\mathbb{T}}$) is defined by $\mathbb{T}M(n) = \coprod_{t \in \mathrm{T}(n)} M(t)$ (resp. $\tilde{\mathbb{T}}M(n) = \coprod_{t \in \tilde{\mathrm{T}}(n)} M(t)$).
3. The *weight grading* $\mathbb{T}M(n) = \coprod_{w \geq 0} \mathbb{T}^{(w)}M(n)$ is given by $\mathbb{T}^{(w)}M(n) \coloneqq \coprod_{|\mathrm{vert}(t)| = w} M(t)$.

The object $\mathbb{T}M(X)$ can be thought of as a space of trees whose leaves are labeled by $X$ and vertices with arity $n$ are labeled by $n$-ary operations of $M$.

*Remark* 1.4.2. For low-weight components, we have explicit descriptions as follows:

- The weight 0 component $\mathbb{T}^{(0)}M(n)$ is $I$ when $n = 1$ and 0 otherwise. Using this weight grading $\tilde{\mathbb{T}}M$ is precisely the positive-weight part of $\mathbb{T}M$.
- Since there is only one tree with a single vertex and $n$ leaves (called *corolla*) for each $n$, the weight 1 component $\mathbb{T}^{(1)}M$ can be identified with $M$.
- The weight 2 component $\mathbb{T}^{(2)}M$ can be considered as a special case $M \circ_{(1)} M$ of the following definition.

[2]If we go further in this direction, we can characterize (colored) operads among dendroidal sets in a similar way as we characterize categories among simplicial sets, and also can give a definition of $\infty$-operads. See [4]

**Definition 1.4.3.** For $M, N \in \mathcal{M}^{\Sigma^{\mathrm{op}}}$, we define the *infinitesimal (or linearized) composition product* $M \circ_{(1)} N \subset M \circ (I \amalg N) = \int^{k \in \Sigma} M(k) \otimes (I \amalg N)^{\otimes k}$ by the sum of components coming from $I \otimes \cdots \otimes N \otimes \cdots \otimes I \subset (I \amalg N)^{\otimes k}$ (contains exactly one $N$). This product is linear in the sense that $\circ_{(1)}$ is distributive with respect to $\amalg$ on both sides.

**Proposition 1.4.4.** *The functor $\mathbb{T}$ admits a monad structure $(\mathbb{T}, \alpha, \iota)$, where $\alpha : \mathbb{T} \circ \mathbb{T} \to \mathbb{T}$ is given by substitution of trees into vertices, in other words, by considering "a tree of trees" as a single tree, and $\iota : \mathrm{id} \to \mathbb{T}$ is the inclusion into the weight 1 part. We can put a monad structure on $\tilde{\mathbb{T}}$ in a similar way.*

**Proposition 1.4.5.** *Let $M$ be a $\Sigma$-module.*

1. *Operad structure on $M$ bijectively corresponds to algebra structures on $M$ over the monad $\mathbb{T}$.*
2. *augmented operad structures on $M \amalg I$ bijectively corresponds to algebra structures on $M$ over the monad $\tilde{\mathbb{T}}$.*

*Proof.* Unwinding the definition, the restriction $\mathbb{T}^{(2)}M \to M$ of the $\mathbb{T}$-algebra structure map determines the partial composition of operads and $I \to \mathbb{T}M$ determines the identity operation. It is also straightforward to check the other direction and the augmented case. □

Using this characterization, we get the construction of *free (augmented) operads* for free:

**Corollary 1.4.6.** *For a $\Sigma$-module $M$, the operad $\mathbb{T}(M)$ (resp. $\tilde{\mathbb{T}}(M) \oplus I$) is free in the category* $\mathbf{Op}(\mathcal{M})$ *(resp.* $\mathbf{aug\text{-}Op}(\mathcal{M})$*). We denote the free augmented operad $\tilde{\mathbb{T}}(M) \oplus I$ on $M$ by $\mathcal{T}M$.*

### 1.5 non-symmetric operad

Once we forget about the symmetric group action and use $\mathbb{N}$, the discrete category of positive integers, instead of $\Sigma$, then we get a notion of non-symmetric (ns for short) operads. Equivalently, it is an algebra over the *planar* tree monad. This notion is simpler[3], but it can only encode algebras that do not need any permutation in their definition[4]. The basic example is As, the associative (ns-)operad, which is defined by $\mathrm{As}(n) = I$ for all $n \geq 1$. In most cases, theorems about operads hold in a parallel manner for ns-operads, and As is the simplest non-trivial example that we should keep in mind.

### 1.6 Algebras as operads

The monoid object (or unital associative algebra) in $\mathcal{M}$ can be identified with operads with only unary operations. Equivalently, if we employ *ladders* instead of trees we recover the notion of monoids. This reduction is often enlightening and should be always kept in mind since almost all constructions in this note for (co)operads recovers the simpler construction for (co)algebras.

## 2 Cooperads

The notion of cooperads is the "dual" of that of operads, in the sense that cooperads are about decomposing operations, while operads are about composing operations. Though the notion of cooperads can be defined on the same line as operads, it involves some subtleties:

*Warning.* In each definition of operads, everything was colimit: even the monoidal product was the notion of this side. Thus, the equivalence of the definitions are completely formal, basically relies on coend calculus. Here in the definition of cooperads, we have some choices about to what extent we dualize these colimits to limits. For defining exact dual we should dualize everything, but the monoidal product does not behave like limits in non-cartesian examples.

Here, for the sake of simplicity, we compromise by restricting ourselves to examples $\mathcal{M} = (\mathbf{Mod}_{\mathbb{K}}, \otimes, \mathbb{K})$, $(\mathbf{grMod}_{\mathbb{K}}, \otimes, \mathbb{K})$, $(\mathbf{dgMod}_{\mathbb{K}}, \otimes, \mathbb{K})$ where $\mathbb{K}$ is a field of characteristic 0. These categories have very special properties that certain limits and colimits coincide: it has biproducts $\oplus$ and for any finite group $G$, the natural map $X^G \to X_G$ between invariants and coinvariants is an isomorphism. Also note that, by Maschke's theorem, $\mathbb{K}[G]$ is semisimple and therefore all $\mathbb{K}[G]$-module are automatically projective and injective, and in particular, we have Künneth isomorphism. Most of the theory can be generalized to modules over arbitrary commutative rings, under some assumptions such as projectivity of modules that appear (see [3] for details).

[3] And in the following characteristic zero assumption of the base field is unnecessary for nonsymmetric operads

[4] For example, to express the Jacobi identity $[x, [y, z]] + [y, [z, x]] + [z, [x, y]]$ we need cyclic permutation of variables.

