1. (10 points) Evaluate the following integral. (Hint: Convert to Spherical coordinates.)

\[ \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} \frac{1}{\sqrt{x^2 + y^2 + z^2}} \, dz \, dy \, dx \]

We are given as a hint to convert to spherical coordinates. To do this we use the transformation:

- \( x = \rho \cos \theta \sin \phi \)
- \( y = \rho \sin \theta \sin \phi \)
- \( z = \rho \cos \phi \)

We use the following formula to convert:

\[ \iiint_D f(x, y, z) \, dx \, dy \, dz = \iiint_{D^*} f(x(\rho, \theta, \phi), y(\rho, \theta, \phi), z(\rho, \theta, \phi)) |J| \, d\rho \, d\theta \, d\phi \]

In our case, \( D \) is the sphere, and \( D^* \) is a box in its coordinate system. \(|J|\) is the Jacobian of our transformation, which, since we are using spherical coordinates, is \( \rho^2 \sin \phi \). If we forget this, we can calculate it explicitly:

\[
J = \det \begin{pmatrix}
\cos \theta \sin \phi & -\rho \sin \theta \sin \phi & \rho \cos \theta \cos \phi \\
\sin \theta \sin \phi & \rho \cos \theta \sin \phi & \rho \sin \theta \cos \phi \\
\cos \phi & 0 & -\rho \sin \phi
\end{pmatrix} = -\rho^2 \cos^2 \theta \sin^3 \phi + \rho \sin \phi \sin \theta (-\rho \sin^2 \phi \sin \theta - \rho^2 \sin \theta \cos^2 \phi) - \rho^2 \sin \phi \cos^2 \theta \cos^2 \phi \\
= -\rho^2 \cos^2 \theta \sin^3 \phi + \rho \sin \phi \sin \theta (-\rho \sin \theta) - \rho^2 \sin \phi \cos^2 \theta \cos^2 \phi \\
= -\rho^2 \sin \phi \cos^2 \theta \sin^2 \phi - \rho^2 \sin \phi \sin^2 \theta - \rho^2 \sin \phi \cos^2 \theta \cos^2 \phi \\
= -\rho^2 \sin \phi (\cos^2 \theta) - \rho^2 \sin \phi (\sin^2 \theta) \\
= -\rho^2 \sin \phi
\]

But, since we don’t care about the sign, we get \(|J| = \rho^2 \sin \phi\).
Now, if we look at $f$, we notice that $\rho = \sqrt{x^2 + y^2 + z^2}$, so we get:

$$
\int_{-1}^{1} \int_{\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} \frac{1}{\sqrt{x^2 + y^2 + z^2}} dz dy dx
$$

$$
= \int_{0}^{2\pi} \int_{0}^{\pi} \int_{0}^{1} \rho \sin \phi d\rho d\phi d\theta
$$

$$
= \int_{0}^{2\pi} \int_{0}^{\pi} \sin \phi (\frac{1}{2} \rho^2) \Big|_{0}^{1} d\phi d\theta
$$

$$
= \int_{0}^{2\pi} \int_{0}^{\pi} \frac{1}{2} \sin \phi d\phi d\theta
$$

$$
= \int_{0}^{2\pi} \frac{1}{2} (-\cos \phi) \Big|_{0}^{\pi} d\theta
$$

$$
= \int_{0}^{2\pi} 1 d\theta
$$

$$
= \theta \Big|_{0}^{2\pi} = 2\pi
$$

So, our answer is $2\pi$.

Bonus: Third of a 4 part series on mammals:
Name a mammal of the Australian Outback (NOT Kangaroo!).