SOLUTIONS FOR: Explorations Unlimited Round – Additive Number Theory
Saturday, February 18, 2013

1. Sequences

Problem 1 (Sequence Problem: Euler's Continued Fraction)**.** Consider the monotone decreasing sequence given by $a_0 = 1$, $a_n = \frac{1}{1+a_{n-1}}$. As n becomes very large (goes to infinity), what happens to a_n? *You do not need calculus to solve this problem*: this is the same as solving for x in

$$1 + \frac{1}{2 + \frac{1}{2+\frac{1}{2+\dots}}} = x$$

SOLUTION:

We can write this as:

$$1 + \frac{1}{1 + 1 + \frac{1}{2+\frac{1}{2+\dots}}} = x$$

And since $x = \frac{1}{2+\frac{1}{2+\frac{1}{2+\dots}}}$, we can write this as:

$$1 + \frac{1}{1+x} = x$$

Putting everything to the same denominator, we get:

$$\frac{(1+x+1)}{1+x} = x$$
$$x + 2 = x + x^2$$
$$0 = x^2 - 2$$
$$2 = x^2$$
$$x = \sqrt{2}$$

So we know that $1 + \frac{1}{2+\frac{1}{2+\frac{1}{2+\dots}}}$ goes to (converges to) $\sqrt{2}$.

Problem 2 (Uniqueness of Infimum)**.** Prove that if a sequence S has an infimum, which we will call a_n, then this infimum is unique.

SOLUTION:
Let a be an infimum of a sequence S.
Assume, for the sake of contradiction, that the infimum of S is not unique; that is, there is a b that is also an infimum of a sequence S.
Then $a = b$, or $a < b$, or $a > b$. If $a < b$, then b is not a lower bound of S. If $a > b$, then a is not a lower bound of S. So $a = b$, but then the lower bound is unique, a contradiction. So the infimum of a sequence S is unique.

Problem 3. Let $S^{(1)} = \{0, 1\}$; $S^{(2)} = \{0, 2\}$. Compute $S^{(1)} + S^{(2)}$.

SOLUTION:
$S^{(1)} + S^{(2)} = \{0, 1, 2, 3\}$

They **do not** need to show work.
We had $0 + 0 = 0$, $0 + 2 = 2$, $1 + 0 = 1$, $1 + 2 = 3$.

Problem 4 (Sumset Lower Bound)**.** Let A and B be non-empty sets.

- Prove that $min(|A|, |B|) \leq |A + B| \leq |A||B|$.
- Explicitly give two sets $A \subset \mathbb{N}$ and $B \subset \mathbb{N}$ such that $|A + B| = min(|A|, |B|)$.

SOLUTION:

- We compute every addition of the form $a + b$ with $a \in A$ and $b \in B$, and we **keep** repeats, so we will have at least $min(|A|, |B|)$ elements. The upper bound is obtained by noting that there are $|A|$ choices for a and $|B|$ choices for b, giving a maximum number of permutations at $|A||B|$.
- For example, $A = \{0\}$ and $B = \{0\}$. We can check: $A + B = \{0\}$, so $|A + B| = 1$. $|A| = 1$, $|B| = 1$, $min(|A|, |B|) = 1$.

2. Basis and Density of a Sequence

Problem 5. Prove that $0 \leq \frac{A(n)}{n} \leq 1$ for all $n \neq 0$.

SOLUTION: $A(n)$ is at most n, because A is a sequence of natural numbers and $A(n)$ is the number of natural numbers in the sequence S which are smaller or equal to n, which can only be equal to or less than the number of natural numbers in the interval $[0, n]$. So $\frac{A(n)}{n}$ is at most 1. If A is the empty sequence, then $A(n) = 0$, and this is as low as $A(n)$ can go. So $\frac{A(n)}{n} = \frac{0}{n} = 0$ is as low as $\frac{A(n)}{n}$ can go. So $0 \leq \frac{A(n)}{n} \leq 1$.

Problem 6. Prove that if S is a basis of order k and $0 \in S$, then mS is a basis for any $m \geq k$.