## 2.1 Definitions of cooperads and conilpotence

**Definition 2.1.1.**

1. A *cooperad* $(\mathcal{C}, \Delta, \varepsilon)$ in $\mathcal{M}$ is a comonoid in $(\mathcal{M}^{\Sigma^{\mathrm{op}}}, \circ, I)$, whose structure maps are $\Delta : \mathcal{C} \to \mathcal{C} \circ \mathcal{C}$ and $\varepsilon : \mathcal{C} \to I$.
2. A cooperad $\mathcal{C}$ is *coaugmented* when it is equipped with a morphism of cooperads $\eta : I \to \mathcal{C}$ and therefore a splitting $\mathcal{C} \cong \bar{\mathcal{C}} \oplus I$. The image under $\eta$ of the generator (in an appropriate sense) of $I$ is often called identity cooperation and denoted by id.
3. A coaugmented cooperad is *conilpotent* if any successive nontrivial (i.e. not of the form $(\mathrm{id}; \mu)$ or $(\mu; \mathrm{id}, \ldots, \mathrm{id})$) decomposition of any cooperation in $\mathcal{C}$ terminates; see section 5.8.5 of [1] for the precise definition.

For our purpose, conilpotence can be better described by the following characterization:

**Proposition 2.1.2.**

1. *The reduced tree functor $\tilde{\mathbb{T}}$ admits a comonad structure $(\tilde{\mathbb{T}}, \Delta, \varepsilon)$, where $\Delta$ is given by sending a tree to the sum of all possible decomposition as a "tree of nontrivial trees", and $\varepsilon$ is given by the projection onto the weight 1 part.*
2. *For a $\Sigma$-module $M$, The conilpotent cooperad structure on $M \oplus I$ is equivalent to the coalgebra structure on $M$ over the comonad $\tilde{\mathbb{T}}$.*

*Proof.* This is the dual of the Proposition 1.4.5. Conilponence is encoded in the fact that any tree can be decomposed as a tree of nontrivial trees only for finite times. □

**Corollary 2.1.3.** *For a $\Sigma$-Module $M$, the conilpotent cooperad $\tilde{\mathbb{T}}(M) \oplus I$ is cofree in the category* **conil-coOp**, *which we will denote by $\mathcal{T}^c M$.*

# 3 dg-operads and twisting morphisms

dg-operads (resp. graded operads) are operads in the category $(\mathbf{dgMod}_k, \otimes, k)$ (resp. $(\mathbf{grMod}_k, \otimes, k)$).

## 3.1 General remarks on signs and differentials

*Remark* 3.1.1. For dg-modules, we will use the homological degree which is indicated by subscripts. We assume that dg-modules are nonnegatively graded (so that the spectral sequence in the black box works well). The degree of a homogenous element $x$ is denoted by $|x|$. For the symmetric monoidal structure, we apply the Koszul sign rule as follows:

- The differential on the tensor product $(A \otimes B)_n = \bigoplus_{0 \le k \le n} A_k \otimes B_{n-k}$ is given by $d_{A \otimes B}(a \otimes b) = d_A a \otimes b + (-1)^{|a|} a \otimes d_B b$.
- The symmetry isomorphism is given by $a \otimes b \mapsto (-1)^{|a||b|} b \otimes a$.
- The hom set $\mathrm{Hom}_{\mathbf{dgMod}_k}(A, B)$ is the set of (degree 0) chain maps. This is enriched to the internal hom $\underline{\mathrm{Hom}}(A, B) = (\underline{\mathrm{Hom}}(A, B)_\bullet, \partial)$ is given by $\underline{\mathrm{Hom}}(A, B)_n = \prod_{k \ge 0} \mathrm{Hom}_{\mathbf{Mod}_k}(A_k, B_{k+n})$ and $\partial f := [d, f] = d_B \circ f - (-1)^{|f|} f \circ d_A$.

The generality from Section 1.3 applies and gives the symmetric monoidal categories of dg and graded $\Sigma$-modules. By construction, the composition product of two dg $\Sigma$-modules agrees with the composition product of the underlying graded $\Sigma$-modules as graded $\Sigma$-modules. For describing Leibniz rule in dg operads, it is convenient to introduce the following notation:

**Definition 3.1.2.** For maps between $\Sigma$-modules $f : M \to M'$ and $g : N \to N$, we define $f \circ' g : M \circ N \to M' \circ N$ by $\sum_{i+j=n-1} f \otimes_{\Sigma_n} (\mathrm{id}^{\otimes i} \otimes g \otimes \mathrm{id}^{\otimes j})$.

The differential on the composite product of $(M, d_M), (N, d_N)$ is given by the Leibniz rule $d_{M\circ N} = d_M \circ \mathrm{id}_N + \mathrm{id}_M \circ' d_N$. Written plainly, it looks like the "usual" Leibniz rule with Koszul signs involved:

$$(\mu; \nu_1, \ldots, \nu_k) \mapsto (d_M\mu; \nu_1, \ldots, \nu_k) + \sum_{i=1}^{k} (-1)^{|\mu|+|\nu_1|+\cdots+|\nu_{i-1}|}(\mu; \nu_1, \ldots, d_N\nu_i, \ldots, \nu_k),$$

where $(\mu; \nu_1, \ldots, \nu_k) \in M(k) \otimes_{\Sigma_k} (N(i_1) \otimes \cdots \otimes N(i_k)) \otimes_{\Sigma_{i_1}\times\cdots\times\Sigma_{i_k}} k[\Sigma_n]$. The differential of a dg-(co)operad can be considered as an additional structure on the underlying graded (co)operad.

**Definition 3.1.3.** Let $(\mathcal{P}, \gamma, \eta)$ (resp. $(\mathcal{C}, \Delta, \varepsilon)$) be a graded operad (resp. cooperad). We say that a homogenous map $d_{\mathcal{P}} : \mathcal{P} \to \mathcal{P}$ (resp. $d_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$)is a derivation (resp. coderivation) if the following diagram commutes:

$$\begin{tikzcd} \mathcal{P}\circ\mathcal{P} \arrow[r, "\gamma"] \arrow[d, "d_{\mathcal{P}\circ\mathcal{P}}"'] & \mathcal{P} \arrow[d, "d_{\mathcal{P}}"] & \mathcal{C} \arrow[r, "\Delta"] \arrow[d, "d_{\mathcal{C}}"'] & \mathcal{C}\circ\mathcal{C} \arrow[d, "d_{\mathcal{C}\circ\mathcal{C}}"] \\ \mathcal{P}\circ\mathcal{P} \arrow[r, "\gamma"] & \mathcal{P} & \mathcal{C} \arrow[r, "\Delta"] & \mathcal{C}\circ\mathcal{C}. \end{tikzcd}$$

We denote the linear space of derivations on $\mathcal{P}$ (resp. coderivations on $\mathcal{C}$) by $\mathrm{Der}(\mathcal{P})$ (resp. $\mathrm{coDer}(\mathcal{C})$).

