Section 7.6
(C)
Using what you learned in problem (A) and Section 7.6, solve the 2nd Order equation

\[ x'' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} x \]

Make sure your final answer is \( \mathbb{R} \) valued.

Section 7.8
(D)
In this problem we consider using techniques for the 2 dimensional 1st order equation that deal with repeated roots to solve such a 2nd order equation.

(i) Show that \( x = \zeta t e^{\lambda t} + \eta e^{\lambda t} \) (where \( \zeta \) and \( \eta \) are constant vectors) is a solution to

\[ x'' = Ax \]

if and only if

\[ A\zeta = \lambda^2 \zeta \]
\[ (A - \lambda^2)\eta = 2\lambda \zeta \]

From the first equation we have \( \zeta \) is an eigenvector of \( A \) with eigenvalue \( \lambda^2 \).

(ii) Now use what you learned in part (i) to solve

\[ x'' = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} x \]

Hint: you should get 4 solutions for your fundamental set of solutions. 2 of these solutions will come from the method used in problem (A).

(E)
Now we deal with 2 dimensional 2nd order equations of the form

\[ t^2 x'' = Ax \]

where \( A \) has a repeated root which gives only one linearly independent solution. Let \( t > 0 \)
(i) Show that \( x = \zeta t^r \ln(t) + \eta t^r \) where \( \zeta \) and \( \eta \) are constant vectors is a solution to
\[
 t^2 x'' = Ax
\]
if and only if
\[
 A\zeta = r(r - 1)\zeta \\
 (A - r(r - 1)I)\eta = (2r - 1)\zeta
\]
The first equation gives us that \( \zeta \) is an eigenvector of \( A \) with eigenvalue \( r(r - 1) \).

(ii) Using what you know from part (i) solve
\[
 t^2 x'' = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} x
\]
Hint: you should get 4 solutions for your fundamental set of solutions. 2 of these solutions will come from the method used in problem (B).