SOLUTION:
S is a basis of order k, so kS contains all the natural numbers including 0. Now $c \in \{kS + S\}$ is of the form $c = a + b$ with $a \in kS$ and $b \in S$. Since $0 \in S$, let $b = 0$; we now have all elements in kS, in $kS + S$. So $kS + S$ has all the natural numbers. Because $0 \in S$, we can keep adding S and we never lose the elements that are in kS. It follows that $kS + nS$ has all natural numbers, or, written differently, that mS is a basis for any $m \geq k$.

Problem 7 (Density and natural numbers)**.** Prove that a sequence S contains every natural number if and only if $d(S) = 1$.

SOLUTION:
If S contains every natural number, then $S(n) = n$ for all n. Then for all n, $\frac{S(n)}{n} = \frac{n}{n} = 1$. So $d(S) = 1$.
Now assume $d(S) = 1$. Then by definition there exists an $m \in S$ such that $\frac{S(m)}{m} = d(S)$, and this m minimizes $\frac{S(m)}{m}$ by definition of density. So we have: $1 = \frac{S(m)}{m}$, and this is the smallest value of $\frac{S(n)}{n}$ possible. So $S(n) \geq n$, and by definition of $S(n)$ this means that there are at least n natural numbers in every interval from 0 to n for every n, so S contains every natural number.
So S contains every natural number if and only if $d(S) = 1$.

3. Several Results by Schnirelmann

Problem 8 (Schnirelmann's Inequality)**.** To streamline notation, let: $d(A) = \alpha, d(B) = \beta, A + B = C, d(C) = \gamma,\ l = a_{k+1} - a_k - 1$. Let $B(l)$ be the number of numbers of B in the segment $[1, l]$. Let $A(n)$ be the number of natural numbers in A that appear in the segment $[1, n]$. The key to this proof is to break down the segment $[1, n]$ into two parts: elements in A, and segments between such elements (which are not in A).

- Prove that $C(n) \geq A(n) + \Sigma B(l)$, with $\Sigma B(l)$ meaning the sum of $B(l)$'s for every segment of elements that are not in A that are between two elements in A. (Remember, we broke up the interval $[1, n]$ into different parts.)
- Prove that $B(l) \geq \beta l$.
- Use this to prove that $C(n) \geq A(n) + (n - A(n))\beta$.
- Use a similar method from (2) and (3) to prove that: $C(n) \geq \alpha n(1 - \beta) + n\beta$.
- Use this to prove that $\frac{C(n)}{n} \geq \alpha + \beta - \alpha\beta$, which completes our proof.

SOLUTION:

- Consider the segment $[1, n]$, $n \in \mathbb{N}$.
  $[1, n]$ has elements in A, and intervals of elements not in A in between.
  This has $A(n)$ numbers from A, also appearing in C.
  Now consider $[a_k, a_{k+1}]$. This has at least $B(l)$ numbers from B, because all numbers of the form $a_k + r$ where r is in B appear in this interval (they are in C and are in this interval), and there are as many as there are numbers of B in the segment $[1, l]$: $a_{k+1}, a_{k+2}, ..., a_{k+l}$.
  So $C(n) \geq A(n) + \Sigma B(l)$ with the summation being over all segments which are free of the numbers appearing in A.
- By definition, $B(l)$ is the number of natural numbers in B which do not exceed l, so $B(l) \leq l$.
  $d(B) = \beta$ is the density of the sequence B. Let h be the number such that $\frac{B(h)}{h} = d(B) = \beta$. Consider: $d(B) = \frac{B(h)}{h}(l)$. We know that for any density, we have: $0 \leq d(B) \leq 1$. So $\frac{B(h)}{h}(l) \leq l$. And we had $B(l) \leq 1$, so we have $\frac{B(h)}{h}(l) \leq B(l)$. Re-packing the notation on the left side gives us our desired result: $\beta l \leq B(l)$.
- We have $C(n) \geq A(n) + \Sigma B(l)$ and $B(l) \geq \beta(l)$, so $C(n) \geq A(n) + \beta\Sigma l$.
  And Σl is the sum of the lengths of all the segments free of elements in A, which is equal to the number of numbers of the segment $[1, n]$ which are not in A. This is given by $n - A(n)$. So $C(n) \geq A(n) + (n - A(n))\beta$.
- We just need to prove $\alpha n \leq A(n)$. We proceed just like in 2).
  By definition, $A(n)$ is the number of natural numbers in A which do not exceed n, so $A(n) \leq n$.
  $d(A) = \alpha$ is the density of the sequence A. Let y be the number such that $\frac{A(y)}{y} = d(A) = \alpha$.
  Consider: $d(A) = \frac{A(y)}{y}(n)$. We know that for any density, we have: $0 \leq d(A) \leq 1$. So $\frac{A(y)}{y}(n) \leq n$.
  And we had $A(n) \leq n$, so we have $\frac{A(y)}{y}(n) \leq A(n)$. Re-packing the notation on the left side, we have: $\alpha n \leq A(n)$.
  Now with that last inequality and the one in 3), we get: $C(n) \geq \alpha n - \alpha\beta n + n\beta$, which after factoring, gives us $C(n) \geq \alpha n(1 - \beta) + n\beta$.
- We start with: $C(n) \geq \alpha n(1 - \beta) + n\beta$.
  Since the sequences are composed of natural numbers, $n > 0$. [this is very important, make sure they have this for full points]
  So $\frac{C(n)}{n} \geq \alpha(1 - \beta) + \beta$. Expanding, we have: $\frac{C(n)}{n} \geq \alpha - \alpha\beta + \beta$, which is what we needed.