This condition again looks like a usual Leibniz rule if we see $\gamma$ as a product.

*Remark* 3.1.4. $d_{\mathcal{P}}$ defines a structure of a dg-operad on the graded operad $(\mathcal{P}, \gamma, \eta)$ if it is a square-zero coderivation. A similar statement also holds for cooperads.

The following lemma is convenient when we define (co)derivations on (co)free (co)operads in terms of generators.

**Lemma 3.1.5.** *For any graded $\Sigma$-module $E$, we have the following isomorphisms:*

1. $\mathrm{Der}(\mathcal{T}(E)) \xrightarrow{\cong} \mathrm{Hom}_{\mathbf{grMod}^{\Sigma^{\mathrm{op}}}}(E, \mathcal{T}(E))$ *which is given by restriction to the generators.*
2. $\mathrm{coDer}(\mathcal{T}^c(E)) \xrightarrow{\cong} \mathrm{Hom}_{\mathbf{grMod}^{\Sigma^{\mathrm{op}}}}(\mathcal{T}^c(E), E)$ *which is given by composition with the projection $\mathcal{T}^c(E) \to E$.*

## 3.2 Twisting morphisms

Let $\mathcal{P}$ be an augmented dg-operad, $\mathcal{C}$ be a coaugmented dg-cooperad. We define a space of $\Sigma$-equivariant maps by

$$\mathrm{Hom}_\Sigma(\bar{\mathcal{C}}, \bar{\mathcal{P}}) \coloneqq \prod_{n\geq 0} \mathrm{Hom}_{\Sigma_n}(\bar{\mathcal{C}}(n), \bar{\mathcal{P}}(n)).$$

Then we define a "linearized convolution product" $f \star g$ of $f, g \in \mathrm{Hom}_\Sigma(\bar{\mathcal{C}}, \bar{\mathcal{P}})$ by

$$f \star g : \mathcal{C} \xrightarrow{\Delta_{(1)}} \mathcal{C} \circ_{(1)} \mathcal{C} \xrightarrow{f \circ_{(1)} g} \mathcal{P} \circ_{(1)} \mathcal{P} \xrightarrow{\gamma_{(1)}} \mathcal{P}.$$

This is a pre-Lie product in the following sense:

**Definition 3.2.1.**

1. A dg-pre-Lie algebra $(L, \partial, \star)$ is a dg-module $(L, \partial)$ with a binary product $\star : L \otimes L \to L$ which commute with the differentials and whose associator is right commutative i.e.,
$$(x \star y) \star z - x \star (y \star z) = (-1)^{|y||z|}((x \star z) \star y - x \star (z \star y)).$$
2. A dg-Lie algebra $(L, \partial, [\text{-},\text{-}])$ is a dg-module $(L, \partial)$ with a binary product $[-, -] : A \otimes A \to A$ which commute with the differentials and satisfies the anticommutativity $[x, y] = -(-1)^{|x||y|}[y, x]$ and $\mathrm{ad}_z \coloneqq [z, \text{-}]$ is a derivation for all $z$, i.e., $[z, [x, y]] = [[z, x], y] + (-1)^{|x||z|}[x, [z, y]]$.

The notion of pre-Lie algebras is a generalization of that of associative algebras. It is straightforward to check that the antisymmetrization of a dg-pre-Lie algebra $[f, g] = x \star y - (-1)^{|x||y|} y \star x$ gives a dg-Lie algebra $(L, [\text{-},\text{-}])$. The following definition plays an important role in deformation theory:

**Definition 3.2.2.** A Maurer-Cartan element of a dg-Lie (resp. pre-Lie) algebra is an element $\alpha \in L_{-1}$ which satisfies the Maurer-Cartan equation $\partial\alpha + \frac{1}{2}[\alpha, \alpha] = 0$ (resp. $\partial\alpha + \alpha \star \alpha = 0$).

A Maurer-Cartan element of the convolution dg-pre-Lie algebra $(\mathrm{Hom}_\Sigma(\mathcal{C}, \mathcal{P}), \partial, \star)$ whose compositions with augmentation and coaugmentation are 0 is called a *twisting morphism* from $\mathcal{C}$ to $\mathcal{P}$. The set of twisting morphism is denoted by $\mathrm{Tw}(\mathcal{C}, \mathcal{P}) \subset \mathrm{Hom}_\Sigma(\bar{\mathcal{C}}, \bar{\mathcal{P}})_{-1}$. One can readily see that $\mathrm{Tw}(\mathcal{C}, \mathcal{P})$ is functorial in $\mathcal{C}$ and $\mathcal{P}$. It will be proven in the next section that they are representable in both variables.

# 4 Bar and cobar construction

In this section, we establish a pair of functors $\Omega : \textbf{conil-dgcoOp} \leftrightarrows \textbf{aug-dgOp} : \mathrm{B}$ with natural isomorphisms

$$\mathrm{Hom}_{\textbf{aug-dgOp}}(\Omega\mathcal{C}, \mathcal{P}) \cong \mathrm{Tw}(\mathcal{C}, \mathcal{P}) \cong \mathrm{Hom}_{\textbf{conil-dgcoOp}}(\mathcal{C}, \mathrm{B}\mathcal{P}).$$

Define the suspension (resp. desuspension) of a dg $\Sigma$-module $M$ by $sM \coloneqq s\mathbb{K} \otimes M$ (resp. $s^{-1}M \coloneqq s^{-1}\mathbb{K} \otimes M$[5]), where $s\mathbb{K}$ (resp. $s^{-1}\mathbb{K}$) is the $\Sigma$-module concentrated in arity 1 and degree $|s| = 1$. Observe that we have a sequence of natural isomorphisms

$$\begin{aligned} \mathrm{Hom}_{\textbf{aug-grOp}}(\mathcal{T}(s^{-1}\bar{\mathcal{C}}), \mathcal{P}) &\cong \mathrm{Hom}_{\Sigma}(s^{-1}\bar{\mathcal{C}}, \bar{\mathcal{P}})_0 \\ &\cong \mathrm{Hom}_{\Sigma}(\bar{\mathcal{C}}, \bar{\mathcal{P}})_{-1} \\ &\cong \mathrm{Hom}_{\Sigma}(\bar{\mathcal{C}}, s\bar{\mathcal{P}})_0 \cong \mathrm{Hom}_{\textbf{conil-grcoOp}}(\mathcal{C}, \mathcal{T}^c(s\bar{\mathcal{P}})). \end{aligned}$$

We will define a square-zero (co)derivation respectively on $\mathcal{T}(s^{-1}\bar{\mathcal{C}})$ and $\mathcal{T}^c(s\bar{\mathcal{P}})$ to define dg-(co)operads $\Omega\mathcal{C}$ and $\mathrm{B}\mathcal{P}$ so that the above isomorphisms restricts to the desired adjunction.

