Proofs 1 Answer Key

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Question 1

We need to show that $\mathbb{C}$ is not an ordered field. Suppose it were, then either $i > 0$ or $i < 0$.

Case 1: $i > 0$
If $i > 0$ then by Definition 2 part ii $i * i > 0$ but $i * i = -1 < 0$ so $i > 0$ is impossible.

Case 2: $i < 0$
If $i < 0$ then $-i > 0$ so by Definition 2 part ii $(-i) * (-i) > 0$ but $(-i) * (-i) = i * i = -1 < 0$ so $i < 0$ is impossible.

Since both cases are impossible we have a contradiction so $\mathbb{C}$ cannot be an ordered field, hence $A \neq \mathbb{C}$

Question 2

We want to find an element of A that is not in $\mathbb{Q}$. We contend that $\sqrt{2}$ is such an element. It is a basic fact of algebra that $\sqrt{2} \notin \mathbb{Q}$. On the other hand $\mathbb{Q} \subset A$, So let $X = \{x \in \mathbb{Q} : x^2 < 2\}$, since $X \subset \mathbb{Q}$ and $\mathbb{Q} \subset A$ then $X \subset A$. Since X is clearly bounded above, and A satisfies the Least Upper Bound Property we must have the least upper bound of X in A. The least upper bound of X is $\sqrt{2}$. So $\sqrt{2} \notin \mathbb{Q}$ but $\sqrt{2} \in A$ so $\mathbb{Q} \neq A$

Question 3

First we show that if $x, y \in A$ and $x \neq 0$ there is an integer n such that $nx > y$. Suppose there isn't then $\{nx : x \in A \ n \in \mathbb{Z}\}$ is bounded above by y. So using the least upper bound property we may find a least upper bound for this set, call it M. Then, since $M - x < M$, we may use Definition 3 ii to find and n such that $M - x < nx < M$. But if $M - x < nx$ then $M < nx + x = (n + 1)x$ which means that M is not actually upper bound for $\{nx : x \in A \ n \in \mathbb{Z}\}$. This is a

contradiction, So there is some n so that $nx > y$.
Now, $b - a \neq 0$. So there is some integer n so that $n(b-a) > 1$. So nb and na differ by more than 1 So there is some integer q in between them. $na < q < b$ so $a < \frac{q}{n} < b$. But now we can find infinitely many by repeating the argument replacing a with $\frac{q}{n}$.

Question 4

Let $B \subset A$ be a set that is bounded from below. Let C=$\{-x : x \in B\}$. Since B is bounded below it follows that C is bounded above, hence has a least upper bound M. it follows that -M is the greatest lower bound of B.

Question 5

Let $X = \{a_i\}$ and $Y = \{b_i\}$. For the remainder of this problem we will write $I_n = I_{a_n b_n}$. Notice that since for any n $I_n \supset I_{n+1}$ we must have that
$a_1 < a_2 < ...$ and $b_1 > b_2 > ....$ This also implies that b_1 is greater than a_i for any i and a_1 is less than b_i for any i. So X is bounded above, and Y is bounded below. We may use the result of Question 4, and the least upper bound property of A to find a least upper bound of X and greatest lower bound of Y. Call them a and b respectively. We will show that $\bigcap_{n=1}^{\infty} I_n \supset I_{ab}$.

Suppose $x \in I_{ab}$. Then $a \leq x \leq b$ Since a is the least upper bound of X and b is the greatest lower bound of Y, then for any n, we have:

$$a_n \leq a \leq x \leq b \leq b_n \tag{1}$$

Hence $x \in I_n$ for any n so $x \in \bigcap_{n=1}^{\infty} I_n$ so $\bigcap_{n=1}^{\infty} I_n \supset I_{ab} \neq \emptyset$

Question 6

$$0 = \lim_{n\to\infty} Length(I_{a_n b_n}) = \lim_{n\to\infty}(b_n - a_n) = \lim_{n\to\infty} b_n - \lim_{n\to\infty} a_n \tag{2}$$

So $\lim_{n\to\infty} b_n = \lim_{n\to\infty} a_n$ hence in the notation of our answer to question 5 $a = b$ so I_{ab} is a single point.

Question 7

Let $X \subset B$ be bounded above. Let $a_1 \in X$. If a_1 is an upper bound of X then we are done since this would mean that a_1 is a least upper bound of X. So we assume that a_1 is not an upper bound of X. Now, since X is bounded above let b_1 be any upper bound. Again, if b_1 is the least upper bound we are done so we may assume that it is not. Let $I_1 = I_{a_1 b_1}$. Let $x = \frac{b_1 - a_1}{2}$. If x is an upper bound of X then let $I_2 = I_{a_1x}$. If x is not an upper bound of X then let a_2 be an element of X such that $a_2 > x$ and let $I_2 = I_{a_2b_1}$. In either case notice that $Length(I_2) < \frac{1}{2}Length(1_1)$. We repeat this process indefinitely, relabeling the greatest and smallest element of each interval to a_n and b_n, creating a chain of intervals

$$I_{a_1b_1} \supset I_{a_2b_2} \supset I_{a_3b_3} \supset ... \tag{3}$$

Such that $\lim_{n\to\infty} Length(I_{a_nb_n}) = 0$
It follows from question 6 that $\bigcap\limits_{n=1}^{\infty} I_{a_nb_n}$ is a single point $z \in B$. We wish to show that z is in fact the least upper bound of X.
Let $a \in X$, and $a \neq z$. Since $a \neq z$ we must have that $a \notin \bigcap\limits_{n=1}^{\infty} I_{a_nb_n}$ so there is some n so that $I_{a_nb_n}$ so that $a \notin I_{a_nb_n}$. Since b_n is an upper bound of X, so $a < b_n$ so we must have $a < a_n$ So we must have $a < z$, for any $a \in X$ so z is an upper bound of X. Furthermore, notice that we have actually shown z satisfies property ii of Definition 3 for least upper bounds so z is the least upper bound of X. Hence B satisfies the least upper bound property.