Problem 9 (Generalized Schnirelmann's Inequality)**.** Note that we can rearrange Schnirelmann's Inequality as such: $1 - d(A + B) \leq (1 - d(A))\,(1 - d(B))$
Prove the case for an arbitrary number of summands: $1 - d(A_1 + A_2 + ... + A_k) \leq \Pi_{i=1}^{k}\,(1 - d(A_i))$

SOLUTION:
We take care of a special case, and then we proceed by induction on k.
First note that if $1 - d(A_i) = 0$, then we have $0 = 0$, so $1 - d(A_1 + A_2 + ... + A_k) \leq \Pi_{i=1}^{k}\,(1 - d(A_i))$. So we can assume $1 - d(A_i)$, for all i, is non-zero.
Base case $k = 2$: this is Schnirelmann's Inequality, which we have just proven.
Induction hypothesis: $1 - d(A_1 + A_2 + ... + A_k) \leq \Pi_{i=1}^{k}\,(1 - d(A_i))$
Induction step $k + 1$: We can write $A_1 + A_2 + ... + A_k + A_{k+1} = A'_k + A_{k+1}$. By base case, $A'_k + A_{k+1} \leq (1 - d(A'_k))(1 - d(A_{k+1})$. By induction hypothesis, $1 - d(A'_k) \leq \Pi_{i=1}^{k}\,(1 - d(A_i))$. Combining both inequalities gives us: $A'_k + A_{k+1} \leq \Pi_{i=1}^{k}\,(1 - d(A_i))\,(1 - d(A_{k+1}))$. Which gives us our desired result, $1 - d(A_1 + A_2 + ... + A_k + A_{k+1}) \leq \Pi_{i=1}^{k+1}\,(1 - d(A_i))$.
This completes our inductive step, which completes our proof, giving us:
$1 - d(A_1 + A_2 + ... + A_k) \leq \Pi_{i=1}^{k}\,(1 - d(A_i))$

Problem 10 (Schnirelmann's Lemma)**.** Prove that for two sequences A and B with $0 \in A \cap B$, if $A(n) + B(n) \geq n - 1$, then n occurs in $A + B$. Remember that if n is in A or B, then n is in $A + B$.

SOLUTION:
Because $0 \in A \cap B$ and elements in A+B are of the form a+b with $a \in A$ and $b\ inB$, if $n \in A$ or $n \in B$, then $n \in A + B$.

So assume $n \notin A \cup B$.
We now assume, to begin our proof, that $A(n) + B(n) > n - 1$.
Since $n \notin A \cup B$, $A(n) = A(n-1) and B(n) = B(n-1)$, so we have $A(n-1) + B(n-1) > n - 1$.
Let us call the elements in A and in $[1, n]$: $a_1, a_2, ..., a_r$.
Similarly, let us call the elements in B and in $[1, n]$: $b_1, b_2, ..., b_s$.
By definition, we now have $A(n-1) = r$ and $B(n-1) = s$.
Now note that since $b_1, b_2, ...b_s$ are all in $[1, n]$ and all less than n, we have $n - b_1$, $n - b_2$,...$n - b_s$ in $[1, n]$.
And as already mentioned, we have $a_1, a_2, ..., a_r$ also in $[1, n]$.
There are $r + s = A(n-1) + B(n-1)$ of these numbers, and we know $r + s > n - 1$ by assumption, so one of the $n - b_i$'s is equal to one of the a_j's (with $i \leq r$ and $j \leq s$). So we have: $a_j = n - b_i$, so $a_j + b_i = n$. So n is in $A + B$, which completes our proof.