## 4.1 Differential on bar construction

Let $d_1$ be the differential on the cofree conilpotent dg-cooperad $\mathcal{T}^c\bar{\mathcal{P}}$. We define $d_2$ to be a unique coderivation which lifts the following map:

$$d_2 : \mathcal{T}^c(s\bar{\mathcal{P}}) \twoheadrightarrow \mathcal{T}^c(s\bar{\mathcal{P}})^{(2)} \cong s\bar{\mathcal{P}} \circ_{(1)} s\bar{\mathcal{P}} \to s\bar{\mathcal{P}},$$

where the last map is given by $(s\mu; \mathrm{id}, \dots, s\nu, \dots, \mathrm{id}) \mapsto (-1)^{|\mu|} s\mu \circ_i \nu$.

**Lemma 4.1.1.** $d_1^2 = d_2^2 = 0$, $d_1 d_2 + d_2 d_1 = 0$, $(d_1 + d_2)^2 = 0$.

*Proof.* The first two points are verified directly using the Koszul sign rule. The third point follows from the first two. □

As in the classical bar construction for dg-algebras, the differential $d_1$ is the differential coming from the original differential on each term, whereas $d_2$ corresponds to "removing the bar | (this was the shorthand notation for $\otimes$ in tensor coalgebra)."

**Definition 4.1.2.** We define the *bar construction* of an augmented operad $\mathcal{P}$ as the dg cooperad whose underlying graded cooperad is $\mathcal{T}^c(s\mathcal{P})$ and whose differential is $d = d_1 + d_2$ above. We denote this dg cooperad by $\mathrm{B}\mathcal{P}$.

**Proposition 4.1.3.** *Under the isomorphism* $\mathrm{Hom}_{\textbf{conil-grcoOp}}(\mathcal{C}, \mathcal{T}^c(s\bar{\mathcal{P}})) \to \mathrm{Hom}_{\Sigma}(\bar{\mathcal{C}}, \bar{\mathcal{P}})_{-1}; f \mapsto \mathrm{pr} \circ f|_{\bar{\mathcal{C}}}$*, a graded morphism $f$ is sent to a twisting morphism if and only if it commutes with the differentials, i.e. it is a morphism of conilpotent dg-cooperads $\mathcal{C} \to \mathrm{B}\mathcal{P}$.*

## 4.2 Differential on cobar construction

Dually, we can define the cobar construction $\Omega\mathcal{C}$ for a conilpotent cooperad $\mathcal{C}$. It is an augmented dg-operad whose underlying graded operad is $\mathcal{T}(s^{-1}\bar{\mathcal{C}})$ and the differential is $d = d_1 + d_2$ defined in the dual way.

**Proposition 4.2.1.** *Under the isomorphism* $\mathrm{Hom}_{\textbf{aug-grOp}}(\mathcal{T}(s^{-1}\bar{\mathcal{C}}), \mathcal{P}) \xrightarrow{\cong} \mathrm{Hom}_{\Sigma}(\bar{\mathcal{C}}, \bar{\mathcal{P}})_{-1}; f \to f|_{\bar{\mathcal{C}}}$*, a graded morphism $g$ is sent to a twisting morphism if and only if it commutes with the differentials, i.e. it is a morphism of augmented dg-operads $\Omega\mathcal{C} \to \mathcal{P}$.*

# 5 Homotopy transfer: $\Omega\mathcal{C}$-algebra structure is homotopy invariant

The reason why we care about bar and cobar construction is that it gives a resolution of an operad in a suitable sense and therefore any algebra structure on cobar construction is "homotopy invariant."

[5] More precisely, we discard the degree 0 part of $M$ when we desuspend.

**Definition 5.1.1.** Let $(V, d_V)$ and $(W, d_W)$ be dg vector spaces. We say $V$ is a *homotopy retract* of $W$ when they are given maps

$$\begin{tikzcd} (W, d_W) \arrow[loop left, "h"] \arrow[r, shift left, "p"] & (V, d_V) \arrow[l, shift left, "i"] \end{tikzcd}$$

where $i$ is a quasi-isomorphism and $h$ satisfies $\mathrm{id}_W - ip = d_W h + h d_W$.

**Example.**
1. If $V$ and $W$ are homotopy equivalent, then they are homotopy retracts of one another.
2. Since we are working with field coefficients, for any $(V, d_V) \in \mathbf{dgMod}_k$, the homology $(H_\bullet(V), 0)$ is a homotopy retract of $V$.

**Theorem 5.1.2.** *Let $\mathcal{C}$ be a conilpotent cooperad and $V$ be a homotopy retract of $W$. Any $\Omega\mathcal{C}$-algebra structure on $W$ can be transferred into a $\Omega\mathcal{C}$-algebra structure on $V$.*

*Proof.* Using the bar-cobar adjunction $\mathrm{Hom}(\Omega\mathcal{C}, \mathrm{End}_W) \cong \mathrm{Hom}(\mathcal{C}, \mathrm{BEnd}_W)$, it is enough to construct a morphism of cooperads $\mathrm{BEnd}_W \to \mathrm{BEnd}_V$ using the data of homotopy retract as follows:

1. First, we define a map $\mathcal{T}^c(s\mathrm{End}_W) \to s\mathrm{End}_V$ of $\Sigma$-modules. Elements in the domain are generated by elements of treewise tensor product $s\mu_1 \otimes \cdots \otimes s\mu_k \in (s\mathrm{End}_W)(t)$ for some nontrivial tree $t$ with $k$ vertices and $n$ leaves. We may assume that $\mu_1$ labels the root vertex. We send such a tree to the suspension of the operation
$$V^{\otimes n} \xrightarrow{i^{\otimes n}} W^{\otimes n} \xrightarrow{\gamma(\mu_1 \otimes h\mu_2 \otimes \cdots \otimes h\mu_k)} W \xrightarrow{p} V,$$
where $\gamma : (s\mathrm{End}_W)(t) \to s\mathrm{End}_W(n)$ is the treewise composition map. Equivalently, we obtain such operation on $V$ by decorating all leaves by $i$, the root by $p$, and all internal edges by $h$.
2. We lift the map in (1) to a morphism of graded cooperads $\mathcal{T}^c(s\mathrm{End}_W) \to \mathcal{T}^c(s\mathrm{End}_V)$.
3. It can be directly proven that the map in (2) commutes with the differentials.

