

Time Limit: 60 minutes.

Instructions: This test contains 10 short answer questions. All answers must be expressed in simplest form unless specified otherwise. Only answers written inside the boxes on the answer sheet will be considered for grading.

No Calculators.

Useful Formulas:

$$\sum_{n=1}^{\infty} \frac{x^{n-1}}{n!} = \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2\pi^2}\right) \quad \text{and} \quad \lim_{n \rightarrow \infty} n! = \lim_{n \rightarrow \infty} \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$$

1. A circle of radius 1 is tangent to the parabola $y = x^2 - 2x + 1$ at two points. Find the coordinates of the center of this circle.

Answer: $(1, 5/4)$

2. Johnny Bravo is building a head board with the outline of the function $y = x - x^2$ bounded by the x axis. He wants the head board to be half green and half red, but splitting the board down the middle would be too easy! Johnny wants a diagonal line starting at the origin that will divide the headboard in half. Find the equation of this line.

Answer: $y = (1 - \frac{1}{2^{\frac{1}{3}}})x$

3. Compute the following integral:

$$\int_0^{\pi} \frac{x \sin(x)}{a + \cos^2 x} dx$$

Answer: $(\frac{\pi}{2})^2$

4. Compute the following integral:

$$\int_0^{\infty} \frac{dx}{(x^2 + 1)(x^k + 1)}$$

Answer: $\frac{\pi}{4}$

5. Find $\lim_{n \rightarrow \infty} \sqrt{n^{200} + n^{100} + 1} - n^{100}$

Answer: $\frac{1}{2}$

6. Define:

$$f(x) = x^{x^x}$$

find $\frac{f'(e)}{f(e)}$

Answer: $e^{e-1} + 2e^e = e^{e-1}(1 + 2e)$

7. Compute the following product:

$$\prod_{n=1}^{\infty} \left(e \left(\frac{n}{n+1} \right)^n \sqrt{\frac{n}{n+1}} \right)$$

Answer: $\frac{\sqrt{2\pi}}{e}$

8. Compute the following integral

$$\int_{-2}^0 \frac{x^2 + x - 5}{(x-1)^2} e^x dx$$

Answer: -2

9. Compute the following sum:

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n}$$

Answer: 6

10. Compute the following product:

$$\prod_{n=1}^{\infty} \left(\frac{2n}{2n-1} \right) \left(\frac{2n}{2n+1} \right)$$

Answer: $\frac{\pi}{2}$