Problem 11 (Schnirelmann's Theorem)**.** Let $kA = A'$. It can be proven that if $d(A) > 0$, then $d(A') > \frac{1}{2}$ for sufficiently large k.
Using the previous results and this given fact, prove that any sequence A with $d(A) > 0$ is a basis for the sequence of natural numbers.

SOLUTION:
Let $A' = kA$.
By the given fact, we have: $\frac{A'(n)}{n} > \frac{1}{2}$
$A'(n) > \frac{n}{2}$
$A'(n) > \frac{1}{2}(n-1)$
So $A'(n) + A'(n) \geq n - 1$
And by Schnirelmann's Lemma, this means that n appears in $A' + A'$, and this n is arbitrary. So $kA + kA = 2kA$ contains every natural number, meaning A is a basis for the sequence of natural numbers.

4. Mann's Theorem

Problem 12 (Dyson Transform Property 1)**.** Prove that $0 \in A' \cup B'$.

SOLUTION:

$0 \in A \cap B$, so $0 \in A$ and $0 \in B$ by definition.
And $A' = A \cup \{B + \{e\}\}$, so $0 \in A'$. So $0 \in A' \cup B'$.

Problem 13 (Dyson Transform Property 2)**.** Prove that $d(B') \leq d(B)$.

SOLUTION:

If $b \in B \cap \{A - \{e\}\}$, $b \in B$ by definition.
If $b \notin B, b \not\in B \cap \{A - \{e\}\}$ by definition.
So the subsequence $\{1, ..., m\}$ with m s.t. $d_m(B) = d(B)$ can only lose elements after the Dyson transform.
So $d(B') \leq d(B)$.

Problem 14 (Dyson Transform Property 3)**.** Prove that $|A'| + |B'| = |A| + |B|$. (assuming A and B are finite sequences, so our definition of cardinality works)

SOLUTION:

We need to know how many elements are in $\{B + \{e\}\}$ and in A at the same time, so in $A \cap \{B + \{e\}\}$.
By problem 14), we know that this is at most $|B| - |B'|$.
So we can expand and simplify: $|A'| + |B'| = (|A| + |B| - |B'|) + |B'| = |A| + |B|$.

Problem 15 (Dyson Transform Property 4)**.** Prove that $A' + B' \subset A + B$.

SOLUTION:
For $a\ inA$, $b \in B$, $c \in A + B$ is of the form $c = a + b$.
For $a' \in A'$, $b' \in B'$, $c' \in A' + B'$ is of the form $c' = a' + b'$.
Now $b' \in B$ by definition of B', so let us only consider elements of the form $a' \notin A$.
And $b' = a - e$. If $a' \notin A$, then $a' \in \{B + \{e\}\}$ by definition of A'. So $a' = b + e$.
And so $a' + b' = b + e + a - e = a + b$. So any element in $A' + B'$ will also be in $A + B$.

Problem 16 (A piece of Mann's Theorem)**.** Assume $\frac{A(m)+B(m)}{m} \geq 1$. Prove Mann's Theorem for this case.

SOLUTION:

Assume $\frac{A(m)+B(m)}{m} \geq 1$.
Of course, $\frac{A(m)+B(m)}{m} = \frac{A(m)}{m} + \frac{B(m)}{m} = d(A) + d(B)$. So we have: $\frac{A(m)+B(m)}{m} \geq 1$. By Problem 10 (Schnirelmann's Lemma), all $n \geq 1$ is in $A + B$ (and we already know 0 is in $A + B$ since $0 \in A$ and $0 \in B$). So $A + B$ contains all natural numbers. So by Problem 7, $d(A + B) = 1$. Since $1 \geq 1$, we're done.