□

Therefore in order to find an appropriate "homotopy invariant version" of $\mathcal{P}$-algebra, we want to find a quasi-isomorphism $\Omega\mathcal{C} \xrightarrow{\simeq} \mathcal{P}$, which is called a resolution. This will be achieved in two ways in the following sections: the bar-cobar resolution and the Koszul resolution.

# 6 Twisted composition product

## 6.1 Twisted composition product

Let $\mathcal{C}$ be a conilpotent cooperad and $\mathcal{P}$ be an augmented operad, $\alpha \in \mathrm{Tw}(\mathcal{C}, \mathcal{P})$ be a twisting morphism. We can form a *twisted* composition products $\mathcal{C} \circ_\alpha \mathcal{P}$ and $\mathcal{P} \circ_\alpha \mathcal{C}$, which is analogous to the geometric construction of twisted products, i.e. bundles[6]. See [5] for the original geometric idea.

Let $(\mathcal{P}, \gamma_\mathcal{P}, \eta_\mathcal{P})$ be a dg-operad. A left graded $\mathcal{P}$-module is a module over the monoid $\mathcal{P}$ in **grMod**, i.e. graded $\Sigma$-module $M$ with module structure maps $\gamma_M : \mathcal{P} \circ M$ which satisfies obvious action properties. We say that a map $d_M : M \to M$ is a derivation if the following diagram commutes:

$$\begin{tikzcd} \mathcal{P} \circ M \arrow[r, "\gamma_M"] \arrow[d, "d_{\mathcal{P}\circ M} = d_\mathcal{P} \circ \mathrm{id}_M + \mathrm{id}_\mathcal{P} \circ' d_M"'] & M \arrow[d, "d_M"] \\ M \arrow[r, "\gamma_M"] & M. \end{tikzcd}$$

We define similarly right graded $\mathcal{P}$-modules and its derivations. We denote both spaces of derivations by $\mathrm{Der}_\mathcal{P}(M)$. The following is an analog of 3.1.5 for free modules:

[6] I guess there is an analogous way to construct a twisted bundle from a flat connection on a bundle, where the flatness corresponds to the cocycle condition of pasting via parallel transport, but I'm not sure. cf. Grothendieck connection.

**Lemma 6.1.1.** *For any graded $\Sigma$-module $M$, the derivation on the free graded $\mathcal{P}$-modules $\mathcal{P} \circ M$ and $M \circ \mathcal{P}$ are characterized by the restriction on the generators, i.e. the following restrictions are isomorphisms:*

$$\mathrm{Der}_{\mathcal{P}}(\mathcal{P} \circ M) \xrightarrow{\cong} \mathrm{Hom}(M, \mathcal{P} \circ M), \quad \mathrm{Der}_{\mathcal{P}}(M \circ \mathcal{P}) \xrightarrow{\cong} \mathrm{Hom}(M, M \circ \mathcal{P}).$$

Let $\alpha \in \mathrm{Hom}_{\Sigma}(\bar{\mathcal{C}}, \bar{\mathcal{P}})_{-1}$ be any map of degree $-1$. First, we define $d_\alpha^l \in \mathrm{Der}_{\mathcal{P}}(\mathcal{P} \circ \mathcal{C})$ to be the unique derivation on the free left module which extends $\mathcal{C} \xrightarrow{\Delta} \mathcal{C} \circ \mathcal{C} \xrightarrow{\alpha \circ \mathrm{id}_{\mathcal{C}}} \mathcal{P} \circ \mathcal{C}$. Similarly, we define $d_\alpha^r \in \mathrm{Der}_{\mathcal{P}}(\mathcal{C} \circ \mathcal{P})$ as a unique derivation the free right module which extends $\mathcal{C} \xrightarrow{\Delta_{(1)}} \mathcal{C} \circ_{(1)} \mathcal{C} \xrightarrow{\mathrm{id}_{\mathcal{C}} \circ_{(1)} \alpha} \mathcal{C} \circ_{(1)} \mathcal{P} \to \mathcal{C} \circ \mathcal{P}$. We add this "twisting terms" to the ordinary differentials on the composition products to get $d_\alpha := d_{\mathcal{P} \circ \mathcal{C}} + d_\alpha^l$ on $\mathcal{P} \circ \mathcal{C}$ and $d_\alpha := d_{\mathcal{C} \circ \mathcal{P}} + d_\alpha^r$ on $\mathcal{C} \circ \mathcal{P}$.

**Lemma 6.1.2.** *When $\alpha$ is a twisting morphism, $d_\alpha$ is a differential (i.e. square-zero derivation of degree $-1$) on composition products $\mathcal{C} \circ \mathcal{P}$ and $\mathcal{P} \circ \mathcal{C}$. We denote this composition product with twisted differential by $\mathcal{C} \circ_\alpha \mathcal{P}$ and $\mathcal{P} \circ_\alpha \mathcal{C}$.*

*Proof.* Since $d_\alpha$ is a derivation of degree $-1$ by construction, it is enough to prove that it squares to zero. This follows from the equations $d_\alpha = d^l_{\partial(\alpha)+\alpha\star\alpha}$ (resp. $d^r_{\partial(\alpha)+\alpha\star\alpha}$) on $\mathcal{P} \circ \mathcal{C}$ (resp. $\mathcal{C} \circ \mathcal{P}$). The proof of this equation is based on the observation that $[d_\alpha^l, d_\beta^l] = d^l_{[\alpha,\beta]}$ and $[d_\alpha^r, d_\beta^r] = d^r_{[\alpha,\beta]}$. □

## 6.2 The black box(es) of the day

Here are two technically important lemmas that I will not even sketch the proof, which would involve some intricate argument using spectral sequences.

### 6.2.1 The first black box

In the natural isomorphism $\mathrm{Hom}_{\textbf{aug-dgOp}}(\Omega\mathcal{C}, \Omega\mathcal{C}) \cong \mathrm{Tw}(\mathcal{C}, \Omega\mathcal{C})$, the twisting morphism corresponding to $\mathrm{id}_{\Omega_\mathcal{C}}$ is called the *universal twisting morphism* and is denoted by $\iota$. Dually, in $\mathrm{Tw}(\mathrm{B}\mathcal{P}, \mathcal{P}) \cong \mathrm{Hom}_{\textbf{conil-dgcoOp}}(\mathrm{B}\mathcal{P}, \mathrm{B}\mathcal{P})$, the corresponding universal twisting morphism to $\mathrm{id}_{\mathrm{B}\mathcal{P}}$ is denoted by $\pi$.

*Remark* 6.2.1.

1. These are universal in the sense that any twisting morphism $\alpha : \mathcal{C} \to \mathcal{P}$ factors uniquely through $\iota$ and $\pi$, i.e. $\alpha = \pi \circ f_\alpha = g_\alpha \circ \iota$ for a unique $f_\alpha \in \mathrm{Hom}_{\textbf{aug-dgOp}}(\Omega\mathcal{C}, \mathcal{P})$ and $g_\alpha \in \mathrm{Hom}_{\textbf{conil-dgcoOp}}(\mathcal{C}, \mathrm{B}\mathcal{P})$.
2. These have explicit descriptions: $\pi$ is the map

$$\mathcal{T}^c(s\bar{\mathcal{P}}) \twoheadrightarrow s\bar{\mathcal{P}} \xrightarrow{s^{-1}} \bar{\mathcal{P}} \hookrightarrow \mathcal{P},$$

and $\iota$ is the map

$$\mathcal{C} \twoheadrightarrow \bar{\mathcal{C}} \xrightarrow{s^{-1}} s^{-1}\mathcal{C} \hookrightarrow \mathcal{T}(s^{-1}\mathcal{C}).$$

**Lemma 6.2.2** (acyclicity of the universal twisted composite products)**.** *For universal twisting morphisms $\pi : \mathrm{B}\mathcal{P} \to \mathcal{P}$ and $\iota : \mathcal{C} \to \Omega\mathcal{C}$, twisted composition products $\mathrm{B}\mathcal{P} \circ_\pi \mathcal{P}$, $\mathcal{P} \circ_\pi \mathrm{B}\mathcal{P}$, $\mathcal{C} \circ_\iota \Omega\mathcal{C}$, $\Omega\mathcal{C} \circ_\iota \mathcal{C}$ are acyclic.*

Geometric intuition behind the scene is that these twisted composition products are analogs of the fibrations $\Omega X \to \mathrm{Path}(X) \simeq * \to X$ and $G \to EG \simeq * \to BG$.

### 6.2.2 The second black box

The construction of the twisted composite product is functorial in the following sense:

*Remark* 6.2.3. Let $\alpha \in \mathrm{Tw}(\mathcal{C}, \mathcal{P})$ and $\alpha' \in \mathrm{Tw}(\mathcal{C}', \mathcal{P}')$ be twisting morphisms. Let $(f, g)$ be a morphism of twisting morphisms, i.e. $f : \mathcal{C} \to \mathcal{C}'$ and $g : \mathcal{P} \to \mathcal{P}'$ are morphisms of (co)operads and compatible twisting morphisms: $\alpha' \circ f = g \circ \alpha$. In this case, the composite products $f \circ g : \mathcal{C} \circ_\alpha \mathcal{P} \to \mathcal{C}' \circ_{\alpha'} \mathcal{P}'$ and $g \circ f : \mathcal{P} \circ_\alpha \mathcal{C} \to \mathcal{P}' \circ_{\alpha'} \mathcal{C}'$ are morphisms of dg-$\Sigma$-modules.

We need extra grading for our key technical lemma.

**Definition 6.2.4.** We say a dg-operad $\mathcal{P}$ is *weight-graded* if it is a sum of sub-dg-$\Sigma$-modules $\mathcal{P} = \bigoplus_{w \geq 0} \mathcal{P}^{(w)}$ whose weight grading is preserved by the operad structure maps (i.e. the weight of $\gamma(\mu; \nu_1, \ldots, \nu_k)$ is the sum of the weights of $\mu$ and $\nu$'s, and the weight of id is zero). A weight-graded dg-operad is *connected* if $\mathcal{P}^{(0)} = \mathbb{K}\mathrm{id}$, i.e. it is augmented and $\bar{\mathcal{P}} = \bigoplus_{w \geq 1} \mathcal{P}^{(w)}$. We similarly define (connected) weight-graded dg-cooperads.

Examples of weight-graded (co)operads include (co)free (conilpotent) graded (co)operads with differentials (such (co)operads are said to be *quasi-free*), and quadratic (co)operads which will be the main subject of Section 7.

**Lemma 6.2.5** (comparison lemma)**.** *Let $(f,g)$ be a morphism of weight-preserving twisting morphisms $\mathcal{C} \xrightarrow{\alpha} \mathcal{P}$, $\mathcal{C}' \xrightarrow{\alpha} \mathcal{P}'$ between weight-graded connected dg-(co)operads. We have the following two-out-of-three properties:*

1. *If two of $f$, $g$, $f \circ g : \mathcal{C} \circ_\alpha \mathcal{P} \to \mathcal{C}' \circ_\alpha \mathcal{P}'$ are quasi-isomorphism, then so is the third.*
2. *If two of $f$, $g$, $g \circ f : \mathcal{P} \circ_\alpha \mathcal{C} \to \mathcal{P}' \circ_\alpha \mathcal{C}'$ are quasi-isomorphism, then so is the third.*

The geometric intuition is the following: consider the morphism of fibrations

$$\begin{tikzcd} F \arrow[r, "f"] \arrow[d] & F' \arrow[d] \\ E \arrow[r, "e"] \arrow[d] & E' \arrow[d] \\ B \arrow[r, "b"] & B' \end{tikzcd}$$

of simply connected spaces. Then if two of $f, e, b$ induce homology isomorphisms, so is the third, proven either by homotopy long exact sequence or by comparison of Serre spectral sequences. In the algebraic context, only the latter option is available.

## 6.3 Fundamental theorem and bar-cobar resolution

Combining these two black boxes we obtain the fundamental theorem.

**Theorem 6.3.1** (The fundamental theorem of twisting morphisms)**.** *Let $\mathcal{P}$ and $\mathcal{C}$ be connected weight-graded dg-(co)operads and $\alpha : \mathcal{C} \to \mathcal{P}$ be a weight-preserving twisting morphism. The following are equivalent:*

1. *$\mathcal{C} \circ_\alpha \mathcal{P}$ is acyclic,*
2. *$\mathcal{P} \circ_\alpha \mathcal{C}$ is acyclic,*
3. *$f_\alpha : \mathcal{C} \to \mathrm{B}\mathcal{P}$ in Remark 6.2.1 is a quasi-isomorphism,*
4. *$g_\alpha : \Omega\mathcal{C} \to \mathcal{P}$ is a quasi-isomorphism.*

*Proof.* It suffices to prove (1)⇔(3), (1)⇔(4), (2)⇔(3), (1)⇔(4). Each of these follows immediately by comparing with the universal case 6.2.2, using the comparison lemma 6.2.5. ☐

**Definition 6.3.2.** We say that a twisting morphism $\alpha : \mathcal{C} \to \mathcal{P}$ is *Koszul* when either $\mathcal{C} \circ_\alpha \mathcal{P}$ or $\mathcal{P} \circ_\alpha \mathcal{C}$ is acyclic. We denote the set of Koszul morphisms by $\mathrm{Kos}(\mathcal{C}, \mathcal{P})$.

The universal twisting morphisms are Koszul.

**Theorem 6.3.3** (Bar-cobar resolution)**.** *For any augmented operad $\mathcal{P}$, the counit $\Omega\mathrm{B}\mathcal{P} \xrightarrow{\sim} \mathcal{P}$ is a quasi-isomorphism of dg-operads. Dually, for any conilpotent cooperad $\mathcal{C}$, the unit $\mathcal{C} \xrightarrow{\sim} \mathrm{B}\Omega\mathcal{C}$ is a quasi-isomorphism of dg-cooperads.*

*Proof.* In the weight-graded case, this is an obvious corollary of the fundamental theorem, but this holds in general. See [3] or the theorem below. ☐

*Warning.* It is not generally true that $\Omega$ preserves the quasi-isomorphism. It is true if $f : \mathcal{C} \to \mathcal{C}'$ is a quasi-isomorphism between *simply-connected* cooperads, i.e. $\bar{\mathcal{C}}_n, \bar{\mathcal{C}}'_n = 0$ when $n = 0, 1$. This roughly corresponds to the geometric situation, the homology-isomorphism on base and total space on fibration does not necessarily induce the homology-isomorphism on the fiber when the base is not simply-connected.

A more general and stronger result is the following [2]:

**Theorem 6.3.4.** *There exists a Quillen equivalence of presentable model categories*

$$\Omega_u : \{\text{curved conilpotent cooperads}\} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \{\text{(not necessarily reduced) dg-operads}\} : \mathrm{B}_u \ , \textit{where}$$

1. *The curved conilpotent cooperads are graded operad with "connection" and "curvature" instead of square-zero derivation,*
2. *dg-operads can have nullary operations,*
3. *The category of dg-operads is endowed with the projective model structure, i.e. fibrations are arity-wise surjection and weak equivalences are quasi-isomorphisms,*
4. *The category of curved conilpotent cooperads is endowed with right a model structure whose cofibrations and weak equivalences are created by $\Omega_u$. In fact, the cofibrations are precisely the degreewise injections and weak equivalence implies quasi-isomorphism.*

*In particular, all operads are fibrant and all cooperads are cofibrant, and therefore the counit gives a cofibrant replacement of operads, and unit gives a fibrant replacement of cooperads.*

# 7 Koszul duality of operads

For many graded operads we find in nature, a more refined theory of resolution is available.

## 7.1 Quadratic (co)operads

**Definition 7.1.1.** A *quadratic data* is a pair $(E, R)$, where $E \in \mathbf{grMod}^{\Sigma^{\mathrm{op}}}$ and $R \subset \mathbb{T}^{(2)}(E)$. A morphism of quadratic data $f : (E, R) \to (E', R')$ is a morphism of graded $\Sigma$-modules $f : E \to E'$ such that $\mathcal{T}f(R) \subset R'$. We associate a graded operad $\mathcal{P}(E, R)$ and a graded cooperad $\mathcal{C}(E, R)$ to a quadratic data as follows:

- $\mathcal{P}(E, R)$ is the operadic quotient of $\mathcal{T}(E)$ by the operadic ideal generated by $R$. In other words, it is the universal one among operads with a map of operads $\mathcal{T}(E) \to \mathcal{P}(E, R)$ such that the composite $R \hookrightarrow \mathcal{T}(E) \to \mathcal{P}(E, R)$ is zero.
- $\mathcal{C}(E, R)$ is the maximal subcooperad among those the composite $\mathcal{C} \hookrightarrow \mathcal{T}^c(E) \twoheadrightarrow \mathcal{T}^c(E)^{(2)}/(R)$ is zero.

We call $E$ the space of *generating operations* and $R$ the space of *relators*. When an operad $\mathcal{P}$ is isomorphic to $\mathcal{P}(E, R)$, we call $(E, R)$ a quadratic presentation of $\mathcal{P}$, and we call it a quadratic (co)operad.

Since these are defined as quotients or submodules of tree functor construction using homogenous relators, they inherit the weight grading and are connected.

## 7.2 Koszul dual (co)operad of an operad

**Definition 7.2.1.** Let $\mathcal{P} = \mathcal{P}(E, R)$ be a quadratic operad. We define the *Koszul dual cooperad* $\mathcal{P}^{¡}$ by $\mathcal{C}(sE, s^2R)$. Dually, if $\mathcal{C} = \mathcal{C}(E, R)$ be a quadratic cooperad, we define its Koszul dual operad $\mathcal{C}^{¡}$ by $\mathcal{P}(s^{-1}E, s^{-2}R)$.

$\mathcal{C}(sE, s^2R)$ is isomorphic to $\mathcal{C}(E, R)$ as non-graded modules, and $s$ is only shifting1 the degree of the generators. This constrcution obviously gives an equivalence of appropriate categories

$$(\text{-})^{¡} : \{\text{quadratic operads}\} \rightleftarrows \{\text{quadratic cooperdas with } E_0 = 0\} : (\text{-})^{¡} .$$

We can relate $\mathcal{P}^{¡}$ and $\mathrm{B}\mathcal{P}$ as follows:

Recall that the differential on the bar construction $\mathrm{B}\mathcal{P} := \mathcal{T}^c(s\bar{\mathcal{P}})$ was defined by $d = d_1 + d_2$, where $d_1$ is coming from the internal differential and $d_2$ is coming from "contraction of an edge of trees." Since the quadratic operads are graded operads, $d_1 = 0$ and therefore $d = d_2$. Since the operad $\mathcal{P}$ itself is weight-graded, we define the weight-grading on $\mathrm{B}\mathcal{P}$ to be the sum of the weights of the generators, i.e. the number of generating operations of $E$ which constitute the cooperation in $\mathrm{B}\mathcal{P}$. Since the differential preserves the weight, $\mathrm{B}\mathcal{P}$ is, as a dg-$\Sigma$-module, a direct sum of homogenous weight part: $\mathrm{B}\mathcal{P} = \bigoplus_{k \geq 0} (\mathrm{B}\mathcal{P})^{(k)}$. $\mathrm{B}\mathcal{P}$ has yet another grading, called *syzygy degree*. It is defined on $\mathcal{P}$ as (weight) $- 1$, and extended to $\mathrm{B}\mathcal{P}$ additively (equivalently, consider each "bar" in the bar construction have degree $-1$). This is a nonnegative cohomological grading (i.e. $d$ has degree 1). This makes each $(\mathrm{B}\mathcal{P})^{(k)}$ into a cochain complex, and we can consider $H^0(\mathrm{B}\mathcal{P})$.

**Proposition 7.2.2.** *The natural inclusion $\mathcal{P}^{i} \hookrightarrow \mathrm{B}\mathcal{P}$ of cooperads induce an isomorphism $\mathcal{P}^{i} \xrightarrow{\cong} H^0(\mathrm{B}\mathcal{P})$.*

We define the natural twisting morphism $\kappa : \mathcal{C}(sE, s^2R) \to \mathcal{P}(E,R)$ by $\mathcal{C}(sE, s^2R) \twoheadrightarrow sE \xrightarrow{s^{-1}} E \hookrightarrow \mathcal{P}(E,R)$, or equivalently the composite $\mathcal{C}(sE, s^2R) \hookrightarrow \mathrm{B}\mathcal{P}(E,R) \xrightarrow{\pi} \mathcal{P}(E,R)$. To check that it is actually a twisting morphism, it is enough to see that $\kappa \star \kappa = 0$, and it is done by writing down the map explicitly.

**Definition 7.2.3.** A quadratic operad $\mathcal{P}$ is called *Koszul* when the twisting morphism $\kappa : \mathcal{P}^{\text{i}} \to \mathcal{P}$ is Koszul.

By the fundamental theorem, $\mathcal{P}$ is Koszul iff the corresponding morphism of dg-(co)operads $\Omega\mathcal{P}^{\text{i}} \to \mathcal{P}$ (resp. $\mathcal{P}^{\text{i}} \to \mathrm{B}\mathcal{P}$) is a quasi-isomorphism iff the twisted composite product $\mathcal{P} \circ_\kappa \mathcal{P}^{\text{i}}$ or $\mathcal{P}^{\text{i}} \circ_\kappa \mathcal{P}$ (these are called *Koszul complex*) is acyclic. It is also equivalent to saying that $\mathrm{B}\mathcal{P} \to H_0(\mathrm{B}\mathcal{P})$ is quasi-isomorphism.

**Definition 7.2.4.** When $\mathcal{P}$ is a Koszul operad, we denote $\mathcal{P}_\infty := \Omega\mathcal{P}^{\text{i}}$ and the quasi-isomorphism $\mathcal{P}_\infty \xrightarrow{\sim} \mathcal{P}$ is called the *Koszul resolution*.

Koszul resolution is minimal in the following sense:

**Definition 7.2.5.** A *minimal operad* is a dg operad whose underlying graded operad is free $\mathcal{T}E$ with the following condition:

1. the differential $d$ is decomposable, i.e. it is the extension of some map $E \to \bigoplus_{w \geq 2} \mathcal{T}^{(w)}E$, and
2. $E$ admits a decomposition $E = \bigoplus_{k \geq 1} E^{(k)}$ satisfying $d(E^{(k+1)}) \subset \mathcal{T}(\bigoplus_{i=1}^{k} E^{(i)})$.

It is known that, when $\mathcal{P}$ admits a minimal model (i.e. a minimal operad $\mathcal{M}$ with the resolution $\mathcal{M} \to \mathcal{P}$ is), it is unique up to (non-unique) isomorphism. There is also a notion of *Koszul dual operad* of an operad, defined by $\mathcal{P}^!(n) := ((\mathrm{Hom}((s\mathbb{K})^n, s\mathbb{K}))^* \otimes \mathcal{P}^{\text{i}}(n))^*$, where $(\text{-})^*$ is the linear dual which turns cooperads into operads and arity-wise finite dimensional operads into cooperads.

**Proposition 7.2.6.** *1. $\mathcal{P}^!$ is quadratic.*

*2. If the space of generators $E$ is finite-dimensional in each arity, $(\mathcal{P}^!)^! \cong \mathcal{P}$.*

*3. $\mathcal{P}$ is Koszul if and only if $\mathcal{P}^!$ is Koszul.*

# 8 As and $A_\infty$

The nonsymmetric operad As is defined by $\mathrm{As}(n) = 1$ for $n \geq 1$ and $\mathrm{As}(0) = 0$. Operad structure is given by the identification $\mathrm{As}(n) \otimes \mathrm{As}(k_1) \otimes \cdots \otimes \mathrm{As}(k_n) = \mathrm{As}(k_1 + \cdots + k_n)$. In the case of topological or algebraic operads, the point or generator in $\mathrm{As}(n)$ is denoted by $\mu_n$. Algebras over As are nonunital associative algebras and free algebra over $V \in \mathcal{M}$ is given by $\overline{T}(V) = \coprod_{n \geq 1} V^{\otimes n}$.

The operad As is quadratic and Koszul, with $\mathrm{As}^! = \mathrm{As}$. The Koszul resolution $\mathrm{As}_\infty$ is usually denoted by $A_\infty$ and called *A-infinity operad*. $A_\infty(n)$ is isomorphic to the free module spanned by all the planar trees whose vertices have arity $\geq 2$ and with $n$ leaves.

Let $X$ be a topological space. Then it can be readily verified that the singular cohomology (with field coefficient) $A := H^\bullet_{\mathrm{Sing}}(X; \mathbb{K})$ is a homotopy retract of the dg-algebra of singular chains $C^\bullet_{\mathrm{Sing}}(X; \mathbb{K})$. By the homotopy transfer theorem, $A$ admits an $A_\infty$ structure. The product $A^{\otimes 3} \to A$ given by the corolla with three leaves corresponds to the Massey product, which detects, for example, the nontriviality of the complement of the Borromean rings (this cannot be detected by binary product, i.e. the cohomology ring structure).

# References


[1] Jean-Louis Loday and Bruno Vallette, *Algebraic operads*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 346, Springer-Verlag, Berlin, 2012.

[2] Brice Le Grignou, *Homotopy theories of unital algebras and operads*, `https://hal.archives-ouvertes.fr/tel-01375927v2`, Université Côte d'Azur, 2016.

[3] Benoit Fresse, *Koszul duality of operads and homology of partition posets*, Contemp. Math. 346, Amer. Math. Soc., 2004.

[4] Ieke Moerdijk and Bertrand Töen, *Simplicial Methods for Operads and Algebraic Geometry*, Advanced Courses in Mathematics - CRM Barcelona, Birkhäuser Basel, 2010

[5] E. H. Brown, Jr., *Twisted tensor products. I*, Ann. of Math. (2)**69**(1959), 223-246.